Dynamic Convex Duality in Constrained Utility Maximization

Yusong Li* and Harry Zheng†

Abstract

In this paper, we study a constrained utility maximization problem following the convex duality approach. After formulating the primal and dual problems, we construct the necessary and sufficient conditions for both the primal and dual problems in terms of forward and backward stochastic differential equations (FBSDEs) plus some additional conditions. Such formulation then allows us to explicitly characterize the primal optimal control as a function of the adjoint process coming from the dual FBSDEs in a dynamic fashion and vice versa. We also find that the optimal wealth process coincides with the adjoint process of the dual problem and vice versa. Finally we solve three constrained utility maximization problems, which contrasts the simplicity of the duality approach we propose and the technical complexity of solving the primal problem directly.

Keywords: convex duality, primal and dual FBSDEs, utility maximization, convex portfolio constraints

AMS MSC2010: 91G80, 93E20, 49N05, 49N15

1 Introduction

One of the most commonly studied problems in mathematical economics is the optimal investment problem. Such problems have their goal of constructing the investment strategy that maximizes the agent's expected utility of the wealth at the end of the planning horizon. Here we assume that trading strategies take values in a closed convex set which is general enough to include short selling, borrowing, and other trading restrictions, see [13].

There has been extensive research in dynamic portfolio optimization. The stochastic control approach was first introduced in the two landmark papers of Merton [17, 18], which was wedded to the Hamilton-Jacobi-Bellman equation and the requirement of an underlying Markov state process. The optimal investment problem in a non-Markov setting was solved using the martingale method by, among others, Pliska [21], Cox and Huang

*Department of Mathematics, Imperial College, London SW7 2BZ, UK. Email: y.li11@imperial.ac.uk

†Corresponding Author. Department of Mathematics, Imperial College, London SW7 2BZ, UK. Email: h.zheng@imperial.ac.uk

[4, 5], Karatzas et al. [11]. The stochastic duality theory of Bismut [1] was first employed to study the constrained (no-short-selling) optimal investment problem in Shreve and Xu [25]. The effectiveness of the convex duality method was later adopted to tackle more general incomplete market models in the works of, among others, Karatzas et al. [12], Pearson and He [7, 8], Cvitanić and Karatzas [6]. The spirit of this approach is to suitably embed the constrained problem in an appropriate family of unconstrained ones and find a member of this family for which the corresponding optimal policy satisfies the constraints. However, despite the evident power of this approach, it is nevertheless true that obtaining the corresponding dual problem remains a challenge as it often involves clever experimentation and subsequently show to work as desired. To bring some transparency to the dual problem, Labbé and Heunis [16] established a simple synthetic method of arriving at a dual functional, bypassing the need to formulate a fictitious market. It often happens that the dual problem is much nicer than the primal problem in the sense that it is easier to show the existence of a solution and in some cases explicitly obtain a solution to the dual problem than it is to do likewise for the primal problem.

In this paper, we first follow [14] to convert the original primal problem into an equivalent dual problem by the supermartingale approach, then progress, following the approach in [9, 22], to simultaneously characterise the necessary and sufficient optimality conditions for both the primal and dual problems as systems of forward and backward stochastic differential equations (FBSDEs) coupled with additional optimality conditions. Such formulation allows us to characterize the primal optimal control as a function of the adjoint process coming from the dual FBSDEs in a dynamic fashion and vice versa. Moreover, we also find that the optimal wealth process coincides with the adjoint process of the dual problem and vice versa. To the best of our knowledge, this is the first time the dynamic relations of the primal and dual problems have been explicitly established for constrained utility maximization problems under a non-Markov setting. After establishing the optimality conditions and the relations for the primal and dual problems, we solve three constrained utility maximization problems with both Markov and non-Markov setups. Instead of tackling the primal problem directly, we start from the dual problem and then construct the optimal solution to the primal problem from that to the dual problem. All examples contrast the simplicity of the duality approach we propose and the technical complexity of solving the primal problem directly.

The rest of the paper is organised as follows. In Section 2 we set up the market model and formulate the primal and dual problems following the approach in [14]. In Section 3 we state and prove the main results of necessary and sufficient optimality conditions for the primal and dual problems and their connections in a dynamic fashion. In Section 4 we give three examples to demonstrate the effectiveness of the dynamic duality approach in solving constrained utility maximization problems. Section 5 concludes the paper.

2 Market Model and Primal and Dual Problems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is defined a $\mathbb{R}^N$-valued standard Brownian motion $\{W(t), t \in [0,T]\}$ with $T > 0$ denoting a fixed terminal time. Let

$\{\mathcal{F}_t, t \in [0,T]\}$ be the standard filtration induced by W, where

$$\mathcal{F}_t \triangleq \sigma\{W(s), s \in [0,t]\} \bigvee \mathcal{N}(P),\ t \in [0,T],$$

in which $\mathcal{N}(P)$ denotes the collection of all $\mathbb{P}$-null events in $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mathcal{F}^*$ the σ-algebra of $\mathcal{F}_t$ progressively measurable sets on $\Omega \times [0,T]$. For any stochastic process $v : \Omega \times [0,T] \to \mathbb{R}^m,\ m \in \mathbb{N}^+$, we write $v \in \mathcal{F}^*$ to indicate v is $\mathcal{F}^*$ measurable. We introduce the following notation:

$$\mathcal{H}^p(0,T;\mathbb{R}^m) \triangleq \left\{\xi : \Omega \times [0,T] \to \mathbb{R}^m \mid \xi \in \mathcal{F}^*, E\left[\int_0^T |\xi(t)|^p dt\right] < \infty\right\},$$

where $p \geq 1$.

