Constrained Utility Deviation-Risk Optimization and Time-consistent HJB Equation

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Abstract

In this paper we propose a unified utility deviation-risk model which covers both utility maximization and mean-variance analysis as special cases. We derive the time-consistent Hamilton-Jacobi-Bellman (HJB) equation for the equilibrium value function and significantly reduce the number of state variables, which makes the HJB equation derived in this paper much easier to solve than the extended HJB equation in the literature. We illustrate the usefulness of the time-consistent HJB equation with several examples which recover the known results in the literature and go beyond, including mean-variance model with stochastic volatility dependent risk aversion, utility deviation-risk model with state dependent risk aversion and control constraint, and constrained portfolio selection model. The numerical and statistical tests show that the utility and deviation-risk have significant impact on the equilibrium control strategy and the distribution of the terminal wealth.

Keywords: utility deviation-risk optimization, stochastic risk aversion, incomplete market, control constraint, time-consistent dynamic programming equation.

AMS MSC2010: 49L20, 60H30

1. Introduction

Expected utility and mean-variance are two predominant investment decision rules in portfolio selection. There has been huge literature for both approaches. The pioneering work of Merton [17] solves utility maximization problems with the stochastic control method and the dynamic programming principle (DPP). The martingale convex duality method can also solve utility maximization problems in a complete market by using the unique pricing kernel, see Pliska [19]; Karatzas and Shreve [13]; Cox and Huang [6]), and in an incomplete market with closed convex cone control constraints by using the so called minimal pricing kernel, see He and Zhou [10]. For the excellent exposition of these methods and applications, see Pham [18].

The mean-variance analysis of Markowitz [16] has long been recognized as the cornerstone of modern portfolio theory. The one-period model [16] has been greatly extended to general models, for example, the multi-period model in Li and Ng [14], the continuous time model in Zhou and

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Li [22], the incomplete market model in Lim [15], the nonnegative terminal state constrained model in Bielecki et al. [2], the semimartingale model in Xia [21], and many more. The methods of solving these models are different, for example, the "embedding" technique to change the problem into a standard linear quadratic (LQ) problem [22], the stochastic maximum principle to get the forward backward stochastic differential equations [15], the Lagrange multiplier method to accommodate the constraints [2], etc. The key common feature of these models is that the problem is solved at time 0 and the optimal control is the so-called pre-commitment optimal strategy which is not time-consistent, that is, when one solves the same problem at later time $t > 0$, the resulting optimal control is different from the one obtained at time 0.

To address the time-inconsistency issue, Basak and Chabakauri [1] adopt a game theoretic approach to solving a continuous time mean-variance problem with constant risk aversion for the investor who updates her nonlinear mean-variance objective and takes future updates, time-consistently, into account, and derives the equilibrium control. Björk and Murgoci [4] discuss more general time-inconsistent problems in discrete time setting and find the so-called subgame perfect Nash equilibrium strategies and the equilibrium value function that satisfies some extended Bellman equation (a system of highly complicated nonlinear equations). The continuous time counterpart is derived in Björk and Murgoci [3]. Applying the results of [3], Björk et al. [5] solve a continuous time mean-variance problem with state dependent risk aversion which is the key departure from the model in [1] and is "substantially more complicated" and "cannot easily be treated within the framework of [1], but it is a simple special case of the theory developed in [3]". It is still highly difficult to solve the extended HJB equation in [5] as it is a system of fully nonlinear partial differential equations (PDEs), or ordinary differential equations for a "natural choice" of state dependent risk aversion.

Despite great progress in solving mean-variance problems in a time-consistent way, especially ones in [1, 5], there are still many situations which are beyond the scope of the existing literature. Specifically, the mean-variance models in [1, 5] fail to answer some natural questions. The first one is that if we start with the initial wealth level being a non-square integrable random variable, even with the trivial control, the variance of terminal wealth does not exist. The second is that if there are control constraints, then the equilibrium control strategy may be different. The third is that if we consider a constrained portfolio selection which minimizes the variance of the terminal wealth subject to a minimum level of the expected terminal wealth, then with varying starting time, this leads to a family of time-inconsistent problems with time and state dependent risk aversion. The fourth is that if there exist exogeneous factors, it is reasonable that the risk aversion depends on these factors as well as the wealth process. The first two questions are not covered by the existing literature in [1, 3, 5]. The last two questions in theory are covered by the model formulation in [4, 3], with the help of the increased number of state variables, see Remark 2.3, however, the resulting extended HJB equation is highly complex, all but impossible to solve due to the curse of dimensionality.

Motivated by these open questions, we formulate a utility deviation-risk model with control constraints, which covers both utility maximization and mean-variance models as special cases. Specifically, we want to maximize the expected utility of the terminal wealth less the deviation-risk of the terminal wealth, adjusted by the stochastic risk aversion which depends on the initial time and state, and solve the problem time-consistently. This utility deviation-risk setup is similar to the model discussed in Wong et al. [20] with the following fundamental differences: [20] focuses on the existence of a pre-commitment optimal solution in a complete Black-Scholes market and characterizes its solution with a system of algebraic equations, whereas in this paper we discuss a dynamic portfolio optimization problem in a possibly incomplete market with control constraints and solve it with a time-consistent HJB equation, so called to reflect the time-consistent nature of our approach for the equilibrium value function.

We introduce a new deviation-risk measure which depends on a parameter β greater than or equal to one. If β is equal to two, the measure is the variance. If β is equal to one, the measure disappears and the problem becomes a pure utility maximization problem. If β tends to infinity, the measure forces the portfolio investment towards riskfree asset only. The flexibility of choice of β provides the whole spectrum of risk preferences for the investor. Another notable feature is that the terminal wealth is only required to be in the space of L^β, which lessens the integrability condition when β is less than two, compared with L^2 requirement under the variance measure.

The utility deviation risk optimization problem in this paper is a time-inconsistent model as the objective function contains a nonlinear function of the conditional expectation of the terminal wealth from the deviation risk measure, a function of the initial time and state from the risk aversion which can cover many interesting cases, including hyperbolic discounting and state dependent risk aversion. If one wants to use the method in [3, 5] to solve our problem, one would have to solve a system of extended HJB equations with six state variables and one time variable, all but impossible due to high dimensionality and nonlinearity of the extended HJB equations. In sharp contrast, we derive a time-consistent HJB equation with only two state variables and one time variable, which makes possible to find the solution and solve the problem.

The methodology used in this paper is the extension of the "total variance formula (TVF)", in the same spirit as the one in [1], but we give a rigorous treatment and go much beyond to cover general deviation-risk measure and time and state dependent risk aversion. This shows the TVF is a viable approach to solving time-inconsistent problems, partially addresses the criticism in [5] that the TVF "works very nicely in the MV case, but drawback of this particular approach is that it seems quite hard to extend the results to other objective functions than MV". The TVF method in this paper results in a simpler and more transparent and computable time-consistent HJB equation, compared with the extended HJB system in [5], especially with much reduced dimensionality of state variables.

The main contribution of this paper is that we propose a unified utility deviation-risk model with control constraints. We derive the time-consistent HJB equation for the equilibrium value function and recover the known time consistent mean variance results in the literature, especially those in [1, 5], and go beyond in terms of general deviation-risk measure, time and state dependent risk aversion, weak integrability condition, and control constraints. The numerical and statistical tests show that the utility and deviation-risk have significant impact on the equilibrium control strategy and the distribution of the optimal terminal wealth.

The rest of the paper is organized as follows. In Section 2 we formulate the utility deviation-risk portfolio selection problem with stochastic risk aversion and present the main result of the paper, Theorem 2.3, and its proof. In Section 3 we give three examples to illustrate the usefulness of Theorem 2.3 in solving various problems, including mean-variance model with volatility dependent risk aversion (Section 3.1), utility deviation-risk model with state dependent risk aversion (Section 3.2), and constrained portfolio selection model (Section 3.3). In Section 4 we do some numerical and statistical tests to see how the equilibrium control strategies and optimal terminal wealth distributions change with respect to changes of model parameters. Section 5 concludes the paper. Appendix contains the proofs of Proposition 3.1 and Theorems 3.4, 3.5 and 3.6.

2. Model and Main Results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, W a standard Brownian motion, Y a continuous time real-valued observable Markov process. Let $\{\mathcal{F}_t\}$ be the natural filtration generated by W and Y, augmented with all $\mathbb{P}$-null sets.

We study a portfolio decision problem of an investor (asset manager) operating in a financial market with two assets, one cash (B) and one stock (S), satisfying the following dynamics:

$$\begin{aligned} dB(t) &= r(t, Y(t))B(t)dt, & B(0) &= 1, \\ dS(t) &= S(t)(\mu(t, Y(t))dt + \sigma(t, Y(t))dW(t)), & S(0) &= s_0 > 0, \end{aligned}$$

where functions r, μ and $\sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}^+ := [0, \infty)$ are measurable. Hence

$$r(t) := r(t, Y(t)), \mu(t) := \mu(t, Y(t)), \sigma(t) := \sigma(t, Y(t))$$

are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic processes, representing the riskless interest rates, growth rates and volatility rates of the stock, respectively. Assume that

$$\int_0^T |r(t)| + |\mu(t)| + |\sigma(t)|^2 dt < +\infty, \mathbb{P} \text{ a.s.}$$

The dynamics of the investor's wealth satisfy the following controlled stochastic differential equation (SDE):

$$dX(t) = [r(t)X(t) + \theta(t)(\mu(t) - r(t))]dt + \theta(t)\sigma(t)dW(t), \tag{2.1}$$

with the initial endowment $X(0) > 0$, where θ is the dollar amount invested in the stock and $X(0) \in L^{\beta}_{\mathcal{F}_0}$ (the set of all $\mathcal{F}_0$-measurable random variables ξ with $E[|\xi|^{\beta}] < \infty$).

The following definitions and assumptions (S1-S5) are adopted throughout the paper.

S1. Define $\mathcal{U}[t,T] := \{\theta_{[t,T]}(s) : [t,T] \times \Omega \to \mathbb{U} | \theta_{[t,T]}(s)$ is $\{\mathcal{F}_s\}_{s\geq 0}$-adapted for $s \in [t,T]\}$, where $\mathbb{U}$ $(\subseteq \mathbb{R})$ is the control domain.

S2. Utility function u is strictly increasing, concave and twice continuously differentiable on $\mathbb{R}^+$. It is assumed in this paper that u has the following form:

$$u(x) = \frac{1}{\alpha} x^{\alpha}, \alpha \leq 1.$$

S3. Risk aversion process $\{\lambda(t)\}$ is a $\{\mathcal{F}_t\}_{t\geq 0}$-adapted nonnegative stochastic process with a specific form $\lambda(t) := \lambda(t, X(t), Y(t))$, where the map $\lambda : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is measurable.

S4. Define $\mathcal{R}[X|\mathcal{G}]$, the conditional deviation-risk of random variable $X \in L^{\beta}_{\mathcal{F}}$ given a σ-field $\mathcal{G} \subseteq \mathcal{F}$, by

$$\mathcal{R}[X|\mathcal{G}] = E[X^{\beta}|\mathcal{G}] - (E[X|\mathcal{G}])^{\beta}, \beta \geq 1.$$

S5. α, β are not equal to one at the same time, which is to ensure the well-posedness of the model.

Remark 2.1 *For any $\beta > 1$, we have $\mathcal{R}[X|\mathcal{G}] \geq 0$ due to Jensen's inequality. When $\beta = 2$, $\mathcal{R}[X|\mathcal{G}] = Var[X|\mathcal{G}]$, we recover the conditional variance of X given $\mathcal{G}$. As a well-defined deviation-risk measure generalized from variance, $\mathcal{R}[X|\mathcal{G}]$ requires less integrability condition on X when $\beta \in (1, 2)$.*

A state and control pair $(X(.), \theta(.))$ is called admissible for $[t, T]$, if

1. $\theta(.) \in \mathcal{U}[t, T]$, i.e., $\theta(s)$ is $\mathcal{F}_s$-adapted for $s \in [t, T]$;

2. Under the control $\theta(.)$, $X(.)$ is the unique solution to the SDE (2.1) for $[t, T]$;

3. $X(T) \in \mathcal{L}^{\beta}_{\mathcal{F}_T}$.

The set of all admissible controls is denoted by $\mathcal{U}_{ad}[t, T]$. The dynamic optimization problem of a utility deviation-risk investor at time $t \in [0, T]$ is given by

$$J(t) := J(t, X(t), Y(t)) = \max_{\theta_{[t,T]}(.) \in \mathcal{U}_{ad}[t,T]} E[u(X^{\theta_{[t,T]}}(T))|\mathcal{F}_t] - \lambda(t)\mathcal{R}[X^{\theta_{[t,T]}}(T)|\mathcal{F}_t]. \tag{2.2}$$

With varying time $t \in [0, T]$, (2.2) forms a family of optimization problems. Due to the structure of the deviation-risk measure $\mathcal{R}$ (nonlinear function of the conditional expectation $E[X^{\theta_{[t,T]}}(T)|\mathcal{F}_t]$) and the risk aversion process $\lambda(t)$ (dependence on the initial time t and initial states $X(t), Y(t)$), this leads to time-inconsistency. Therefore, we aim at finding the equilibrium control of problem (2.2). Let

$$V^{\theta_{[t,T]}}(t, x, y) = E[u(X^{\theta_{[t,T]}}(T))|\mathcal{F}_t] - \lambda(t)\mathcal{R}[X^{\theta_{[t,T]}}(T)|\mathcal{F}_t],$$

where $X(t) = x, Y(t) = y$. The equilibrium control definition in this paper is the same as that in Björk and Murgoci [3], Ekeland and Lazrak [7], Ekeland and Pirvu [8].