Consider a market consisting of a bank account with price $\{S_0(t)\}$ given by

$$dS_0(t) = r(t)S_0(t)dt,\ 0 \leq t \leq T,\ S_0(0) = 1, \tag{1}$$

and N stocks with prices $\{S_n(t)\},\ n = 1, \cdots, N$, given by

$$dS_n(t) = S_n(t)\{b_n(t)dt + \sum_{m=1}^{N} \sigma_{nm}(t)dW_m(t)\},\ 0 \leq t \leq T,\ S_n(0) > 0. \tag{2}$$

Throughout the paper we assume that the interest rate $\{r(t)\}$, the appreciation rates on stocks denoted by entries of the $\mathbb{R}^N$-valued process $\{b(t)\}$ and the volatility rates denoted by entries of the $N \times N$ matrix valued process $\{\sigma(t)\}$ are uniformly bounded $\{\mathcal{F}_t\}$-progressively measurable scalar processes on $\Omega \times [0,T]$. We also assume that there exists a positive constant k such that

$$z^\intercal \sigma(t)\sigma^\intercal(t)z \geq k|z|^2$$

for all $(z, \omega, t) \in \mathbb{R}^N \times \Omega \times [0,T]$, where $z^\intercal$ is the transpose of z. According to [25, (2.4) and (2.5)], the strong non-degeneracy condition above ensures that matrices $\sigma(t), \sigma^\intercal(t)$ are invertible and uniformly bounded.

Consider a small investor with initial wealth $x_0 > 0$ and a self-financing strategy. Define the set of admissible portfolio strategies by

$$\mathcal{A} := \left\{\pi \in \mathcal{H}^2(0,T;\mathbb{R}^N) : \pi(t) \in K \textit{ for } t \in [0,T] \textit{ a.e.}\right\},$$

where $K \subseteq \mathbb{R}^N$ is a closed convex set with $0 \in K$ and π is a portfolio process with each entry $\pi_n(t)$ defined as the fraction of the wealth invested in the stock n for $n = 1, \ldots, N$ at time t. Given any $\pi \in \mathcal{A}$, the investor's total wealth X^π satisfies the following dynamics

$$\begin{cases} dX^\pi(t) = X^\pi(t)\{[r(t) + \pi^\intercal(t)\sigma(t)\theta(t)]dt + \pi^\intercal(t)\sigma(t)dW(t)\},\ 0 \leq t \leq T, \\ X^\pi(0) = x_0, \end{cases} \tag{3}$$

where $\theta(t) := \sigma^{-1}(t)\,[b(t) - r(t)\mathbf{1}]$ is the market price of risk at time t and is uniformly bounded and $\mathbf{1} \in \mathbb{R}^N$ has all unit entries. A pair (X^π, π) is *admissible* if $\pi \in \mathcal{A}$ and X^π satisfies (3) with control process π.

Remark 1. *Here we define the nth entry of $\pi(t)$ as the fraction of small investor's wealth invested in the stock n at time t. Such setup ensures the positivity of the wealth process X^π, but surrenders the Lipschitz property of the coefficients in both X and π. Hence, the stochastic maximum principle developed in [3, 19] are not directly applicable in our case.*

Let $U : [0, \infty) \to \mathbb{R}$ be a given utility function that is twice continuously differentiable, strictly increasing, strictly concave and satisfies the following conditions:

$$U(0) \triangleq \lim_{x \searrow 0} U(x) > -\infty, \ \lim_{x \searrow 0} U'(x) = \infty, \ \lim_{x \to \infty} U'(x) = 0.$$

Define the value of the expected utility maximization problem as

$$V \triangleq \sup_{\pi \in \mathcal{A}} E\left[U\left(X^\pi(T)\right)\right]. \tag{4}$$

To avoid trivialities, we assume that

$$-\infty < V < +\infty.$$

Any $\hat{\pi} \in \mathcal{A}$ satisfying $E\left[U\left(X^{\hat{\pi}}(T)\right)\right] = V$ is called the *optimal control (portfolio)*, the corresponding $X^{\hat{\pi}}$ is called the *optimal state (wealth) process*.

In the rest of this section, we formulate the dual problem following the approach in [14]. Define the dual function of U by

$$\tilde{U}(y) \triangleq \sup_{x>0}(U(x) - xy).$$

It is clear that $\tilde{U}(y) = \infty$ if $y < 0$ and $\tilde{U}$ is twice continuously differentiable, strictly decreasing and strictly convex on $(0, \infty)$. The dual process Y is a strictly positive process and has the following semimartingale decomposition

$$\begin{cases} dY(t) = Y(t)\{\alpha(t)dt + \beta^\intercal(t)dW(t)\}, \ 0 \le t \le T, \\ Y(0) = y, \end{cases} \tag{5}$$

where processes α and β are chosen such that $X^\pi Y$ is a supermartingale for all admissible control processes $\pi \in \mathcal{A}$. Using Ito's lemma, we have

$$d(X^\pi(t)Y(t)) = X^\pi(t)Y(t)\{[r+\pi^\intercal(t)\sigma(t)\theta(t)+\alpha(t)+\pi^\intercal(t)\sigma(t)\beta(t)]dt+[\pi^\intercal(t)\sigma(t)+\beta^\intercal(t)]dW(t)\}.$$