Definition 2.1 *Consider a control θ^* defined on $[0, T] \times \mathbb{R} \times \mathbb{R}$. Choose a fixed $\theta \in \mathbb{U}$, a fixed real number $h > 0$, sufficiently small. For $0 \leq t < T$, define a control $\hat{\theta}$ by*

$$\hat{\theta}(s, x, y) = \begin{cases} \theta, & t \leq s \leq t + h, x, y \in \mathbb{R}, \\ \theta^*(s, x, y), & t + h \leq s \leq T, x, y \in \mathbb{R}. \end{cases} \tag{2.3}$$

If

$$\liminf_{h \to 0} \frac{V^{\theta^*}(t, x, y) - V^{\hat{\theta}}(t, x, y)}{h} \geq 0$$

for all $\theta \in \mathbb{U}$, we say θ^* is an equilibrium control and the equilibrium value function J is defined by

$$J(t, x, y) = V^{\theta^*}(t, x, y).$$

From the definition, one can see that the equilibrium control is in the class of feedback (close loop) controls. Unlike open loop controls (see Hu et al [11]), the perturbation of the feedback control in $[t, t+h)$ affects the control process in $[t+h, T]$, as the control is adapted to the changing state. This is important in ensuring that the controlled state process X is a Markov process and the conditional expectation with filtration $\mathcal{F}_t$ is a function of initial time t and initial states $X(t), Y(t)$ due to the Markov property. One cannot conclude the same with open loop controls. In view of this, we equalize the two notations $\theta^*(t)$ and $\theta^*(t, x, y)$ for $t \in [0, T]$ with $X(t) = x, Y(t) = y$ and may switch between them in the rest of the paper.

The genuine idea of an equilibrium control is the following:

- If we denote by $\theta^*_{[t,T]}(.)$ the equilibrium control process on [t, T], we have

$$\theta^*_{[t,T]}(s) = \theta^*_{[t+h,T]}(s), t+h \le s \le T.$$

- Any perturbation of θ^* will give a worse value function.

Motivated by this, we define a "strong" equilibrium control as follows:

Definition 2.2 *("Strong" equilibrium control) Consider a control $\theta^*_{[t,T]}(.)$ on time interval $[t, T]$ for $t \in [0, T]$. For a fixed real number $h_0 > 0$ and any real number $h \in (0, h_0)$ and any arbitrary control $\theta(.)$ on $[t, t+h]$, if both of the following conditions hold:*

1. *(Time consistency) $\theta^*_{[t,T]}(s) = \theta^*_{[t+h,T]}(s), t+h \le s \le T$.*
2. *(Optimality) $V^{\theta^*_{[t,T]}}(t, x, y) \ge V^{\hat{\theta}_{[t,T]}}(t, x, y)$ for any $(x, y) \in \mathbb{R} \times \mathbb{R}$ with*

$$\hat{\theta}_{[t,T]}(s, x, y) = \begin{cases} \theta(s, x, y), & t \le s \le t+h, x, y \in \mathbb{R}, \\ \theta^*_{[t+h,T]}(s, x, y), & t+h < s \le T, x, y \in \mathbb{R}, \end{cases}$$

we say $\theta^*_{[t,T]}$ is a "strong" equilibrium control on [t, T].

The definition of "strong" equilibrium control in 2.2 is a similar analogue of that in He and Jiang [9] and Huang and Zhou [12].

Remark 2.2 *Apparently, if control $\theta^*_{[t,T]}(.)$ is a "strong" equilibrium control on $[t, T]$, then $\theta^*_{[t,T]}(.)$ is also an equilibrium control on $[t, T]$. In this paper, we will use this "strong" equilibrium control as an auxiliary tool to derive the HJB equation and verify that the solution of the corresponding HJB equation gives the equilibrium value function.*

Remark 2.3 *We observe that the key technique in the game theoretic method in [4, 3] is to introduce an auxiliary variable "y" which is of the same dimension as that of the state variable "x" in the HJB system. Their basic objective function is:*

$$\max_{\theta(.)} E_{t,x}[F(x, X(T))] + G(x, E_{t,x}[X(T)]),$$

where $F, G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are known functions. The HJB system involves the following functions:

$$f(t, x, y) = E_{t,x}[F(y, X(T))], g(t, x) = E_{t,x}[X(T)].$$

The auxiliary variable y is used to represent the initial state variable, so if there is no dependence of the initial state variable $X(t)$ in the objective function, then y is not needed. If one applies the theory developed in [4, 3] to solve the problem (2.2) in which time variable t as well as initial state variables $X(t), Y(t)$ appear in $\lambda(t)$, since functions F and G do not depend on t explicitly, one would need to group (t, x) as a new extended state variable x', where t has trivial dynamics. As a consequence, one would have to introduce an auxiliary variable $y' := (t', y)$, where t' corresponds to time variable t. The extended HJB system would involve time variable t and extended state variables x', y', which results in a fully nonlinear PDE in $2n+3$ dimensions, too complicated to be solved, even for $n = 1$. Motivated by overcoming this limitation (or curse of dimensionality) in [4, 3], we derive a time-consistent dynamic programming equation in Theorem 2.3, where the dimension of the resulting HJB system is $n+1$. For problem (2.2), the number of state variables is 2 for our time-consistent HJB equation (2.5), in sharp contrast to 6 for the extended HJB equation in [4, 3].

To deal with the time-inconsistent problem (2.2), we denote by $\theta^*_{[t,T]}(.)$ the equilibrium control process on $[t, T]$ and $X^{\theta^*_{[t,T]}}(T)$ the corresponding terminal wealth. We also define maps $U, V, W : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned} U(t,x,y) &= E[X^{\theta^*_{[t,T]}}(T)|\mathcal{F}_t] \triangleq U(t), \\ V(t,x,y) &= \mathcal{R}[X^{\theta^*_{[t,T]}}(T)|\mathcal{F}_t] \triangleq V(t), \\ M(t,x,y) &= E[u(X^{\theta^*_{[t,T]}}(T))|\mathcal{F}_t] \triangleq M(t), \end{aligned} \tag{2.4}$$

where $X(t) = x, Y(t) = y$.

We can now state the main result of the paper.

Theorem 2.3 *Assume there exists a "strong" equilibrium control, then the time-consistent dynamic programming equation of utility deviation-risk optimization problem with stochastic risk aversion (2.2) is given by*

$$0 = \max_{\theta(.)} \{E[\Delta M(t)|\mathcal{F}_t] - \lambda(t)E[\Delta V(t)|\mathcal{F}_t] - \lambda(t)\mathcal{R}[U(t) + \Delta U(t)|\mathcal{F}_t]\} \tag{2.5}$$

with the terminal conditions $U(T, x, y) = x$, $V(T, x, y) = 0$ and $M(T, x, y) = u(x)$, where for some process $H(t)$, $\Delta H(t) := H(t+h) - H(t)$ for some $h > 0$.

Proof: Set a control $\{\hat{\theta}_{[t,T]}\}$ defined as follows:

$$\hat{\theta}_{[t,T]}(s) = \begin{cases} \theta(s), & s \in [t, t+h], \\ \theta^*_{[t+h,T]}(s), & s \in (t+h, T], \end{cases}$$

and its corresponding wealth process becomes:

$$X^{\hat{\theta}_{[t,T]}}(s) = \begin{cases} X(s), & s \in [t, t+h], \\ X^{\theta^*_{[t+h,T]}}(s), & s \in (t+h, T]. \end{cases}$$

where $\{\theta(s)\}$ is any arbitrary control in $[t, t+h]$ and its corresponding wealth process $\{X(s) : t \le s \le t+h\}$. Note that $X^{\theta^*_{[t+h,T]}}(s)$, $s \ge t+h$, denotes the corresponding wealth process of adopting control $\theta^*_{[t+h,T]}$ with starting time $t+h$ and state $X(t+h)$. Hence $X^{\theta^*_{[t+h,T]}}(t+h) = X(t+h)$ and $X^{\hat{\theta}_{[t,T]}}(.)$ is well defined.

We have

$$
(2.6)\qquad
\begin{aligned}
& J(t)\\
\geq\ & E[u(X^{\hat{\theta}_{[t,T]}}(T))|\mathcal{F}_t] - \lambda(t)\mathcal{R}[X^{\hat{\theta}_{[t,T]}}(T)|\mathcal{F}_t]\\
=\ & E[E[u(X^{\hat{\theta}_{[t,T]}}(T))|\mathcal{F}_{t+h}] - \lambda(t+h)\mathcal{R}[X^{\hat{\theta}_{[t,T]}}(T)|\mathcal{F}_{t+h}]|\mathcal{F}_t]\\
& +E[(\lambda(t+h)-\lambda(t))\mathcal{R}[X^{\hat{\theta}_{[t,T]}}(T)|\mathcal{F}_{t+h}]|\mathcal{F}_t] - \lambda(t)\mathcal{R}[E[X^{\hat{\theta}_{[t,T]}}(T)|\mathcal{F}_{t+h}]|\mathcal{F}_t]\\
=\ & E[E[u(X^{\theta^*_{[t+h,T]}}(T))|\mathcal{F}_{t+h}] - \lambda(t+h)\mathcal{R}[X^{\theta^*_{[t+h,T]}}(T)|\mathcal{F}_{t+h}]|\mathcal{F}_t]\\
& +E[(\lambda(t+h)-\lambda(t))(\mathcal{R}[X^{\theta^*_{[t+h,T]}}(T)|\mathcal{F}_{t+h}] - \mathcal{R}[X^{\theta^*_{[t,T]}}(T)|\mathcal{F}_t])|\mathcal{F}_t]\\
& +E[(\lambda(t+h)-\lambda(t))\mathcal{R}[X^{\theta^*_{[t,T]}}(T)|\mathcal{F}_t]|\mathcal{F}_t] - \lambda(t)\mathcal{R}[E[X^{\theta^*_{[t+h,T]}}(T)|\mathcal{F}_{t+h}]|\mathcal{F}_t]\\
=\ & E[J(t+h)|\mathcal{F}_t] + E[(\lambda(t+h)-\lambda(t))(V(t+h)-V(t))|\mathcal{F}_t]\\
& +E[(\lambda(t+h)-\lambda(t))V(t)|\mathcal{F}_t] - \lambda(t)\mathcal{R}[U(t+h)|\mathcal{F}_t].
\end{aligned}
$$

The first equality holds due to

$$
\begin{aligned}
& \mathcal{R}[X|\mathcal{F}_t]\\
=\ & E[X^\beta|\mathcal{F}_t] - (E[X|\mathcal{F}_t])^\beta\\
=\ & E[E[X^\beta|\mathcal{F}_{t+h}]|\mathcal{F}_t] - (E[E[X|\mathcal{F}_{t+h}]|\mathcal{F}_t])^\beta\\
=\ & E[\mathcal{R}[X|\mathcal{F}_{t+h}]|\mathcal{F}_t] + E[(E[X|\mathcal{F}_{t+h}])^\beta|\mathcal{F}_t] - (E[E[X|\mathcal{F}_{t+h}]|\mathcal{F}_t])^\beta\\
=\ & E[\mathcal{R}[X|\mathcal{F}_{t+h}]|\mathcal{F}_t] + \mathcal{R}[E[X|\mathcal{F}_{t+h}]|\mathcal{F}_t].
\end{aligned}
$$

Equation (2.6) can be written as the following inequality:

$$
(2.7)\qquad
\begin{aligned}
0 \geq\ & E[\Delta J(t)|\mathcal{F}_t] + E[\Delta\lambda(t)\Delta V(t)|\mathcal{F}_t]\\
& +E[\Delta\lambda(t)|\mathcal{F}_t]V(t) - \lambda(t)\mathcal{R}[U(t)+\Delta U(t)|\mathcal{F}_t].
\end{aligned}
$$

The equality holds in (2.6) and (2.7) when $\{\theta(s) : t \leq s \leq t+h\} = \{\theta^*_{[t,t+h]}(s) : t \leq s \leq t+h\}$, hence we have

$$
(2.8)\qquad
\begin{aligned}
0 =\ & \max\nolimits_{\theta(.)} E[\Delta J(t)|\mathcal{F}_t] + E[\Delta\lambda(t)\Delta V(t)|\mathcal{F}_t]\\
& +E[\Delta\lambda(t)|\mathcal{F}_t]V(t) - \lambda(t)\mathcal{R}[U(t)+\Delta U(t)|\mathcal{F}_t].
\end{aligned}
$$

Note that $J(t) = M(t) - \lambda(t)V(t)$, so (2.8) is equivalent to (2.5). The proof is complete. □

Remark 2.4 *In [4, 3] setting, there is a function $F(x, X(T))$ in the objective function, a natural question to ask is if we can also extend our utility function from $u(X(T))$ to $u(x, X(T))$. The answer is not entirely clear at this stage. On the one hand, in computing the term $E[u(X(t), X^{\theta^*_{[t+h,T]}}(T))|\mathcal{F}_{t+h}]$, one seems need to introduce an additional variable $\tilde{y}$ into the system:*

$$M(t, x, \tilde{y}) = E[u(X(t), X^{\theta^*_{[t+h,T]}}(T))|\mathcal{F}_{t+h}],$$

where $X(t) = x, X(t+h) = \tilde{y}$. The state variables are doubled as in [4, 3], which would diminish the dimension reduction advantage of this paper. On the other hand, the objective function in (2.2) can be written as

$$J(t) = \max_{\theta_{[t,T]}(.)} E_{t,x,y}[F(t, x, y, X(T))] + G(t, x, y, E_{t,x,y}[X(T)])$$

with $X(t) = x, Y(t) = y$, *where*

$$\begin{aligned} F(t, x, y, X) &= U(X) - \lambda(t, x, y) X^\beta \\ G(t, x, y, X) &= \lambda(t, x, y) X^\beta. \end{aligned}$$

These functions are more general than $F(x, X(T))$ *in [4, 3] as initial time* t *appears in* F, G, *but are also more specific as they have some special structures. We succeed in deriving the time-consistent HJB equation without introducing additional auxiliary variables as in [4, 3] for that particular objective function, but not for general ones, which is still open and we leave it for future research.*

Corollary 2.4 *Assume the Markov process* Y *follows a diffusion process, i.e.,*

$$dY(t) = m(t, Y(t))dt + \nu(t, Y(t))dW_1(t),$$

where W_1 *is a standard Brownian motion and* W, W_1 *have correlation* ρ, *functions* $m, \nu : [0, \infty) \times \mathbb{R} \to \mathbb{R}^+$ *are measurable. Hence* $m(t) := m(t, Y(t))$, $\nu(t) := \nu(t, Y(t))$ *are* $\{\mathcal{F}_t\}_{t \geq 0}$*-adapted stochastic processes. The HJB equation is given by*

(2.9)

$$0 = \max_\theta \left\{ \begin{array}{l} \mathcal{L}(M)(t, x, y; \theta) - \lambda(t)\mathcal{L}(V)(t, x, y; \theta) \\ -\frac{1}{2}\beta(\beta - 1)U^{\beta-2}\lambda(t)[(\theta\sigma(t, y)\partial_x U)^2 + (\nu(t, y)\partial_y U)^2 + 2\rho\theta\sigma(t, y)\nu(t, y)\partial_x U \partial_y U] \end{array} \right\},$$

where

$$\begin{aligned} \mathcal{L}(f)(t, x, y; \theta) = \ & \partial_t f + [r(t, y)x + \theta(\mu(t, y) - r(t, y))]\partial_x f + \tfrac{1}{2}\theta^2\sigma(t, y)^2\partial_{xx} f \\ & + m(t, y)\partial_y f + \tfrac{1}{2}\nu(t, y)^2\partial_{yy} f + \rho\theta\sigma(t, y)\nu(t, y)\partial_{xy} f. \end{aligned} \tag{2.10}$$

Proof: By Itô's lemma, we have

$$\Delta f = \int_t^{t+h} \mathcal{L}(f)(s, X(s), Y(s))ds + \theta(s)\sigma(s)\partial_x f(s, X(s), Y(s))dW(s) + \nu(s, Y(s))\partial_y f(s, X(s), Y(s))dW_1(s).$$

Hence,

$$\begin{aligned} E[\Delta M(t)|\mathcal{F}_t] &= E[\textstyle\int_t^{t+h} \mathcal{L}(M)(s, X(s), Y(s))ds|\mathcal{F}_t], \\ E[\Delta V(t)|\mathcal{F}_t] &= E[\textstyle\int_t^{t+h} \mathcal{L}(V)(s, X(s), Y(s))ds|\mathcal{F}_t]. \end{aligned} \tag{2.11}$$

Define

(2.12)

$$\begin{aligned} Z(v) &= U(t) + E[\textstyle\int_t^{t+h} \mathcal{L}(U)(s, X(s), Y(s))ds|\mathcal{F}_t] \\ &\quad + \textstyle\int_t^v \mathcal{L}(U)(s, X(s), Y(s))ds - E[\int_t^v \mathcal{L}(U)(s, X(s), Y(s))ds|\mathcal{F}_t] \\ &\quad + \textstyle\int_t^v \theta(s)\sigma(s)\partial_x U(s, X(s), Y(s))dW(s) + \int_t^v \nu(s, Y(s))\partial_y U(s, X(s), Y(s))dW_1(s) \end{aligned}$$

for $t \leq v \leq t + h$. Clearly, Z is an $\mathcal{F}_t$ adapted process and

$$\begin{aligned} Z(t) &= U(t) + E[\int_t^{t+h} \mathcal{L}(U)(s, X(s), Y(s))ds|\mathcal{F}_t] \\ Z(t + h) &= U(t + h). \end{aligned}$$

Applying Itô's lemma to process $(Z(s))^\beta$, we have

(2.13)

$$\begin{aligned}
&\mathcal{R}[U(t)+\Delta U(t)|\mathcal{F}_t] \\
&= E[(U(t+h))^\beta|\mathcal{F}_t] - (E[U(t+h)|\mathcal{F}_t])^\beta \\
&= E[(Z(t+h))^\beta - (Z(t))^\beta|\mathcal{F}_t] \\
&= E\left[\int_t^{t+h} \beta Z(s)^{\beta-1}[\mathcal{L}(U)(s) - E[\mathcal{L}(U)(s)|\mathcal{F}_t]]ds|\mathcal{F}_t\right] \\
&+E\left[\int_t^{t+h} \tfrac{1}{2}\beta(\beta-1)Z(s)^{\beta-2}\left(\begin{array}{l}(\theta(s)\sigma(s,Y(s))\partial_x U(s))^2 \\ +(\nu(s,Y(s))\partial_y U(s))^2 \\ +2\rho\theta(s)\sigma(s,Y(s))\nu(s,Y(s))\partial_x U(s)\partial_y U(s)\end{array}\right) ds|\mathcal{F}_t\right].
\end{aligned}$$

Here we have used simplified notation $\mathcal{L}(U)(s) = \mathcal{L}(U)(s, X(s), Y(s))$. The terms $\partial_x U(s)$ and $\partial_y U(s)$ are defined similarly.

Inserting (2.11) and (2.13) into the time-consistent dynamic programming equation (2.5), dividing both sides by h and letting $h \to 0$, we derive the HJB equation (2.9) by noting $X(t) = x, Y(t) = y$. □

Applying the Feynman-Kac formula to functions M and U defined in (2.4), we have PDEs as follows:

$$0 = \mathcal{L}(M)(t,x,y;\theta^*) = \mathcal{L}(U)(t,x,y;\theta^*), \tag{2.14}$$

where $\mathcal{L}$ is defined in (2.10) and θ^* is taken to attain the maxima in HJB equation (2.9).

Proposition 2.5 *(Verification Theorem) Let $M, U, V \in C^{1,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ be a solution of the equations* (2.9) *and (2.14) with terminal conditions $U(T,x,y) = x, M(T,x,y) = u(x), V(T,x,y) = 0$ and θ^* is taken to attain the maxima in HJB equation (2.9). Then θ^* is an equilibrium control on* $[0,T]$.

Proof: Let X^{θ^*} be the corresponding wealth process associated with control θ^*. Since M, U is the solution of (2.14) with terminal conditions $M(T,x,y) = u(x), U(T,x,y) = x$, by the Feynman-Kac formula, we have

$$\begin{aligned}
M(t,x,y) &= E[u(X^{\theta^*}(T))|\mathcal{F}_t] \\
U(t,x,y) &= E[X^{\theta^*}(T)|\mathcal{F}_t],
\end{aligned}$$

where $X(t) = x$, $Y(t) = y$. The HJB equation (2.9) can be written as

$$\begin{aligned}
0 &= \mathcal{L}(M)(t,x,y;\theta^*) - \lambda(t)\mathcal{L}(V)(t,x,y;\theta^*) \\
&\quad -\frac{1}{2}\beta(\beta-1)U^{\beta-2}\lambda(t)[(\theta^*\sigma\partial_x U)^2 + (\nu\partial_y U)^2 + 2\rho\theta^*\sigma\nu\partial_x U\partial_y U].
\end{aligned}$$

Using (2.14) and dividing $\lambda(t)$, we have

$$0 = \mathcal{L}(V)(t,x,y;\theta^*) + \frac{1}{2}\beta(\beta-1)U^{\beta-2}[(\theta^*\sigma\partial_x U)^2 + (\nu\partial_y U)^2 + 2\rho\theta^*\sigma\nu\partial_x U\partial_y U].$$

The Feynman-Kac formula implies

$$V(t,x,y) = E[\int_t^T \frac{1}{2}\beta(\beta-1)U^{\beta-2}[(\theta^*\sigma\partial_x U)^2 + (\nu\partial_y U)^2 + 2\rho\theta^*\sigma\nu\partial_x U\partial_y U]ds|\mathcal{F}_t],$$

where $X(t)=x$, $Y(t)=y$. We also know that

$$\begin{aligned}
\mathcal{R}[X^{\theta^*}(T)|\mathcal{F}_t] &= E[(X^{\theta^*}(T))^\beta - (E[X^{\theta^*}(T)|\mathcal{F}_t])^\beta|\mathcal{F}_t] \\
&= E[(U(T,X^{\theta^*}(T),Y(T))^\beta - (U(t,x,y))^\beta|\mathcal{F}_t] \\
&= E[\int_t^T \beta U^{\beta-1}\mathcal{L}(U)(s,X(s),Y(s);\theta^*) \\
&\quad + \frac{1}{2}\beta(\beta-1)U^{\beta-2}[(\theta^*\sigma\partial_x U)^2 + (\nu\partial_y U)^2 + 2\rho\theta^*\sigma\nu\partial_x U\partial_y U]ds|\mathcal{F}_t] \\
&= V(t,x,y),
\end{aligned}$$

Since

$$\begin{aligned}
V^{\theta^*}(t,x,y) &= E[u(X^{\theta^*}(T))|\mathcal{F}_t] - \lambda(t)\mathcal{R}[X^{\theta^*}(T)|\mathcal{F}_t] \\
&= M(t,x,y) - \lambda(t)V(t,x,y),
\end{aligned}$$

it suffices to prove that

$$\liminf_{h\to 0}\frac{M(t,x,y)-\lambda(t)V(t,x,y)-V^{\widehat{\theta}}(t,x,y)}{h}\geq 0,$$

where $\hat{\theta}$ is defined in (2.3). Note that

$$\begin{aligned}
& V^{\widehat{\theta}}(t,x,y) \\
= {} & E[u(X^{\hat{\theta}}(T))|\mathcal{F}_t] - \lambda(t)\mathcal{R}[X^{\hat{\theta}}(T)|\mathcal{F}_t] \\
= {} & E[E[u(X^{\hat{\theta}}(T))|\mathcal{F}_{t+h}]|\mathcal{F}_t] - \lambda(t)E[\mathcal{R}[X^{\hat{\theta}}(T)|\mathcal{F}_{t+h}]|\mathcal{F}_t] - \lambda(t)\mathcal{R}[E[X^{\hat{\theta}}(T)|\mathcal{F}_{t+h}]|\mathcal{F}_t] \\
= {} & E[M(t+h)|\mathcal{F}_t] - \lambda(t)E[V(t+h)|\mathcal{F}_t] - \lambda(t)\mathcal{R}[U(t+h))|\mathcal{F}_t],
\end{aligned}$$

where $F(t+h)=F(t+h,X(t+h),Y(t+h))$ for $F=U,V,M$ and $X(t+h)$ is the wealth at time $t+h$ by using control θ in $[t,t+h]$. Therefore,

$$\begin{aligned}
& M(t,x,y)-\lambda(t)V(t,x,y)-V^{\widehat{\theta}}(t,x,y) \\
= {} & -E[\Delta M(t)|\mathcal{F}_t] + \lambda(t)E[\Delta V(t)|\mathcal{F}_t] + \lambda(t)\mathcal{R}[U(t)+\Delta U(t)|\mathcal{F}_t].
\end{aligned} \tag{2.15}$$

Inserting (2.11) and (2.13) into (2.15), dividing by h and letting $h\to 0$, we have

(2.16)

$$\begin{aligned}
& \lim_{h\to 0}\tfrac{M(t,x,y)-\lambda(t)V(t,x,y)-V^{\widehat{\theta}}(t,x,y)}{h} \\
= {} & -\mathcal{L}(M)(t,x,y;\theta) + \lambda(t)\mathcal{L}(V)(t,x,y;\theta) + \tfrac{1}{2}\beta(\beta-1)U^{\beta-2}\lambda(t)[(\theta\sigma\partial_x U)^2 + (\nu\partial_y U)^2 + 2\rho\theta\sigma\nu\partial_x U\partial_y U] \\
\geq {} & 0.
\end{aligned}$$

The last inequality is due to the HJB equation (2.9). □

A similar argument technique as in the proof of (2.4) and (2.5) can be used to derive the HJB equation and verify the equilibrium controls in the other cases in the rest of this paper.