To make $X^\pi Y$ a supermartingale, we must have

$$r + \pi^\intercal\sigma(t)\theta(t) + \alpha(t) + \pi^\intercal\sigma(t)\beta(t) \le 0$$

for all $\pi \in K$ a.s. for a.e. $t \in [0, T]$, which is equivalent to

$$r + \alpha(t) + \delta_K(-\sigma(t)(\theta(t) + \beta(t))) \le 0,$$

where $\delta_K(\cdot)$ is the support function of the set $-K$, defined by

$$\delta_K(z) \triangleq \sup_{\pi \in K}\{-\pi^\intercal z\}, z \in \mathbb{R}^N.$$

Define $v(t) \triangleq -\sigma(t)(\theta(t)+\beta(t))$. We have

$$\alpha(t) \le -(r+\delta_K(v(t))), \quad \beta(t) = -(\sigma^{-1}(t)v(t)+\theta(t)). \tag{6}$$

From the definition of the dual function, we have

$$E[U(X^\pi(T))] \le E[\tilde{U}(Y(T))] + E[X^\pi(T)Y(T)] \le E[\tilde{U}(Y(T))] + x_0 y.$$

The second inequality above is due to $X^\pi Y$ being a supermartingale. This leads to

$$\sup_\pi E[U(X^\pi(T))] \le \inf_{y,\alpha,v} (E[\tilde{U}(Y(T))] + x_0 y).$$

For any fixed y and v, the solution Y of the SDE (5) satisfying conditions (6) is bounded above by the process $Y^{(y,v)}$ satisfying the SDE

$$\begin{cases} dY^{(y,v)}(t) = -Y^{(y,v)}(t)\left\{[r(t)+\delta_K(v(t))]dt + [\theta(t)+\sigma^{-1}(t)v(t)]^\intercal\, dW(t)\right\}, \ 0 \le t \le T, \\ Y^{(y,v)}(0) = y, \end{cases} \tag{7}$$

that is, $Y(t) \le Y^{(y,v)}(t)$ a.s. for $0 \le t \le T$. Since $\tilde{U}$ is a strictly decreasing function, we have $E[\tilde{U}(Y(T))] \ge E[\tilde{U}(Y^{(y,v)}(T))]$ for any fixed y and v, which implies the optimal α is determined by $\alpha(t) = -(r+\delta_K(v(t)))$. The process $Y^{(y,v)}$ is a dual process and $v \in \mathcal{D}$ is a dual control process, where the set $\mathcal{D}$ is defined by

$$\mathcal{D} \triangleq \left\{ v \triangleq \Omega \times [0,T] \to \mathbb{R}^N | v \in \mathcal{F}^* \ \textit{and} \ \int_0^T \left[\delta_K(v(t)) + |v(t)|^2\right] dt < \infty \ \textit{a.s.} \right\}.$$

If K is a closed convex cone, then $\delta_K(z) = 0$ if $z \in \tilde{K}$ and ∞ otherwise, where $\tilde{K} = \{z : z^\intercal \pi \ge 0, \forall \pi \in K\}$ is the positive polar cone of K. In that case, the dual process $Y^{(y,v)}$ satisfies the SDE

$$\begin{cases} dY^{(y,v)}(t) = -Y^{(y,v)}(t)\left\{r(t)dt + [\theta(t)+\sigma^{-1}(t)v(t)]^\intercal\, dW(t)\right\}, \ 0 \le t \le T, \\ Y^{(y,v)}(0) = y, \end{cases} \tag{8}$$

where v is square integrable and $v(t) \in \tilde{K}$ a.s. for a.e. $t \in [0,T]$.

The optimal value of the dual minimization problem is defined by

$$\tilde{V} \triangleq \inf_{(y,v)\in(0,\infty)\times\mathcal{D}} \left(x_0 y + E\left[\tilde{U}(Y^{(y,v)}(T))\right]\right). \tag{9}$$

Any $(\hat{y},\hat{v}) \in (0,\infty)\times\mathcal{D}$ satisfying $x_0\hat{y} + E\left[\tilde{U}(Y^{(\hat{y},\hat{v})}(T))\right] = \tilde{V}$ is called the *optimal dual control* and the corresponding $Y^{(\hat{y},\hat{v})}$ is called the *optimal dual process*. Dual problem (9) can be naturally solved in two steps: first, fix y and solve

$$\tilde{u}(y) \triangleq \inf_{v\in\mathcal{D}} E\left[\tilde{U}(Y^{(y,v)}(T))\right],$$

and, second, solve

$$\tilde{V} = \inf_{y>0} (x_0 y + \tilde{u}(y)).$$

If coefficients $r(t), b(t), \sigma(t)$ in (1) and (2) are deterministic, then the wealth process X^π is a Markov controlled process and the stochastic optimal control theory may be used to solve the first stage problem. We illustrate this approach with two examples in Section 4.