Corollary 2.6 *Assume the Markov process Y follows a continuous time Markov chain process taking values in $\mathcal{I}=\{1,2,\dots\}$ with a generator matrix $Q=(q_{ij})_{i,j\in\mathcal{I}}$ satisfying $\sum_{j\in\mathcal{I}} q_{ij}=0$*

for any $i \in \mathcal{I}$. By the time-consistent dynamic programming equation (2.5), we obtain the HJB equation by Itô's lemma,

$$0 = \max_{\theta} \left\{ \begin{array}{l} \mathcal{L}(M_i)(t,x) - \lambda(t)\mathcal{L}(V_i)(t,x) \\ -\frac{1}{2}\beta(\beta-1)U_i^{\beta-2}\lambda(t)(\theta\sigma(t,i)\partial_x U_i)^2 \\ +\sum_{j\in\mathcal{I}} q_{ij}\left(M_j - \lambda(t)(V_j + U_j^{\beta} - \beta U_i^{\beta-1}U_j)\right) \end{array} \right\},$$

where $M_i(t,x) = M(t,x,i), V_i(t,x) = V(t,x,i), U_i(t,x) = U(t,x,i)$, for $i \in \mathcal{I}$ and

$$\mathcal{L}(f_i)(t,x) = \partial_t f_i + [r(t,i)x + \theta(\mu(t,i) - r(t,i))]\partial_x f_i + \frac{1}{2}\theta^2\sigma(t,i)^2\partial_{xx} f_i.$$

Remark 2.5 *We assume the utility function u satisfies the assumption S2, i.e. $u(x) = x^{\alpha}/\alpha$. This is only for the simplicity and convenience of the discussion. u can be a general increasing concave function and the time-consistent dynamic programming equation (2.5) still holds, which is clear as we have not used any particular properties of u in proving (2.5). In particular, if we set $\beta = 1$ or $\lambda = 0$, then (2.5) reduces to*

$$0 = \max_{\theta(.)} E[\Delta M(t)|\mathcal{F}_t],$$

which is the standard dynamic programming equation in utility maximization, so we can recover all known results for general utility functions.

Remark 2.6 *In this paper we assume there is one risky stock in the market. This can be easily extended to multi stock model with correlated Brownian motions and control constraints. The time-consistent dynamic programming equation for the equilibrium value function is the same as that in (2.5). This is due to the wealth process X in (2.1) is still a one-dimensional controlled stochastic process, even though the resulting HJB equation is more complicated due to many additional terms involving control variables.*

3. Examples

In this section we present three examples in applying Theorem 2.3 to solve time-inconsistent problems and recover the known results in the literature, including those in Basak and Chabakauri [1] with stochastic volatility and those in Björk et al. [5] with state dependent risk aversion, and beyond.

3.1 Volatility dependent risk aversion

The first example is the well-known Heston's stochastic volatility model. The dynamics of two assets, a bond (B) and a stock (S) are given by

$$\begin{array}{ll} dB(t) = rB(t)dt, & B(0) = 1, \\ dS(t) = S(t)[(r + \delta z(t))dt + \sqrt{z(t)}dW(t)], & S(0) = s_0 > 0, \end{array}$$

and the instantaneous variance $\nu(t)$ follows a mean reversion square root process (the CIR process):

$$dz(t) = \kappa(\omega - z(t))dt + \xi\sqrt{z(t)}dW_1(t), z(0) = z_0 > 0,$$

where $W(t)$ and $W_1(t)$ are standard Brownian motions with correlation ρ, $r, \delta, \kappa, \omega, \xi$ are positive constants, representing the riskless interest rate, the market price of risk coefficient, the mean-reversion speed, the mean-reversion level, and the volatility of the variance rate, respectively. The investor's wealth process is given by

$$dX(t) = [rX(t) + \theta(t)\delta z(t)]dt + \theta(t)\sqrt{z(t)}dW(t), X(0) = x_0 > 0.$$

We consider the optimization problem:

$$\max_{\theta(s):t\le s\le T} E[X(T)|\mathcal{F}_t] - \gamma z(t)^{\varpi} Var[X(T)|\mathcal{F}_t],$$

where γ, ϖ are positive constants, the control domain $\mathbb{U} = \mathbb{R}$. Note $\alpha = 1, \beta = 2, \lambda(t) = \gamma z(t)^{\varpi}$ for this model.

Since $\alpha = 1$, we have $M(t,x,z) = U(t,x,z)$ from (2.4). Substituting $\beta = 2$ into the time-consistent dynamic programming equation (2.5), we obtain HJB equation:

(3.1)

$$\begin{aligned} 0 = \\ \max_\theta \quad & \left\{\partial_t U + [rx + \theta\delta z]\partial_x U + [\kappa(\omega - z)]\partial_z U + \tfrac{1}{2}\theta^2 z\partial_{xx}U + \tfrac{1}{2}\xi^2 z\partial_{zz}U + \theta\xi z\rho\partial_{xz}U \right.\\ & -\gamma z^{\varpi}\left[\partial_t V + [rx + \theta\delta z]\partial_x V + [\kappa(\omega - z)]\partial_z V + \tfrac{1}{2}\theta^2 z\partial_{xx}V + \tfrac{1}{2}\xi^2 z\partial_{zz}V + \theta\xi z\rho\partial_{xz}V \right.\\ & \left.\left. +(\theta\sqrt{z}\partial_x U)^2 + (\xi\sqrt{z}\partial_z U)^2 + 2\xi\theta z\rho\partial_x U\partial_z U\right]\right\}. \end{aligned}$$

We make an ansatz as follows:

$$\begin{aligned} U(t,x,z) &= e^{r(T-t)}x + A(z,t), \\ V(t,x,z) &= B(z,t), \end{aligned} \tag{3.2}$$

with the terminal conditions $A(z,T) = 0$ and $B(z,T) = 0$. Substituting (3.2) into the equation (3.1), we obtain a simplified equation:

$$\begin{aligned} 0 = \min_\theta \quad & \{\partial_t A + e^{r(T-t)}\theta\delta z + k(\omega - z)\partial_z A + \tfrac{1}{2}\xi^2 z\partial_{zz}A \\ & -\gamma z^{\varpi}[\partial_t B + k(\omega - z)\partial_z B + \tfrac{1}{2}\xi^2 z\partial_{zz}B \\ & + e^{2r(T-t)}z\theta^2 + \xi^2 z(\partial_z A)^2 + 2e^{r(T-t)}\theta\xi z\rho\partial_z A]\}. \end{aligned}$$

First order condition yields

$$\theta^* = \left(\frac{\delta}{2\gamma z^{\varpi}} - \xi\rho\partial_z A\right)e^{-r(T-t)}. \tag{3.3}$$

Applying the Feynman-Kac formula to U yields

$$0 = \partial_t A + e^{r(T-t)}\theta^*\delta z + \partial_z A k(\omega - z) + \frac{1}{2}\partial_{zz}A\xi^2 z. \tag{3.4}$$

Combining (3.3) and (3.4), we have the stochastic representation for A:

$$A(z,t) = E^*\left[\int_t^T \frac{\delta^2}{2\gamma z(s)^{\varpi-1}}ds \; |dle| \; z(t) = z\right],$$

where the expectation E^* is taken under the new probability measure $\mathbb{P}^*$. Under the new measure $\mathbb{P}^*$, $\{z(t)\}$ follows the dynamics

$$dz(t) = [k(\omega - z(t)) - \xi\rho\delta z(t)]\,dt + \xi\sqrt{z}dW^*(t),$$

where $W^*(t)$ is a standard Brownian motion under $\mathbb{P}^*$. We can derive B by solving a parabolic PDE:

$$\partial_t B + k(\omega - z)\partial_z B + \frac{1}{2}\xi^2 z \partial_{zz} B + e^{2r(T-t)} z(\theta^*)^2 + \xi^2 z(\partial_z A)^2 + 2e^{r(T-t)}\theta^* \xi z \rho \partial_z A = 0.$$

Remark 3.1 *By taking $\varpi = 0$, we recover the results of Basak and Chabakauri [1].*

Remark 3.2 *One may use the theory developed in [4] to slove the problem above. In that case, one must solve the extended HJB equation in Definition 2.2 in [5], with a two-dimensional state variable x and a two-dimensional auxiliary variable y, where one component of x, y corresponds to wealth level and the other one to volatility level. The resulting equation is a fully nonlinear PDE in* 5 *dimensions.*

3.2 State dependent risk aversion

The second example is the standard Black-Scholes model. The dynamics of two assets, a bond (B) and a stock (S) are given by

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= S(t)(\mu dt + \sigma dW(t)), & S(0) &= s_0 > 0, \end{aligned}$$

where W is a standard $\mathbb{P}$-Brownian motion, and r, μ, σ are positive constants, representing the riskless interest rate, the stock growth rate with $\mu > r$, and the stock volatility rate, respectively. The wealth process is given by

$$dX(t) = [rX(t) + \theta(t)(\mu - r)]dt + \theta(t)\sigma dW(t), X(0) = \zeta > 0.$$

where $\zeta \in L^{\beta}_{\mathcal{F}_0}$. The expected-utility-deviation-risk optimization becomes:

$$\max_{\theta(s): t \le s \le T} E[u(X(T))|\mathcal{F}_t] - \lambda(t)\mathcal{R}[X(T)|\mathcal{F}_t], \tag{3.5}$$

where the risk aversion is

$$\lambda(t) = \frac{\gamma(t)}{X(t)^{\beta - \alpha}},$$

and γ is a positive and piece-wise continuously differentiable ($\mathcal{C}^{1,pw}$) function: $[0, T] \to \mathbb{R}^+$, and the control domain $\mathbb{U} = \mathbb{R}$. Since there is no external factor process Y, by (2.4), we have

$$\begin{aligned} U(x,t) &= E[X^{\theta^*}(T)|X(t) = x], \\ V(x,t) &= \mathcal{R}[X^{\theta^*}(T)|X(t) = x], \\ M(x,t) &= E[u(X^{\theta^*}(T))|X(t) = x]. \end{aligned} \tag{3.6}$$

The time-consistent dynamic programming equation (2.5) can be applied to (3.6) and we obtain HJB equation:

(3.7)

$$\begin{aligned} 0 = \max_\theta \quad &\left\{ \partial_t M + [rx + \theta(\mu - r)]\partial_x M + \tfrac{1}{2}\theta^2\sigma^2 \partial_{xx} M \right. \\ &\left. -\tfrac{\gamma(t)}{x^{\beta-\alpha}} \left[\partial_t V + [rx + \theta(\mu - r)]\partial_x V + \tfrac{1}{2}\theta^2\sigma^2\partial_{xx} V + \tfrac{1}{2}\beta(\beta - 1)U^{\beta-2}(\theta\sigma\partial_x U)^2 \right] \right\}. \end{aligned}$$

We make an ansatz:

$$U(x,t) = A(t)x,$$
$$V(x,t) = B(t)x^{\beta}, \tag{3.8}$$
$$M(x,t) = C(t)x^{\alpha},$$

with the terminal conditions $A(T) = 1, B(T) = 0, C(T) = \frac{1}{\alpha}$. Substituting (3.8) into (3.7), we have

$$\begin{aligned} 0 = \max_{\theta} \quad & \Big\{ \dot{C}(t)x^2 + \alpha x C(t)[rx + \theta(\mu - r)] + \tfrac{1}{2}\alpha(\alpha-1)C(t)\theta^2\sigma^2 \\ & -\gamma(t)\left[\dot{B}(t)x^2 + \beta B(t)x[rx + \theta(\mu-r)] + \tfrac{1}{2}\beta(\beta-1)(B(t) + A(t)^{\beta})\sigma^2\theta^2\right]\Big\}. \end{aligned} \tag{3.9}$$

First order optimality condition yields

$$\theta^* = \frac{\mu - r}{\sigma^2}\frac{\alpha C(t) - \beta\gamma(t)B(t)}{\gamma(t)\beta(\beta-1)B(t) + \gamma(t)\beta(\beta-1)A(t)^{\beta} - \alpha(\alpha-1)C(t)}x := D(t)x. \tag{3.10}$$

Applying the Feynman-Kac formula to U and M yields

$$\begin{cases} 0 = \dot{A}(t) + [r + D(t)(\mu - r)]A(t), \\ 0 = \dot{C}(t) + \alpha[r + D(t)(\mu-r)]C(t) + \frac{1}{2}\alpha(\alpha-1)\sigma^2 D(t)^2 C(t). \end{cases} \tag{3.11}$$

Combining (3.9), we obtain

$$0 = \dot{B}(t) + \beta[r + D(t)(\mu - r)]B(t) + \frac{1}{2}\beta(\beta-1)(B(t) + A(t)^{\beta})\sigma^2 D(t)^2. \tag{3.12}$$

By (3.11) and (3.12), we can write functions A, B, C in terms of function D:

(3.13)

$$\begin{aligned} A(t) &= \exp\{\textstyle\int_t^T [r + D(s)(\mu - r)]ds\}, \\ B(t) &= \exp\{\beta \textstyle\int_t^T [r + D(s)(\mu-r) + \frac{1}{2}(\beta-1)D(s)^2\sigma^2]ds\} - \exp\{\beta\textstyle\int_t^T [r + D(s)(\mu-r)]ds\}, \\ C(t) &= \tfrac{1}{\alpha}\exp\{\alpha\textstyle\int_t^T [r + D(s)(\mu - r) + \frac{1}{2}(\alpha-1)D(s)^2\sigma^2]ds\}. \end{aligned}$$

The function D solves the following equation:

$$D(t) = \frac{\mu - r}{\sigma^2}\frac{\alpha C(t) - \beta\gamma(t)B(t)}{\gamma(t)\beta(\beta-1)B(t) + \gamma(t)\beta(\beta-1)A(t)^{\beta} - \alpha(\alpha-1)C(t)}, \tag{3.14}$$

with the terminal condition

$$D(T) = \frac{\mu - r}{\sigma^2}\frac{1}{\gamma(T)\beta(\beta-1) - (\alpha-1)}. \tag{3.15}$$

Let $\mathcal{C}^{pw}([0,T])$ denote the set of all piece-wise continuous functions on $[0,T]$.

Proposition 3.1 *If $\gamma(.)$ is positive and belongs to $\mathcal{C}^{1,pw}([0,T])$, then there exists a unique solution to (3.14) in the class $\mathcal{C}^{pw}([0,T])$; if particular, if $\gamma(.) \in \mathcal{C}^1([0,T])$, this unique solution belongs to $\mathcal{C}([0,T])$.*

Proof: See Appendix. □

Remark 3.3 *Note that θ^* in (3.10) is a stationary point of the objective function in (3.9). To ensure θ^* is indeed a maximum point, we need to show the objective function in (3.9) is a concave quadratic function of θ. Tthe second order derivative is given by*

$$\alpha(\alpha-1)C(t)\sigma^2 - \gamma(t)\beta(\beta-1)(B(t)+A(t)^\beta)\sigma^2.$$

From the proof of Proposition 3.1, we know $A_n(.), B_n(.), C_n(.)$ defined in (6.1) converge uniformly to $A(.), B(.), C(.)$, respectively, also noting $\underline{D}_n(.)$ defined in (6.2) and the inequality in (6.3), we get

$$\begin{aligned} & \alpha(\alpha-1)C(t)\sigma^2 - \gamma(t)\beta(\beta-1)(B(t)+A(t)^\beta)\sigma^2 \\ = \; & -\sigma^2 \lim_{n\to+\infty}(\gamma(t)\beta(\beta-1)(B_n(t)+A_n(t)^\beta) - \alpha(\alpha-1)C_n(t)) \\ = \; & -\sigma^2 \lim_{n\to+\infty} \underline{D}_n(t) \\ \leq \; & -\sigma^2 \underline{D}_{min} < 0, \end{aligned}$$

which implies θ^* in (3.10) is the unique maximum point.

Combining the result in Proposition 3.1, we immediately have

Theorem 3.2 *The equilibrium strategy of the optimization problem (3.5) is given by*

$$\theta^*(t) = D(t)X(t),$$

where D is the unique solution to (3.14) in $\mathcal{C}^{pw}([0,T])$ and the optimal terminal wealth is given by

(3.16)
$$X^*(T) = X(t)\exp\left\{\int_t^T [r + D(s)(\mu-r) - \frac{1}{2}D(s)^2\sigma^2]ds + \int_t^T D(s)\sigma dW(s)\right\}, t\in[0,T].$$

Remark 3.4 *We have the optimal terminal wealth, $X^*(T)$, is L^β integrable if and only if $X(0)=\zeta$ is L^β integrable, hence $\mathcal{R}[X^*(T)|\mathcal{F}_t]$ is well defined. The introduction of deviation-risk measure $\mathcal{R}$ for $\beta\in(1,2)$ requires less integrability restriction of the initial wealth level and broaden the admissible control set $\mathcal{U}_{ad}[t,T]$, compared with the traditional variance risk measure.*

By taking $\alpha = 1$ and $\beta = 2$ in Proposition 3.2, we have the following result.

Corollary 3.3 *Assume $\alpha = 1$ and $\beta = 2$. Then problem (3.5) reduces to a mean-variance model with state dependent risk aversion:*

$$\max_{\theta(s):t\leq s\leq T} E[X(T)|\mathcal{F}_t] - \frac{\gamma(t)}{X(t)}Var[X(T)|\mathcal{F}_t]. \tag{3.17}$$

The equilibrium is given by $\theta^*(t) = D(t)X(t)$, where D is the unique solution of the following differential equation in $\mathcal{C}^{pw}([0,T])$:

(3.18)
$$D(t) = \frac{\mu-r}{\sigma^2}\left(\frac{1}{2\gamma(t)}\exp\left\{-\int_t^T r + D(s)(\mu-r) + D(s)^2\sigma^2 ds\right\} + \exp\left\{-\int_t^T D(s)^2\sigma^2 ds\right\} - 1\right)$$

and the optimal terminal wealth $X^*(T)$ is given by (3.16). In particular, if $\gamma(t) = \gamma/2$ and γ is a positive constant, then we recover the result in Björk et al. [5].

Remark 3.5 *Assume $\alpha < 1$ and $\beta = 1$. Then problem (3.5) reduces to a standard utility maximization model with $V(x,t) \equiv 0$, which results in $B(t) \equiv 0$ from (3.8) and $D(t) \equiv \frac{\mu - r}{\sigma^2(1-\alpha)}$ from (3.14). We have recovered the classical Merton's portfolio [17].*

The next two theorems give convergence results when β tends to 1 and $+\infty$. Their proofs are in the Appendix.

Theorem 3.4 *Assume $\alpha < 1$. The equilibrium strategy (3.10) converges to that in the standard utility model when β tends to 1.*

Proof: See Appendix. □

Theorem 3.5 *The equilibrium strategy (3.10) converges to zero along the investment horizon when β tends to $+\infty$.*

Proof: See Appendix. □

Note that $\theta^*(t)$ can be negative for some t when some special parameters are chosen. This implies that the strategy requires short selling a stock. The following theorem shows the strategy when short selling is prohibited, i.e., $\mathbb{U} = \mathbb{R}^+$, whose proof can be found in the Appendix.

Theorem 3.6 *If the control domain $\mathbb{U} = \mathbb{R}^+$, the equilibrium strategy of the optimization problem (3.5) is given by*

$$\theta^*(t) = D(t)X(t),$$

where D is the unique solution in $\mathcal{C}^{pw}([0,T])$ to the following equation:

$$D(t) = \max\left\{\frac{\mu - r}{\sigma^2}\frac{\alpha C(t) - \beta\gamma(t)B(t)}{\gamma(t)\beta(\beta-1)B(t) + \gamma(t)\beta(\beta-1)A(t)^\beta - \alpha(\alpha-1)C(t)}, 0\right\}, \tag{3.19}$$

where $A(t), B(t)$ and $C(t)$ are defined in (3.13).

Proof: See Appendix. □

3.3 Constrained portfolio selection

The third example has the same model setup as that in section 3.2 and deals with a constrained portfolio problem, with continuously monitored constraints (P_c):

$$\begin{aligned} \min_{\theta(s):t\le s\le T} \quad & Var[X(T)|\mathcal{F}_t] \\ \text{subject to} \quad & E[X(T)|\mathcal{F}_t] \ge X(t)\exp\{\eta(T-t)\}, \end{aligned}$$

where $\eta \in [r, \mu]$ is the expected return rate required by the investor.

For a fixed $t \in [0,T]$, we can solve (P_c) by introducing a Lagrange multiplier $\lambda(t) > 0$ and a mean-variance optimization problem (P_a):

$$\max_{\theta(s):t\le s\le T} E[X(T)|\mathcal{F}_t] - \lambda(t)Var[X(T)|\mathcal{F}_t],$$

where the state dependent risk aversion $\lambda(t)$ is a stochastic process adapted to $\{\mathcal{F}_t\}$. To determine the specific form for $\lambda(t)$, we have the following result.

Proposition 3.7 *Suppose a stochastic process $\lambda(t) = \lambda(t, X(t))$, where $\lambda : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a positive measurable function. Suppose the optimal control problem (P_a) admits an equilibrium strategy $\{\theta^*(s) : t \le s \le T\}$ and the corresponding terminal wealth $X^*(T)$ such that*

$$E[X^*(T)|\mathcal{F}_t] = X(t) \exp\{\eta(T-t)\},$$

then $\{\theta^*(s)\}$ is also the equilibrium strategy for (P_c).

Proof: Let us consider problem (P_c) and suppose $X^*(T)$ is the corresponding terminal wealth with control $\theta^*(.)$. Define a control $\{\hat{\theta}\}$ as follows:

$$\hat{\theta}(s) = \begin{cases} \theta(s), & s \in [t, t+h], \\ \theta^*(s), & s \in (t+h, T], \end{cases}$$

which gives associated terminal wealth $\hat{X}(T)$, for any investment policy $\{\theta(s)\}$ in $[t, t+h]$ such that $E[\hat{X}(T)|\mathcal{F}_t] \ge X(t) \exp\{\eta(T-t)\}$. We know

$$\begin{aligned} Var[X^*(T)|\mathcal{F}_t] &= -\tfrac{1}{\lambda(t)}(-\lambda(t)Var[X^*(T)|\mathcal{F}_t]) \\ &= -\tfrac{1}{\lambda(t)}(E[X^*(T)|\mathcal{F}_t] - \lambda(t)Var[X^*(T)|\mathcal{F}_t] - X(t)\exp\{\eta(T-t)\}), \end{aligned}$$

and

$$Var[\hat{X}(T)|\mathcal{F}_t] \ge -\frac{1}{\lambda(t)}(E[\hat{X}(T)|\mathcal{F}_t] - \lambda(t)Var[\hat{X}(T)|\mathcal{F}_t] - X(t)\exp\{\eta(T-t)\}).$$

Thus,

$$\begin{aligned} & \liminf_{h\to 0} \tfrac{Var[X^*(T)|\mathcal{F}_t] - Var[\hat{X}(T)|\mathcal{F}_t]}{h} \\ \le\ & \liminf_{h\to 0} -\tfrac{1}{\lambda(t)} \tfrac{E[X^*(T)|\mathcal{F}_t] - \lambda(t)Var[X^*(T)|\mathcal{F}_t] - \{E[\hat{X}(T)|\mathcal{F}_t] - \lambda(t)Var[\hat{X}(T)|\mathcal{F}_t]\}}{h} \\ \le\ & 0. \end{aligned}$$

The last inequality is due to $\{\theta^*(s)\}$ is an equilibrium strategy of (P_a). In summary, $\{\theta^*(s)\}$ is an equilibrium strategy of (P_c). □

By Proposition 3.7, we know that to find the equilibrium strategy of (P_c), it suffices to find the mapping λ such that the equilibrium strategy of (P_a) also makes the following equality hold:

$$E[X^*(T)|\mathcal{F}_t] = X(t)\exp\{\eta(T-t)\}. \tag{3.20}$$

We make an ansatz:

$$\lambda(t) = \lambda(t, X(t)) = \frac{\gamma(t)}{X(t)}, \tag{3.21}$$

where $\gamma(.)$ is a positive $\mathcal{C}^1$ function. By Corollary 3.3 and (3.16),

$$E[X^*(T)|\mathcal{F}_t] = X(t)\exp\left\{\int_t^T [r + D(s)(\mu - r)]ds\right\},$$

where D solves (3.18). By letting $\exp\{\int_t^T [r + D(s)(\mu - r)]ds = \exp\{\eta(T-t)\}$, we have

$$\begin{cases} D(t) &\equiv \frac{\eta - r}{\mu - r} \triangleq p, \\ \gamma(t) &= \frac{exp\{-(\eta + p^2\sigma^2)(T-t)\}}{2(1 + \frac{\sigma^2 p}{\mu - r} - exp\{-p^2\sigma^2(T-t)\})}. \end{cases} \tag{3.22}$$

Simple calculation yields that $D(t) \equiv p$ for $t \in [0, T]$ solves the equation (3.14) and satisfies the terminal condition (3.15) with $\alpha = 1, \beta = 2$ and $\gamma(t)$ admits the form in (3.22). We can now give a characterization of the optimal solution to the constrained problem (P_c).