In this paper, instead of applying the convex duality method of [1], we use the machinery of the stochastic maximum principle and BSDEs to derive the necessary and sufficient conditions of the primal and dual problems separately. After establishing the optimality conditions as two systems of FBSDEs, we explicitly characterise the primal optimal solution as a function of the adjoint process coming from the dual FBSDEs in a dynamic fashion and vice versa.

Remark 2. *We may also derive the dual process Y and the dual problem following the approach in [16]. The key steps are to first transform the primal dynamic constrained problem (4) into a static unconstrained optimization, then to use convex analysis to find its static dual optimization, and finally to construct the dual dynamic constrained problem (9). We briefly outline the first step to illustrate the idea.*

Given any continuous $\{\mathcal{F}_t\}$ semimartingale process X, we write $X \in \mathcal{S} \triangleq \mathbb{R} \times \mathcal{H}^1(0,T;\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^N)$ if

$$X(t) = X_0 + \int_0^t \dot{X}(s)ds + \int_0^t \Lambda_X^\intercal(s)dW(s),\ 0 \le t \le T,$$

where $(X_0, \dot{X}, \Lambda_X) \in \mathcal{S}$. Define the following penalty functions:

$$\begin{aligned}
l_0(x) &\triangleq \begin{cases} 0, & \text{if } x = x_0, \\ +\infty, & \text{otherwise}, \end{cases} \\
l_T(x) &\triangleq \begin{cases} -U(x), & \text{if } x \in (0,\infty), \\ +\infty, & \text{otherwise}, \end{cases} \\
L(t,x,v,\xi) &\triangleq \begin{cases} 0, & \text{if } x > 0, v = xr(t) + \xi^\intercal\theta(t) \text{ and } x^{-1}[\sigma^\intercal(t)]^{-1}\xi \in K, \\ +\infty, & \text{otherwise}. \end{cases}
\end{aligned}$$

Then problem (4) is equivalent to

$$\inf_{X\in\mathcal{S}} \Big(l_0(X(0)) + E\left[l_T(X(T))\right] + E\Big[\int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t))dt\Big]\Big). \tag{10}$$

Penalty functions l_0 and L ensure only those $X \in \mathcal{S}$ satisfying $X(0) = x_0$ and (3) are used for optimization. Problem (10) is a static convex minimization problem in a Banach space $\mathcal{S}$. We can then use Fenchel conjugate functions of l_0, l_T, L to find its static dual optimization and finally recover the dual process Y in (7) and the dual problem (9), see [16] for details.

For utility maximization with X^π being a wealth process, our way of deriving the dual problem is easier and more straightforward than the alternative method, however, the approach of [16] is more general and may help derive dual problems for more complicated problems.

3 Main Results

In this section, we derive the necessary and sufficient optimality conditions for the primal and dual problems and show the dynamic connection between the optimal solutions through their corresponding FBSDEs.

Given an admissible control $\pi \in \mathcal{A}$ and a solution X^π of the SDE (3), the associated adjoint equation is the following linear BSDE in the unknown processes $p_1 \in \mathcal{H}^2(0,T;\mathbb{R})$ and $q_1 \in \mathcal{H}^2(0,T;\mathbb{R}^N)$:

$$\begin{cases} dp_1(t) = -\left\{ \left[r(t) + \pi^\intercal(t)\sigma(t)\theta(t) \right] p_1(t) + q_1^\intercal(t)\sigma^\intercal(t)\pi(t) \right\} dt + q_1^\intercal(t) dW(t), \\ p_1(T) = -U'(X^\pi(T)). \end{cases} \tag{11}$$

Remark 3. *The BSDE (11) is from the stochastic maximum principle (SMP). Since coefficients of the SDE (3) are not Lipschitz continuous due to unboundedness of X and π and utility function U is not Lipschitz continuous and is only defined on the positive real line, we cannot directly apply the standard SMP in [3, 19]. However, we can formally apply the SMP to find the form of BSDE for adjoint processes and then prove it rigorously. Define the Hamiltonian function*

$$H(t, x, \pi, p_1, q_1) \triangleq x[r(t) + \pi^\intercal\sigma(t)\theta(t)]p_1 + x\pi^\intercal\sigma(t)q_1.$$

Then the adjoint process is a pair of processes (p_1, q_1) satisfying the following BSDE

$$dp_1(t) = -\frac{\partial}{\partial x} H(t, X^\pi(t), \pi(t), p_1(t), q_1(t))dt + q_1^\intercal(t)dW(t)$$

with the terminal condition $p_1(T) = -U'(X^\pi(T))$, which is the BSDE (11).

Assumption 4. *The utility function U satisfies the following conditions:*

(i) $x \to xU'(x)$ is non-decreasing on $(0, \infty)$.

(ii) There exist $\beta \in (0,1)$ and $\gamma \in (1, \infty)$ such that $\beta U'(x) \geq U'(\gamma x)$ for all $x \in (0, \infty)$.

Moreover, for all $\pi \in \mathcal{A}$ and corresponding X^π satisfying the SDE (3), $E[|U(X^\pi(T))|] < \infty$ and $E\left[(U'(X^\pi(T))X^\pi(T))^2 \right] < \infty$.