Proposition 3.8 *The equilibrium strategy of (P_c) is given by $\theta^*(t) = pX(t)$ and the conditional expectation of the optimal terminal wealth is given by the binding constraint (3.20).*

Remark 3.6 *From the specific form (3.21) of $\{\lambda(t)\}$ in the mean-variance model which helps to solve the constrained portfolio optimization problem (P_c), one can see that the recent results in the literature, i.e. the constant risk aversion [1] and the wealth dependent risk aversion [5] cannot help to solve this time-inconsistent constrained problem. It is of essential importance to introduce the time and state dependent risk aversion as proposed in our model. One may use the method in [4, 3] to solve this problem by treating time variable t in the risk aversion $\lambda(t)$ in (3.21) as an additional state variable. Then one would have to solve an extended HJB equation in 5 dimensions, too difficult to solve, see Remark 2.3 for details.*

4. Numerical Studies

In this section we undertake some numerical experiments to illustrate our proposed model with varying parameters. The same model setup is adopted as in Section 3.2. Theorem 3.2 in Section 3.2 claims that the equilibrium strategy of the optimization problem (3.5) is given by $\theta^*(t) = D(t)X(t)$, where D is the unique solution to (3.14) in $\mathcal{C}^{pw}([0, T])$. To obtain a numerical solution to function D, we follow the numerical scheme in (6.1). The proof of Proposition 3.1 shows that the numerical scheme in (6.1) produces a sequence of functions $\{D_n\}$ that converge to the unique continuous function D. This is due to the fact that any subsequence in $\{D_n\}$ has a sub-subsequence that converges to D.

4.1 Varying α and β

We first investigate how $D(t)$ and $X(T)$ change under different pairs of values for α and β. Specifically, we perfrom two sets of statistical tests with the combinations of different α and β. We choose two values for α: 0.5 (power utility) and 1 (mean). In the first test, we choose three values for β: 1.5, 2 and 3, with the benchmark being the mean-variance case ($\alpha = 1, \beta = 2$). In the second test, we choose three values for β: 1, 1.1 and 1.5, with the benchmark being the expected-utility case ($\alpha = 0.5, \beta = 1$), while α and β cannot be 1 at the same time. For other parameters, we fix their values as $\mu = 0.2, r = 0.05, \sigma = 0.2, \gamma(t) \equiv \gamma = 5$ and $T = 1$. Under these different settings, the function D, i.e., the proportion of wealth invested in the stock along time, is plotted in Figure 4.1. One can see that a decreasing value for α or an increasing value for β yields a decreasing proportion invested in the stock. When β is large (we have also done a test with $\beta = 10$ but have not depicted its function D in Figure 4.1), the proportion invested in the stock is almost zero, consistent with the result in Theorem 3.5.

We make statistical comparisons of distributions of optimal terminal wealth with different α and β. The initial wealth $X(0)$ is assumed to be 1. Tables 4.1 and 4.2 list the mean, standard deviation (SD), skewness, quantile values (1% and 99%) of the optimal terminal wealth, the Sharpe

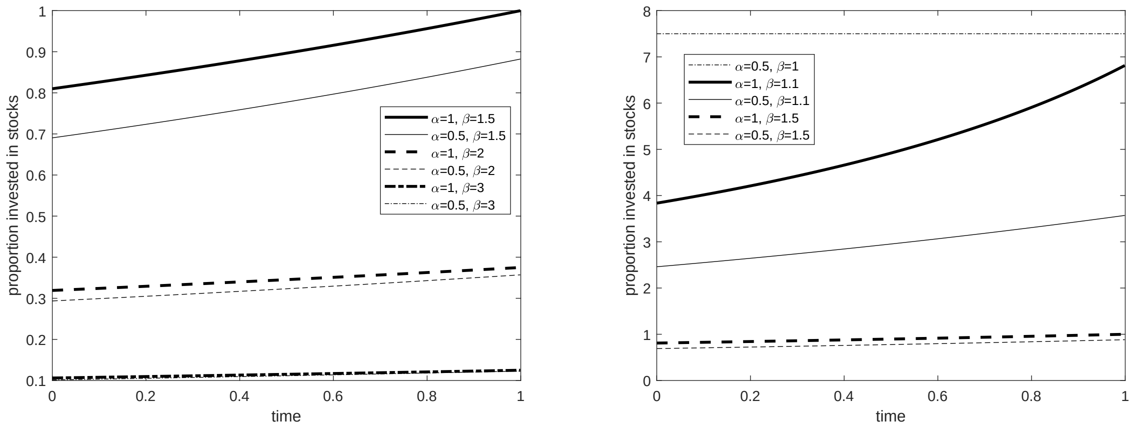

Figure 4.1: Function D with varying α, β.

ratio of the optimal return (i.e., $(E[(X(T)-X(0))/X(0)]-r)/\sqrt{Var[(X(T)-X(0))/X(0)]}$) and the relative errors of these values (written in parentheses) compared with those in the benchmark cases (mean-variance case with $\alpha = 1, \beta = 2$ and expected-utility case with $\alpha = 0.5, \beta = 1$). The tables also list expected optimal terminal utilities and deviation-risks for different α and β.

The statistical results reveal that for a fixed α, a smaller β corresponds to a portfolio with higher profit but larger volatility; for a fixed β, a larger α gives a portfolio with higher profit but larger volatility. As a commonly-used measure to evaluate the performance of portfolio or mutual fund managers, the Sharpe ratio supports the use of a big β. This can be seen by comparing the cases $\alpha = 1, \beta = 2$ and $\alpha = 1, \beta = 3$, a reduction of 3.40% of average terminal wealth results in a reduction of 67.84% of standard derivation-risk, which seems an attractive tradeoff for a less ambitious risk-averse investor. This phenomenon can also be observed from the relations of expected optimal terminal utilities and deviation risks for various α and β. On the other hand, if one is primarily interested in maximizing the expected utility of the terminal wealth, then one should choose small β to increase the exposure to the stock movement. The choice of $\beta = 1$ (corresponding to the standard utility maximization) gives the maximum expected utility of the terminal wealth at the cost of the maximum risk of potential extreme loss compared with any other $\beta > 1$, this can be easily seen from the quantile values in Tables 4.1 and 4.2.

4.2 Varying γ

The function γ in the problem (3.5) plays the role of a penalty level to the deviation-risk measure $\mathcal{R}$. The larger the γ, the heavier the penalty, which would affect the choice of the equilibrium control and result in different optimal expected utilities and deviation-risks for different γ. For fixed α and β, we choose γ to be a constant along the time horizon $[0, T]$ and vary its value to derive the corresponding optimal utility and deviation-risk. Figure 4.2 (left panel) shows the efficient frontiers of optimal utility and deviation-risk with fixed α and β and varying γ. We use the following values for the parameters: $\mu = 0.2, r = 0.05, \sigma = 0.2$ and $T = 1$.

We also consider a case with γ being a time dependent function as $\gamma_0(t) = 3$ for $t \in [0, T/2)$ and $\gamma_0(t) = 7$ for $t \in [T/2, T]$, which may be explained as the changing risk behaviour of

		$\beta = 1.5$	$\beta = 2$	$\beta = 3$
$\alpha = 1$	Mean	1.2031 (0.0865)	1.1073	1.0697 (-0.0340)
	SD	0.2185 (1.8451)	0.0768	0.0247 (-0.6784)
	Skewness	0.0058 (60.4974)	$9.4313^{\times 10^{-5}}$	$1.0408^{\times 10^{-6}}$ (-0.9890)
	Sharpe Ratio	0.7005 (-0.0607)	0.7458	0.7941 (0.0648)
	Quantile (1%)	0.7798 (-0.1698)	0.9393	1.0128 (0.0782)
	Quantile (99%)	1.7828 (0.3727)	1.2988	1.1272 (-0.1321)
	Expected Utility	1.2031	1.1073	1.0696
	Deviation Risk	0.0162	0.0059	0.0020
$\alpha = 0.5$	Mean	1.1818 (0.0673)	1.1036 (-0.0033)	1.0690 (-0.0345)
	SD	0.1860 (1.4219)	0.0717 (-0.0664)	0.0239 (-0.6888)
	Skewness	0.0031 (31.8693)	$7.1852^{\times 10^{-5}}$ (-0.2382)	$9.2059^{\times 10^{-7}}$ (-0.9902)
	Sharpe Ratio	0.7085 (-0.0500)	0.7479 (0.0028)	0.7956 (0.0668)
	Quantile (1%)	0.8141 (-0.1333)	0.9490 (0.0103)	1.0140 (0.0795)
	Quantile (99%)	1.6753 (0.2899)	1.2814 (-0.0134)	1.1262 (-0.1329)
	Expected Utility	2.1676	2.0999	2.0678
	Deviation Risk	0.0118	0.0051	0.0018

Table 4.1: Statistics of the optimal portfolio with mean-variance model as benchmark.

		$\beta = 1$	$\beta = 1.1$	$\beta = 1.5$
$\alpha = 1$	Mean		2.2421 (-0.3076)	1.2031 (-0.6285)
	SD		3.0516(-0.6765)	0.2185 (-0.9768)
	Skewness		187.6903 (-0.9933)	0.0058 (-1.0000)
	Sharpe Ratio		0.3906 (0.06843)	0.7005 (2.0207)
	Quantile (1%)		0.1227 (2.8344)	0.7690 (23.0313)
	Quantile (99%)		14.6833 (-0.5736)	1.7979 (-0.9478)
	Expected Utility		2.2421	1.2031
	Deviation Risk		0.1442	0.0162
$\alpha = 0.5$	Mean	3.2381	1.6421 (-0.4929)	1.1818 (-0.6350)
	SD	9.4339	1.0768 (-0.8859)	0.1860 (-0.9803)
	Skewness	$2.8100^{\times 10^{4}}$	2.8079 (-0.9999)	0.0031 (-1.0000)
	Sharpe Ratio	0.2319	0.5499 (1.3713)	0.7085 (2.0552)
	Quantile (1%)	0.0320	0.3265 (9.2031)	0.8125 (24.3906)
	Quantile (99%)	34.4318	5.5152 (-0.8398)	1.6971 (-0.9507)
	Expected Utility	2.7166	2.4508	2.1676
	Deviation Risk	0	0.0343	0.0118

Table 4.2: Statistics of the optimal portfolio with expected-utility model as benchmark.

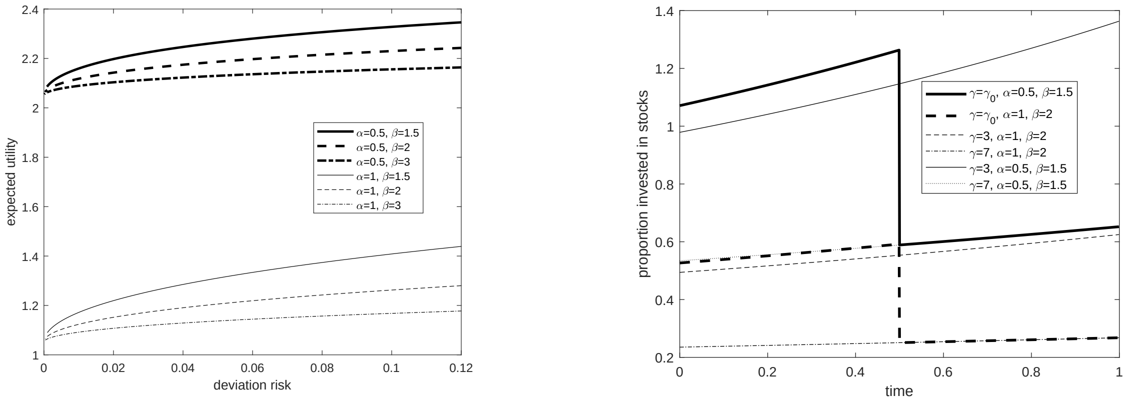

Figure 4.2: Efficient frontiers (left panel) and function D with various γ function (right panel).

an investor half way in the investment from less to more risk averse. We depict the optimal proportion invested in stocks in Figure 4.2 (right panel) for constant γ and time-dependent γ. The following values for the parameters are used: $\mu = 0.2, r = 0.05, \sigma = 0.2$ and $T = 1$. From the numerical study, one can see that a sharp increase in γ results in a simultaneous decrease in the proportion invested in stocks. We may also interpret function γ as a measure to characterize business cycles: in "expansion" days, the risk aversion is smaller for all investors, people have higher willingness for investment in risky asset; while in "recession" days, risk aversion is greater for all investors, people have lower willingness for investment in risky asset.

5. Conclusions

In this paper we propose a unified model for utility deviation-risk portfolio selection with control constraints, which recovers the known results in the literature and goes beyond. In particular, we extend the variance to a more general deviation-risk measure to accommodate investors' risk aptitude and we allow the risk aversion to be time, state and factor dependent. We derive the time-consistent HJB equation and use it to solve a number of problems which are not covered or covered but essentially cannot be solved by the existing literature in [1, 3, 5]. We also perform numerical and statistical studies to illustrate the performance of our proposed model. There remain some open questions which have not been answered in this paper. For example, can the utility function depend on initial state as well as terminal state? is the solution to the time-consistent HJB equation unique? We hope to address these and other questions in our future work.

6. Appendix

6.1 Proof of Proposition 3.1

A similar counterpart of the proof can be found in [5].

When $\alpha < 1$ and $\beta = 1$, equation (3.14) admits a unique solution $D(t) \equiv \frac{\mu - r}{\sigma^2(1-\alpha)}$. In the rest of this proof, we consider the case $\alpha \leq 1, \beta > 1$.

Firstly, we assume $\gamma(.) \in \mathcal{C}^1([0,T])$ in the following steps 1, 2 and 3. We construct a sequence of functions $A_n, B_n, C_n, D_n \in \mathcal{C}([0,T])$ by the following:

(6.1)

$$\begin{array}{rcl} D_0(t) & = & 1, \\ A_n(t) & = & \exp\{\int_t^T [r + D_n(s)(\mu - r)]ds\}, \\ B_n(t) & = & \exp\{\beta \int_t^T [r + D_n(s)(\mu - r) + \frac{1}{2}(\beta - 1)D_n(s)^2\sigma^2]ds\} - \exp\{\beta \int_t^T [r + D_n(s)(\mu - r)]ds\}, \\ C_n(t) & = & \frac{1}{\alpha}\exp\{\alpha \int_t^T [r + D_n(s)(\mu - r) + \frac{1}{2}(\alpha - 1)D_n(s)^2\sigma^2]ds\}, \\ D_{n+1}(t) & = & \frac{\mu - r}{\sigma^2}\frac{\alpha C_n(t) - \beta\gamma(t)B_n(t)}{\gamma(t)\beta(\beta-1)B_n(t) + \gamma(t)\beta(\beta-1)A_n(t)^\beta - \alpha(\alpha-1)C_n(t)}, \end{array}$$

for $n = 0, 1, 2, \cdots$.

Step 1: We prove that $\{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$ are uniformly bounded in $\mathcal{C}([0,T])$. In particular, $\exists\ k_1, k_2 \in \mathbb{R}^+$, such that $0 < k_1 \leq A_n(t), \alpha C_n(t) \leq k_2$, $\forall t \in [0,T]$, for $n = 0, 1, \cdots$.

Note that $A_n(\cdot), \alpha C_n(\cdot), \gamma(\cdot)$ are positive functions in $\mathcal{C}([0,T])$, and $B_n(\cdot)$ is nonnegative function in $\mathcal{C}([0,T])$, for $n = 0, 1, 2, \cdots$, and $\beta > 1, \alpha \leq 1$. We have

$$\begin{array}{rcl} D_n(t) & \geq & \frac{\mu - r}{\sigma^2}\frac{-\beta\gamma(t)B_{n-1}(t)}{\gamma(t)\beta(\beta-1)B_{n-1}(t) + \gamma(t)\beta(\beta-1)A_{n-1}(t)^\beta - \alpha(\alpha-1)C_{n-1}(t)} \\ & = & \frac{\mu - r}{\sigma^2}\frac{-1}{(\beta-1) + (\beta-1)\frac{A_{n-1}(t)^\beta}{B_{n-1}(t)} - \frac{(\alpha-1)}{\beta\gamma(t)}\frac{\alpha C_{n-1}(t)}{B_{n-1}(t)}} \\ & \geq & -\frac{\mu - r}{\sigma^2(\beta-1)} \triangleq D_{min} \end{array}$$

for $n = 1, 2, \cdots$.

$$\begin{array}{rcl} D_n(t) & \leq & \frac{\mu - r}{\sigma^2}\frac{\alpha C_{n-1}(t)}{\gamma(t)\beta(\beta-1)B_{n-1}(t) + \gamma(t)\beta(\beta-1)A_{n-1}(t)^\beta - \alpha(\alpha-1)C_{n-1}(t)} \\ & = & \frac{\mu - r}{\sigma^2}\frac{1}{\gamma(t)\beta(\beta-1)\frac{B_{n-1}(t)}{\alpha C_{n-1}(t)} + \gamma(t)\beta(\beta-1)\frac{A_{n-1}(t)^\beta}{\alpha C_{n-1}(t)} - (\alpha-1)} \\ & \leq & \frac{\mu - r}{\sigma^2}\frac{1}{\gamma(t)\beta(\beta-1)\frac{A_{n-1}(t)^\beta}{\alpha C_{n-1}(t)}\mathcal{I}_{\{\alpha=1\}} + (1-\alpha)\mathcal{I}_{\{\alpha<1\}}} \\ & \leq & \frac{\mu - r}{\sigma^2}\frac{1}{\left(\min\limits_{t\in[0,T]}\gamma(t)\right)\beta(\beta-1)\exp\{(\beta-1)\int_t^T [r + D_{n-1}(s)(\mu - r)]ds\}\mathcal{I}_{\{\alpha=1\}} + (1-\alpha)\mathcal{I}_{\{\alpha<1\}}} \\ & \leq & \frac{\mu - r}{\sigma^2}\frac{1}{\left(\min\limits_{t\in[0,T]}\gamma(t)\right)\beta(\beta-1)\exp\{-\frac{(\mu-r)^2}{\sigma^2}T\}\mathcal{I}_{\{\alpha=1\}} + (1-\alpha)\mathcal{I}_{\{\alpha<1\}}} \triangleq D_{max}, \end{array}$$

where the last inequality is due to $D_n(t) \geq D_{min} = -\frac{\mu - r}{\sigma^2(\beta-1)}$.

We have proved that

$$D_{min} \leq D_n(t) \leq D_{max}, \forall t \in [0,T], n = 1, 2, \cdots$$

Hence $\{A_n\}, \{B_n\}, \{C_n\}$ are also uniformly bounded in $\mathcal{C}([0,T])$ by definition. In particular, $\exists$ $k_1, k_2 \in \mathbb{R}^+$ such that

$$0 < k_1 \leq A_n(t), \alpha C_n(t) \leq k_2, \forall t \in [0,T], n = 1, 2, \cdots$$

Step 2: We prove that $\{\dot{D}_n\}$ is uniformly bounded in $\mathcal{C}([0,T])$.

Let

$$\underline{D}_n(t) = \gamma(t)\beta(\beta-1)B_n(t) + \gamma(t)\beta(\beta-1)A_n(t)^\beta - \alpha(\alpha-1)C_n(t). \tag{6.2}$$

By step 1, We know $\exists\ \underline{D}_{max}, \underline{D}_{min} \in \mathbb{R}^+$ such that

$$0 < \underline{D}_{min} \leq \underline{D}_n(t) \leq \underline{D}_{max}, \forall t \in [0,T], n = 0, 1, \cdots. \tag{6.3}$$

Note that

$$\begin{aligned}
\dot{A}_n(t) &= -A_n(t)[r + D_n(t)(\mu - r)] \\
\dot{C}_n(t) &= -\alpha C_n(t)[r + D_n(t)(\mu - r) + \frac{1}{2}(\alpha-1)D_n(t)^2\sigma^2] \\
\dot{B}_n(t) &= -(B_n(t) + A_n(t)^\beta)\beta[r + D_n(t)(\mu - r) + \frac{1}{2}(\beta-1)D_n(t)^2\sigma^2] + A_n(t)^\beta\beta[r + D_n(t)(\mu - r)]
\end{aligned}$$

Hence by step 1, $\{\dot{A}_n\}, \{\dot{B}_n\}, \{\dot{C}_n\}$ are uniformly bounded in $\mathcal{C}([0,T])$, for $n = 0, 1, \cdots$. As $\gamma(\cdot) \in \mathcal{C}^1([0,T])$, $\{\underline{\dot{D}}_n(.)\}$ is also uniformly bounded in $\mathcal{C}([0,T])$. Since,

$$\dot{D}_n(t) = \frac{\mu - r}{\sigma^2}\frac{[\alpha\dot{C}_{n-1}(t) - \beta\dot{\gamma}(t)B_{n-1}(t) - \beta\gamma(t)\dot{B}_{n-1}(t)]\underline{D}_{n-1}(t) - \underline{\dot{D}}_{n-1}(t)[\alpha C_{n-1}(t) - \beta\gamma(t)B_{n-1}(t)]}{(\underline{D}_{n-1}(t))^2}$$

we can conclude $\{\dot{D}_n\}$ is uniformly bounded in $\mathcal{C}([0,T])$.

Step 3: Prove existence and uniqueness for the function D.

For any $s, t \in [0,T]$,

$$\begin{aligned}
|D_n(t) - D_n(s)| &= \left|(t-s)\int_0^1 \dot{D}_n(s + u(t-s))du\right| \\
&\leq k(t-s)
\end{aligned}$$

where k is a constant independent of n. Hence $\{D_n\}$ is equicontinuous and we already show it is uniformly bounded, the Arzela-Ascoli Theorem implies that there exists a function $D \in \mathcal{C}([0,T])$ and sequence D_{n_i} such that $D_{n_i} \to D$. Therefore, equation (3.14) has a solution D.

To prove uniqueness, assume $D_1, D_2 \in \mathcal{C}([0,T])$ are two solutions to (3.14). Define accordingly, A_1, A_2, B_1, B_2 and $C_1, C_2 \in \mathcal{C}([0,T])$ and

$$\begin{aligned}
\underline{D}_i(t) &= \gamma(t)\beta(\beta-1)B_i(t) + \gamma(t)\beta(\beta-1)A_i(t)^\beta - \alpha(\alpha-1)C_i(t) \\
\overline{D}_i(t) &= \frac{\mu - r}{\sigma^2}(\alpha C_i(t) - \beta\gamma(t)B_i(t)),
\end{aligned}$$

for $i = 1, 2$. Due to the function $\varphi(x) = e^x$ is globally Lipschitz on any bounded set, there exists a constant $K > 0$ such that,

$$|\underline{D}_1(t) - \underline{D}_2(t)| \leq K\int_t^T |D_1(s) - D_2(s)|ds,$$

and

$$|\overline{D}_1(t) - \overline{D}_2(t)| \leq K\int_t^T |D_1(s) - D_2(s)|ds.$$

Therefore, there exists a constant $\overline{K} > 0$ such that,

$$\begin{aligned} |D_1(t) - D_2(t)| = \left|\frac{\overline{D}_1(t)}{\underline{D}_1(t)} - \frac{\overline{D}_2(t)}{\underline{D}_2(t)}\right| &\leq \frac{|\overline{D}_1(t)||\underline{D}_1(t)-\underline{D}_2(t)|+|\underline{D}_1(t)||\overline{D}_1(t)-\overline{D}_2(t)|}{|\underline{D}_1(t)\underline{D}_2(t)|} \\ &\leq \overline{K}\int_t^T |D_1(s) - D_2(s)|ds, \end{aligned}$$

with $D_1(T) = D_2(T)$. The Grönwall's inequality implies $D_1 \equiv D_2$.

When considering a piece-wise $\mathcal{C}^1([0,T])$ function γ, one can repeat the above three steps in the proof for the right-most sub-interval, and then the second right-most sub-interval, etc. Eventually, we can show that there exists an unique piece-wise continuous function D solves equation (3.14).

6.2 Proof of Theorem 3.4

Assume $\{\beta_n\}_{n=0}^{+\infty}$ is a decreasing sequence of real numbers such that

$$\beta_n \to 1, n \to +\infty.$$

Let $\{D_n\}$ be the unique solution of the following equation in $\mathcal{C}^{pw}([0,T])$:

(6.4)
$$D_n(t) = \frac{\mu - r}{\sigma^2}\frac{\alpha C_n(t) - \beta_n\gamma(t)B_n(t)}{\gamma(t)\beta_n(\beta_n-1)B_n(t) + \gamma(t)\beta_n(\beta_n-1)A_n(t)^{\beta_n} - \alpha(\alpha-1)C_n(t)},$$

where

(6.5)
$$\begin{aligned} A_n(t) &= \exp\{\textstyle\int_t^T [r + D_n(t)(\mu - r)]ds\}, \\ B_n(t) &= \exp\{\beta_n \textstyle\int_t^T [r + D_n(s)(\mu-r) + \frac{1}{2}(\beta_n - 1)D_n(s)^2\sigma^2]ds\} - \exp\{\beta_n \int_t^T [r + D_n(s)(\mu - r)]ds\}, \\ C_n(t) &= \tfrac{1}{\alpha}\exp\{\alpha \textstyle\int_t^T [r + D_n(s)(\mu - r) + \frac{1}{2}(\alpha-1)D_n(s)^2\sigma^2]ds\}. \end{aligned}$$

We define $\tilde{A}_n(t), \tilde{B}_n(t), \tilde{C}_n(t)$ as $A_n(t), B_n(t), C_n(t)$ in (6.5) but replacing $D_n(.)$ by $\tilde{D}_n(.)$, where

(6.6)
$$\tilde{D}_n(t) = \max\left\{\frac{\mu - r}{\sigma^2}\frac{\alpha\tilde{C}_n(t) - \beta_n\gamma(t)\tilde{B}_n(t)}{\gamma(t)\beta_n(\beta_n-1)\tilde{B}_n(t) + \gamma(t)\beta_n(\beta_n-1)\tilde{A}_n(t)^{\beta_n} - \alpha(\alpha-1)\tilde{C}_n(t)}, 0\right\}.$$