Remark 5. *Assumption 4 corresponds to [13, Remark 3.4.4]. Assumption 4(i) implies that the Arrow-Pratt measure of relative risk aversion $R(x) \triangleq -xU''(x)/U'(x)$ does not exceed 1 and $z \to \tilde{U}(e^z)$ is convex when $\tilde{U}$ is the dual function of U. Assumption 4(ii) is equivalent to $\tilde{U}'(\beta y) \geq \gamma \tilde{U}'(y)$ for all $y \in (0, \infty)$.*

Lemma 6. *Let $\hat{\pi} \in \mathcal{A}$ and the strictly positive, adapted process $X^{\hat{\pi}}$ satisfy the SDE (3). Then there exists a unique solution $(\hat{p}_1, \hat{q}_1)$ to the adjoint BSDE (11).*

Proof. According to Assumption 4, the process defined as

$$\alpha(t) \triangleq E\left[-X^{\hat{\pi}}(T)U'(X^{\hat{\pi}}(T) \middle| \mathcal{F}_t \right], \; t \in [0, T]$$

is square integrable. In addition, it is the unique solution of the BSDE

$$\alpha(t) = -X^{\hat{\pi}}(T)U'(X^{\hat{\pi}}(T)) - \int_t^T \beta^\intercal(t)dW(t),\ t \in [0,T], \tag{12}$$

where β is an adapted, square integrable process with values in $\mathbb{R}^N$. Applying Ito's lemma to $\alpha(t)/X^{\hat{\pi}}(t)$, we have

$$\begin{aligned} d\frac{\alpha(t)}{X^{\hat{\pi}}(t)} &= \frac{\beta^\intercal(t)}{X^{\hat{\pi}}(t)}dW(t) - \frac{\alpha(t)}{X^{\hat{\pi}}(t)}\left\{[r(t)+\hat{\pi}^\intercal(t)\sigma(t)\theta(t)]dt + \hat{\pi}^\intercal(t)\sigma(t)dW(t) - |\hat{\pi}^\intercal(t)\sigma(t)|^2dt\right\} \\ &\quad - \frac{\hat{\pi}^\intercal(t)\sigma(t)\beta(t)}{X^{\hat{\pi}}(t)}dt \\ &= -\left\{[r(t)+\hat{\pi}^\intercal(t)\sigma(t)\theta(t)]\hat{p}_1(t) + \hat{q}_1^\intercal(t)\sigma^\intercal(t)\hat{\pi}(t)\right\}dt + \hat{q}_1^\intercal(t)dW(t), \end{aligned}$$

where

$$\hat{p}_1(t) \triangleq \frac{\alpha(t)}{X^{\hat{\pi}}(t)}, \quad \hat{q}_1(t) \triangleq \frac{\beta(t)}{X^{\hat{\pi}}(t)} - \frac{\alpha(t)\sigma^\intercal(t)\hat{\pi}(t)}{X^{\hat{\pi}}(t)}. \tag{13}$$

Hence, we conclude that $(\hat{p}_1, \hat{q}_1)$ solves the adjoint BSDE (11). □

3.1 Necessary and sufficient conditions for primal problems

We now state the necessary and sufficient optimality conditions for the primal problem.

Theorem 7. *(Primal problem and associated FBSDE) Let $\hat{\pi} \in \mathcal{A}$. Then $\hat{\pi}$ is optimal for the primal problem if and only if the solution $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ of FBSDE*

$$\begin{cases} dX^{\hat{\pi}}(t) = X^{\hat{\pi}}(t)\{[r(t)+\hat{\pi}^\intercal(t)\sigma(t)\theta(t)]dt + \hat{\pi}^\intercal(t)\sigma(t)dW(t)\}, \\ X^{\hat{\pi}}(0) = x_0, \\ d\hat{p}_1(t) = -\left\{[r(t)+\hat{\pi}^\intercal(t)\sigma(t)\theta(t)]\,\hat{p}_1(t) + \hat{q}_1^\intercal(t)\sigma^\intercal(t)\hat{\pi}(t)\right\}dt + \hat{q}_1^\intercal(t)dW(t), \\ \hat{p}_1(T) = -U'(X^{\hat{\pi}}(T)) \end{cases} \tag{14}$$

satisfies the condition

$$-X^{\hat{\pi}}(t)\sigma(t)\left[\hat{p}_1(t)\theta(t) + \hat{q}_1(t)\right] \in N_K(\hat{\pi}(t)),\ \forall t \in [0,T],\ \mathbb{P}-a.s., \tag{15}$$

where $N_K(x)$ is the normal cone of the closed convex set K at $x \in K$, defined as

$$N_K(x) \triangleq \left\{y \in \mathbb{R}^N : \forall x^* \in K, y^\intercal(x^*-x) \le 0\right\}.$$

Proof. Let $\tilde{\pi} \in \mathcal{A}$ be an admissible control and $\rho \triangleq \tilde{\pi} - \hat{\pi}$. Let

$$\tau_n \triangleq T \wedge \inf\left\{t \ge 0, \int_0^t |\rho(s)\sigma(s)|^2ds \ge n \ or \ \int_0^t |\rho^\intercal(s)\sigma(s)\sigma^\intercal(s)\hat{\pi}(s)|^2ds \ge n\right\}.$$

Hence, $\lim_{n\to\infty}\tau_n = T$ almost surely. Define $\rho_n(t) \triangleq \rho(t)1_{\{t\le\tau_n\}}$ for $0 \le t \le T$ and

$$\phi_n(\varepsilon) \triangleq U\left(X^{\hat{\pi}+\varepsilon\rho_n}(T)\right)$$

for $0 \le \varepsilon \le 1$. Setting $G(x) \triangleq U(x_0 e^x)$ and taking derivatives, we have

$$\begin{aligned} G'(x) &= U'(x_0 e^x) x_0 e^x \ge 0, \\ G''(x) &= x_0 e^x \left(U'(x_0 e^x) + U'(x_0 e^x) x_0 e^x \right) \le 0, \end{aligned}$$

by Assumption 4. Differentiating ϕ_n on $(0,1)$, we have

$$\begin{aligned} \phi_n'(\varepsilon) =& G'(\cdot) \left[\int_0^T \left(\rho_n^\intercal(t)\sigma(t)\theta(t) - \rho_n^\intercal(t)\sigma(t)\sigma^\intercal(t)\left(\hat{\pi}(t) + \varepsilon\rho_n(t)\right)\right) dt + \int_0^T \rho_n^\intercal(t)\sigma(t) dW(t) \right], \\ \phi_n''(\varepsilon) =& G''(\cdot) \left[\int_0^T \int_0^T \left(\rho_n^\intercal(t)\sigma(t)\theta(t) - \rho_n^\intercal(t)\sigma(t)\sigma^\intercal(t)\left(\hat{\pi}(t) + \varepsilon\rho_n(t)\right)\right) dt + \int_0^T \rho_n^\intercal(t)\sigma(t) dW(t) \right]^2 \\ & - G'(\cdot) \left[\int_0^T \rho_n^\intercal(t)\sigma(t)\sigma^\intercal(t)\rho_n(t) dt \right] \le 0. \end{aligned}$$

Hence we conclude that the function $\Phi_n(\varepsilon) \triangleq (\phi_n(\varepsilon) - \phi(0))/\varepsilon$ is a decreasing function and we have

$$\lim_{\varepsilon \to 0} \Phi_n(\varepsilon) = U'(X^{\hat{\pi}}(T)) X^{\hat{\pi}}(T) H_n^\rho(T),$$

where $H_n^\rho(t) \triangleq \int_0^t \left(\rho_n^\intercal(s)\sigma(s)\theta(s) - \rho_n^\intercal(s)\sigma(s)\sigma^\intercal(s)\hat{\pi}(s)\right) ds + \int_0^t \rho_n^\intercal(s)\sigma(s) dW(s)$. Moreover, we obtain

$$E\left[\left| U'(X^{\hat{\pi}}(T)) X^{\hat{\pi}}(T) H_n^\rho(T) \right| \right] \le E\left[\left(U'(X^{\hat{\pi}}(T)) X^{\hat{\pi}}(T) \right)^2 \right]^{\frac{1}{2}} E\left[H_n^\rho(T)^2 \right]^{\frac{1}{2}} < \infty.$$

Note that for $\varepsilon \in [0,1], \Phi_n(\varepsilon) \ge \Phi_n(1) = U(X^{\hat{\pi}+\rho_n}(T)) - U(X^{\hat{\pi}}(T))$ with $E\left[\Phi_n(1)\right] < \infty$. Therefore the sequence $\Phi_n(\varepsilon)$ is bounded from below. By the Monotone Convergence Theorem, we have

$$\lim_{\varepsilon \to 0} \frac{E\left[U(X^{\hat{\pi}+\varepsilon\rho_n}(T)) \right] - E\left[U(X^{\hat{\pi}}(T)) \right]}{\varepsilon} = E\left[U'(X^{\hat{\pi}}(T)) X^{\hat{\pi}}(T) H_n^\rho(T) \right].$$

In addition, since $\hat{\pi}$ is optimal, we conclude

$$E\left[U'(X^{\hat{\pi}}(T)) X^{\hat{\pi}}(T) H_n^\rho(T) \right] \le 0. \tag{16}$$

Let (α, β) be defined as in (12) and $(\hat{p}_2), \hat{q}_2)$ be the adjoint process corresponding to $\hat{\pi}$. Applying Ito's lemma to $-\alpha(t) H_n^\rho(t)$ and simplifying the terms using (12) and (13), we obtain

$$-d\alpha(t) H_n^\rho(t) = -X^{\hat{\pi}}(t) \rho_n^\intercal(t)\sigma(t)(\hat{p}_1(t)\theta(t) + \hat{q}_1(t)) dt + \left[\beta^\intercal(t) H_n^\rho(t) - \alpha(t)\rho_n^\intercal(t)\sigma(t) \right] dW(t). \tag{17}$$

Next, we prove that the local martingale $\int_0^t (\beta^\intercal(s) H^\rho(s) - \alpha(s)\rho^\intercal(s)\sigma(s)) dW(s)$ is a true martingale. We have