By a similar argument in the proof of Proposition 3.1 and noting $\alpha < 1$, we can show that

$$0 \leq \tilde{D}_n(t) \leq \frac{\mu - r}{\sigma^2(1-\alpha)}.$$

Therefore, one can show that $\{\tilde{A}_n\}, \{\tilde{B}_n\}, \{\tilde{C}_n\}, \{\tilde{D}_n\}$ are uniformly bounded. In particular, $\exists$ $k_1, k_2 \in \mathbb{R}^+$, such that $0 < k_1 \leq \tilde{A}_n(t), \alpha\tilde{C}_n(t) \leq k_2$, $\forall t \in [0,T]$, for $n = 0, 2, \cdots$. Note that,

$$0 \leq \beta_n(\beta_n - 1)\frac{\tilde{A}_n(t)^{\beta_n}}{\alpha\tilde{C}_n(t)} \leq \beta_n(\beta_n-1)\frac{k_2^{\beta_0}}{k_1},$$

and

$$\begin{aligned} 0 \leq \frac{\tilde{B}_n(t)}{\alpha\tilde{C}_n(t)} &= \frac{\tilde{A}_n(t)^{\beta_n}}{\alpha\tilde{C}_n(t)}\left(\exp\left\{\tfrac{1}{2}\beta_n(\beta_n-1)\textstyle\int_t^T \tilde{D}_n(s)^2\sigma^2]ds\right\} - 1\right) \\ &\leq \frac{k_2^{\beta_0}}{k_1}\left(\exp\left\{\tfrac{1}{2}\beta_n(\beta_n-1)\textstyle\int_t^T \tilde{D}_n(s)^2\sigma^2]ds\right\} - 1\right) \\ &\leq \frac{k_2^{\beta_0}}{k_1}\left(\exp\left\{\tfrac{1}{2}\beta_n(\beta_n-1)\frac{(\mu-r)^2T}{\sigma^2(1-\alpha)^2}\right\} - 1\right). \end{aligned}$$

We know $\tilde{D}_n(t)$ is given by (6.6), thus, when $n \to +\infty$, $\tilde{D}_n(t)$ converges to $\frac{\mu-r}{\sigma^2(1-\alpha)}$ uniformly in $[0,T]$. Therefore, there exists a $N \in \mathbb{N}^+$ such that when $n > N$, $\tilde{D}_n(t)$ is positive for all $t \in [0,T]$, that is to say $D_n(t) = \tilde{D}_n(t)$ for $t \in [0,T]$. In particular, $D_n(t)$ converges to $\frac{\mu-r}{\sigma^2(1-\alpha)}$ uniformly in $[0,T]$.

Therefore, the equilibrium control for the utility deviation-risk model (3.5) is given by

$$\theta(t) = D(t)X(t) \to \frac{\mu - r}{\sigma^2(1-\alpha)} X(t),$$

which gives the Merton's portfolio, when $\beta \to 1$.

6.3 Proof to Theorem 3.5

Assume $\{\beta_n\}_{n=0}^{+\infty}$ is an increasing sequence of real numbers such that $\beta_n > 1$ for $n \in \mathbb{N}^+$ and

$$\beta_n \to +\infty, \; n \to +\infty.$$

Let $\{D_n\}$ be the unique solution in $\mathcal{C}^{pw}([0,T])$ of the equation (6.4) with $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ defined in (6.5). We have

$$\begin{aligned} D_n(t) &\geq \frac{\mu - r}{\sigma^2} \frac{-\beta_n \gamma(t) B_n(t)}{\gamma(t)\beta_n(\beta_n - 1)B_n(t) + \gamma(t)\beta_n(\beta_n-1)A_n^{\beta_n} - \alpha(\alpha-1)C_n(t)} \\ &\geq -\frac{\mu - r}{\sigma^2} \frac{1}{(\beta_n - 1)}. \end{aligned}$$

Hence,

$$D_n(t) \geq -\frac{k_1}{\beta_n},$$

where k_1, k_2 are positive constants independent of n.

On the other hand,

$$\begin{aligned} D_n(t) &\leq \frac{\mu - r}{\sigma^2} \frac{\alpha C_n(t)}{\gamma(t)\beta_n(\beta_n - 1)B_n(t) + \gamma(t)\beta_n(\beta_n-1)A_n^{\beta_n}(t) - \alpha(\alpha-1)C_n(t)} \\ &\leq \frac{\mu - r}{\sigma^2} \frac{\alpha C_n(t)}{\gamma(t)\beta_n(\beta_n-1)A_n^{\beta_n}(t)} \\ &= \frac{\mu - r}{\sigma^2} \frac{1}{\gamma(t)\beta_n(\beta_n-1)} \exp\left\{ \int_t^T (\alpha - \beta_n)[r + D_n(s)(\mu - r)] + \frac{1}{2}\alpha(\alpha-1)D_n(s)^2\sigma^2 ds \right\} \\ &\leq \frac{k_3}{\beta_n^2} \exp\left\{ \int_t^T -r\beta_n + k_4\beta_n|D_n(s)| + k_4 D_n(s)^2 ds \right\}, \end{aligned}$$

where k_3, k_4 are positive constants independent of n.

Therefore, we have

$$-\frac{k}{\beta_n} < D_n(t) < \frac{k}{\beta_n^2} \exp\left\{ \int_t^T -r\beta_n + k\beta_n|D_n(s)| + k\beta_n^2 D_n(s)^2 ds \right\},$$

where k is a positive constant independent of n. Let $\widetilde{D_n}(t) = \beta_n D_n(t)$ and we have

$$-k < \widetilde{D_n}(t) < \frac{k}{\beta_n} \exp\left\{ \int_t^T -r\beta_n + k|\widetilde{D_n}(s)| + k|\widetilde{D_n}(s)|^2 ds \right\},$$

with $-k < \widetilde{D_n}(T) < \frac{k}{\beta_n}$.

We prove $\widetilde{D_n}(t) \le \frac{k}{\beta_n}$, for all $t \in [0,T]$, for sufficiently large n. For any fixed n, we assume that there exists a $t \in [0,T]$ such that $\widetilde{D_n}(t) > \frac{k}{\beta_n}$. Let $\tau := \sup\{t \in [0,T] : \widetilde{D_n}(t) > \frac{k}{\beta_n}\}$. Note $t < \tau < T$. Then

$$\exp\left\{\int_\tau^T -r\beta_n + k|\widetilde{D_n}(s)| + k|\widetilde{D_n}(s)|^2 ds\right\} \ge 1.$$

There must exist a $s \in [\tau, T]$ such that,

$$-r\beta_n + k|\widetilde{D_n}(s)| + k|\widetilde{D_n}(s)|^2 \ge 0. \tag{6.7}$$

As $-k < \widetilde{D_n}(s) \le \frac{k}{\beta_n} < k$, (6.7) cannot be true when n is large.

Hence, we can draw a conclusion that when n is sufficiently large, $\widetilde{D_n}(t) \le \frac{k}{\beta_n}$ and $D_n(t) \le \frac{k}{\beta_n^2}$ for $t \in [0,T]$. Also, we have $-k/\beta_n < D_n(t)$ for $t \in [0,T]$. Thus, $D_n(t)$ uniformly converges to zero, when $n \to +\infty$.

6.4 Proof of Theorem 3.6

The HJB equation (3.9) holds true where $\theta \in \mathbb{U} = \mathbb{R}^+$. Therefore, we have

$$\theta^* = \max\{\frac{\mu - r}{\sigma^2}\frac{\alpha C(t) - \beta\gamma(t)B(t)}{\gamma(t)\beta(\beta-1)B(t) + \gamma(t)\beta(\beta-1)A(t)^\beta - \alpha(\alpha-1)C(t)}x, 0\} := D(t)x,$$

where $A(t), B(t)$ and $C(t)$ are defined in (3.13) with boundary condition $A(T) = 1, B(T) = 0, C(T) = \frac{1}{\alpha}$. It suffices to prove that there exists a unique solution D in $\mathcal{C}^{pw}([0,T])$ to equation (3.19), for which we use a similar argument in Proposition 3.1.

When $\alpha < 1$ and $\beta = 1$, equation (3.19) admits a unique solution $D(t) \equiv \frac{\mu - r}{\sigma^2(1-\alpha)}$. In the rest of this proof, we consider the case $\alpha \le 1, \beta < 1$.

Firstly, we assume $\gamma(.) \in \mathcal{C}^1([0,T])$. We construct a sequence of functions $A_n, B_n, C_n, D_n \in \mathcal{C}([0,T])$ by the following:

(6.8)
$$\begin{aligned}
D_0(t) &= 1,\\
A_n(t) &= \exp\{\textstyle\int_t^T [r + D_n(s)(\mu - r)]ds\},\\
B_n(t) &= \exp\{\beta \textstyle\int_t^T [r + D_n(s)(\mu - r) + \frac{1}{2}(\beta - 1)D_n(s)^2\sigma^2]ds\} - \exp\{\beta \int_t^T [r + D_n(s)(\mu - r)]ds\},\\
C_n(t) &= \tfrac{1}{\alpha}\exp\{\alpha \textstyle\int_t^T [r + D_n(s)(\mu - r) + \frac{1}{2}(\alpha - 1)D_n(s)^2\sigma^2]ds\},\\
D_{n+1}(t) &= \max\{\tfrac{\mu - r}{\sigma^2}\tfrac{\alpha C_n(t) - \beta\gamma(t)B_n(t)}{\gamma(t)\beta(\beta-1)B_n(t) + \gamma(t)\beta(\beta-1)A_n(t)^\beta - \alpha(\alpha-1)C_n(t)}, 0\},
\end{aligned}$$

for $n = 0, 1, \cdots$.

As a similar argument in Proportion 3.1 we prove that $\{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$ are uniformly bounded in $\mathcal{C}([0,T])$. In particular, $\exists\ k_1, k_2 \in \mathbb{R}^+$, such that $0 < k_1 \le A_n(t), \alpha C_n(t) \le k_2$, $\forall t \in [0,T]$, for $n = 0, 1, \cdots$.

Denote by

$$\mathfrak{D}_{n+1}(t) = \frac{\mu - r}{\sigma^2}\frac{\alpha C_n(t) - \beta\gamma(t)B_n(t)}{\gamma(t)\beta(\beta-1)B_n(t) + \gamma(t)\beta(\beta-1)A_n(t)^\beta - \alpha(\alpha-1)C_n(t)}.$$

Then $D_n(t) = \max\{\mathfrak{D}_n(t), 0\}$. By a similar proof in Proposition 3.1, one can show that $\{\mathfrak{D}_n\}$ is differentiable in $[0,T]$ and $\{\dot{\mathfrak{D}}_n\}$ is uniformly bounded.

For any $s,t\in[0,T]$,

$$\begin{aligned}|D_n(t)-D_n(s)| &\le |\mathfrak{D}_n(t)-\mathfrak{D}_n(s)|\\ &= \left|(t-s)\int_0^1 \dot{\mathfrak{D}}_n(s+u(t-s))du\right|\\ &\le k(t-s),\end{aligned}$$

where k is a constant independent of n. Hence $\{D_n\}$ is equicontinuous and we already show it is uniformly bounded, the Arzela-Ascoli Theorem implies that there exists a function $D\in\mathcal{C}([0,T])$ and sequence D_{n_i} such that $D_{n_i}\to D$. Therefore, equation (3.19) has a solution D.

To prove uniqueness, suppose we have two solutions to (3.19) D_1, D_2 in $\mathcal{C}([0,T])$. By adopting the notation of $\overline{D}_i(t), \underline{D}_i(t)$ in the proof of Proposition 3.1, we obtain the following.

Case I: for any $t\in[0,T]$ such that $D_1(t), D_2(t)>0$, we have

$$|D_1(t)-D_2(t)| = \left|\frac{\overline{D}_1(t)}{\underline{D}_1(t)}-\frac{\overline{D}_2(t)}{\underline{D}_2(t)}\right| \le \overline{K}\int_t^T |D_1(s)-D_2(s)|ds.$$

Case II: for any $t\in[0,T]$ such that $D_1(t)>0, D_2(t)=0$, that is $\frac{\overline{D}_1(t)}{\underline{D}_1(t)}>0$ and $\frac{\overline{D}_2(t)}{\underline{D}_2(t)}\le 0$,

$$|D_1(t)-D_2(t)| = \left|\frac{\overline{D}_1(t)}{\underline{D}_1(t)}-0\right| \le \left|\frac{\overline{D}_1(t)}{\underline{D}_1(t)}-\frac{\overline{D}_2(t)}{\underline{D}_2(t)}\right| \le \overline{K}\int_t^T |D_1(s)-D_2(s)|ds.$$

Case III: for any $t\in[0,T]$ such that $D_1(t)=0, D_2(t)>0$, same argument in Case II gives

$$|D_1(t)-D_2(t)| \le \overline{K}\int_t^T |D_1(s)-D_2(s)|ds.$$

Case IV: for any $t\in[0,T]$ such that $D_1(t), D_2(t)=0$,

$$|D_1(t)-D_2(t)| = 0 \le \overline{K}\int_t^T |D_1(s)-D_2(s)|ds.$$

Therefore, Grönwall's inequality gives $D_1\equiv D_2$.

When considering a piece-wise $\mathcal{C}^1([0,T])$ function γ, one can repeat the above arguments in the proof for the right-most sub-interval, and then the second right-most sub-interval, etc. Eventually, we can show that there exists a unique piece-wise continuous function D solves equation (3.19).

Acknowledgments. The authors are grateful to two anonymous reviewers whose constructive comments and suggestions have helped to improve the paper of the previous two versions. This research of J.W. Gu was supported in part by the NSF of China under Grant 11801262.

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