$$
\begin{aligned}
& E\left[\sup_{t\in[0,T]} |H_n^\rho(t)|^2\right] \\
= \;& E\left[\sup_{t\in[0,T]} \left|\int_0^t \left(\rho_n^\intercal(s)\sigma(s)\theta(s) - \rho_n^\intercal(s)\sigma(s)\sigma^\intercal(s)\hat{\pi}(s)\right) ds + \int_0^t \rho_n^\intercal(s)\sigma(s)dW(s)\right|^2\right] \\
\leq \;& C\Bigg\{ E\left[\sup_{t\in[0,T]} \left|\int_0^t \rho_n^\intercal(s)\sigma(s)dW(s)\right|^2\right] \\
& + E\left[\sup_{t\in[0,T]} \left|\int_0^t \left(\rho_n^\intercal(s)\sigma(s)\theta(s) - \rho_n^\intercal(s)\sigma(s)\sigma^\intercal(s)\hat{\pi}(s)\right) ds\right|^2\right] \Bigg\} \\
\leq \;& C\left\{ E\left[\int_0^T |\rho_n^\intercal(s)\sigma(s)|^2 ds\right] + E\left[\int_0^T |\rho_n^\intercal(s)\sigma(s)\theta(s)|^2 ds\right] + E\left[\int_0^T |\rho_n^\intercal(s)\sigma(s)\sigma^\intercal(s)\hat{\pi}(s)|^2 ds\right]\right\} \\
< \;& \infty.
\end{aligned}
$$

Here we have used the Burkholder-Davis-Gundy inequality in the second last inequality above. In addition, we have

$$
E\left[\int_0^T |\alpha(s)\rho_n^\intercal(s)\sigma(s)|^2 ds\right] < \infty.
$$

Hence, (16) can be reduced to the following

$$
E\left[\int_0^{\tau_n} -X^{\hat{\pi}}(t)\rho_n^\intercal(t)\sigma(t)\Big(\hat{p}_1(t)\theta(t) + \hat{q}_1(t)\Big) dt\right] \leq 0, \ \forall n \in \mathbb{N}. \tag{18}
$$

Define the following sets:

$$
B \triangleq \left\{(t,\omega) \in [0,T] \times \Omega : (\pi^\intercal - \hat{\pi}^\intercal(t))\,\sigma(t)\,(\hat{p}_1(t)\theta(t) + \hat{q}_1(t)) < 0, \ \textit{for } \forall \pi \in K\right\},
$$

and, for any $\pi \in K$,

$$
B^\pi \triangleq \left\{(t,\omega) \in [0,T] \times \Omega : (\pi^\intercal - \hat{\pi}^\intercal(t))\,\sigma(t)\,(\hat{p}_1(t)\theta(t) + \hat{q}_1(t)) < 0\right\}.
$$

Obviously for each $t \in [0,T]$, $B_t^\pi \in \mathcal{F}_t$. Consider the control $\tilde{\pi} : [0,T] \times \Omega \to K$, defined by

$$
\tilde{\pi}(t,\omega) \triangleq \begin{cases} \pi, & \textit{if } (t,\omega) \in B^\pi \\ \hat{\pi}(t,\omega), & \textit{otherwise.} \end{cases}
$$

Then $\tilde{\pi}$ is adapted and there exists $n^* \in \mathbb{N}$ such that

$$
E\left[\int_0^{\tau_n} X^{\hat{\pi}}(t)\,(\tilde{\pi}^\intercal(t) - \hat{\pi}^\intercal(t))\,\sigma(t)\,(\hat{p}_1(t)\theta(t) + \hat{q}_1(t))\,dt\right] < 0, \ \forall n > n^*,
$$

contradicting (18), unless $(Leb \otimes \mathbb{P})\{B^\pi\} = 0$ for all $\pi \in K$. Since $\mathbb{R}^N$ is a separable metric space, we can find a countable dense subset $\{\pi_n\}$ of K. Denote by $\hat{B} = \cup_{n=1}^\infty B^{\pi_n}$. Then $(Leb \otimes \mathbb{P})\{\hat{B}\} = (Leb \otimes \mathbb{P})\{\cup_{n=1}^\infty B^{\pi_n}\} \leq \sum_{n=1}^\infty (Leb \otimes \mathbb{P})\{B^{\pi_n}\} = 0$. Hence, we conclude that

$$
-X^{\hat{\pi}}(t)\sigma(t)\left[\hat{p}_1(t)\theta(t) + \hat{q}_1(t)\right] \in N_K(\hat{\pi}(t)), \ \forall t \in [0,T], \ \mathbb{P} - a.s.
$$

We have proved the necessary condition.

Now we prove the sufficient condition. Let $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ be a solution to the FBSDE (14) and satisfy condition (15). Applying Ito's lemma, we have

$$
\begin{aligned}
&\left(X^{\hat{\pi}}(t) - X^{\pi}(t)\right) \hat{p}_1(t) \\
=& \int_0^t \left(X^{\hat{\pi}}(s) - X^{\pi}(s)\right) \{-\left[\left(r(s) + \hat{\pi}^{\intercal}(s)\sigma(s)\theta(s)\right) \hat{p}_1(s) + \hat{q}_1^{\intercal}(t)\sigma^{\intercal}(t)\pi(t)\right] dt + \hat{q}_1^{\intercal}(t) dW(t)\} \\
&+ \int_0^t \hat{p}_1^{\intercal}(s) \Big\{ \left[X^{\hat{\pi}}(t)\left(r(s) + \hat{\pi}^{\intercal}(s)\sigma(s)\theta(s)\right) - X^{\pi}(s)\left(r(s) + \pi^{\intercal}(s)\sigma(s)\theta(s)\right)\right] ds \\
&+ \left[X^{\hat{\pi}}(s)\hat{\pi}^{\intercal}(s)\sigma(s) - X^{\hat{\pi}}(s)\pi^{\intercal}(s)\sigma(s)\right] dW(s) \Big\} \\
&+ \int_0^t \left[X^{\hat{\pi}}(s)\hat{\pi}^{\intercal}(s)\sigma(s) - X^{\pi}(s)\pi^{\intercal}(s)\sigma(s)\right] \hat{q}_1(s) ds.
\end{aligned}
$$

Rearranging the above equation, we have

$$
\begin{aligned}
&\left(X^{\hat{\pi}}(t) - X^{\pi}(t)\right) \hat{p}_1(t) \\
=& \int_0^t \left(\hat{\pi}^{\intercal}(s) - \pi^{\intercal}(s)\right) X^{\hat{\pi}}(s)\sigma(s) \left[\hat{p}_1(s)\theta(s) + \hat{q}_1(s)\right] ds \\
&+ \int_0^t \left[\left(X^{\hat{\pi}}(s) - X^{\pi}(s)\right) q^{\intercal}(s) + X^{\hat{\pi}}(s)\left(\hat{\pi}^{\intercal}(t) - \pi^{\intercal}(t)\right)\sigma(s)\right] dW(s).
\end{aligned}
$$

Hence, by condition (15) and the definition of a normal cone, taking expectation of the above, we have

$$
E\left[\left(X^{\hat{\pi}}(T) - X^{\pi}(T)\right) \hat{p}_1(T)\right] \leq 0.
$$

Combining with the concavity of U gives us

$$
\begin{aligned}
E\left[U\left(X^{\pi}(T)\right) - U\left(X^{\hat{\pi}}(T)\right)\right] &\leq E\left[\left(X^{\pi}(T) - X^{\hat{\pi}}(T)\right) U'\left(X^{\hat{\pi}}(T)\right)\right] \\
&= E\left[\left(X^{\hat{\pi}}(T) - X^{\pi}(T)\right) \hat{p}_1(T)\right] \\
&\leq 0.
\end{aligned}
$$

Hence $\hat{\pi}$ is indeed an optimal control. □

3.2 Necessary and sufficient conditions for dual problems

Next we address the dual problem. To ensure the existence of an optimal solution, we impose the following condition:

Assumption 8. *([16, Condition 4.14]) For any $(y, v) \in (0, \infty) \times \mathcal{D}$, we have*

$$
E\left[\tilde{U}\left(Y^{(y,v)}(T)\right)^2\right] < \infty. \tag{19}
$$

According to [16, Proposition 4.15], there exists some $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ such that $\tilde{V} = x_0\hat{y} + E[\tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)]$. Given an admissible dual control $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ with the dual process $Y^{(\hat{y},\hat{v})}$ that solves the SDE (7) and condition (19) holds with $(y, v) = (\hat{y}, \hat{v})$, the associated adjoint equation for the dual problem is the following linear BSDE in the unknown processes $\hat{p}_2 \in \mathcal{H}^2(0,T;\mathbb{R})$ and $\hat{q}_2 \in \mathcal{H}^2(0,T;\mathbb{R}^N)$:

$$\begin{cases} d\hat{p}_2(t) = \{[r(t)+\delta_K(\hat{v}(t))]\,\hat{p}_2(t) + \hat{q}_2^\intercal(t)\,[\theta(t)+\sigma^{-1}(t)\hat{v}(t)]\}\,dt + \hat{q}_2^\intercal(t)dW(t), \\ \hat{p}_2(T) = -\tilde{U}'(Y^{(\hat{y},\hat{v})}(T)). \end{cases} \tag{20}$$

Since $\hat{p}_2 Y^{(\hat{y},\hat{v})}$ is a martingale, we can find $\hat{p}_2(t)$, $0 \le t \le T$, from the relation

$$\hat{p}_2(t)Y^{(\hat{y},\hat{v})}(t) = E[\hat{p}_2(T)Y^{(\hat{y},\hat{v})}(T)|\mathcal{F}_t] = -E[\tilde{U}'(Y^{(\hat{y},\hat{v})}(T))Y^{(\hat{y},\hat{v})}(T)|\mathcal{F}_t]. \tag{21}$$

Lemma 9. *Let $(y,v) \in (0,\infty)\times\mathcal{D}$ and $Y^{(y,v)}$ be the corresponding state process satisfying the SDE* (7) *and condition (19). Then the random variable $Y^{(y,v)}(T)\tilde{U}'(Y^{(y,v)}(T))$ is square integrable and there exists a solution to the adjont BSDE* (20)*.*

Proof. From condition (19) and following similar arguments as in [13, page 290], we have that since $\tilde{U}$ is a decreasing function,

$$\tilde{U}(\eta) - \tilde{U}(\infty) \ge \tilde{U}(\eta) - \tilde{U}(\frac{\eta}{\beta}) = \int_\eta^{\frac{\eta}{\beta}} I(u)du \ge \left(\frac{\eta}{\beta} - \eta\right) I\left(\frac{\eta}{\beta}\right) \ge \frac{1-\beta}{\beta\gamma}\eta I(\eta),$$

for $0 < \eta < \infty$, where $\beta \in (0,1)$ and $\gamma \in (1,\infty)$ are as in Condition 8. Since $\tilde{U}(\infty) = U(0)$ is finite, we conclude that the random variable $Y^{(\hat{y},\hat{v})}(T)\tilde{U}'(Y^{(\hat{y},\hat{v})}(T))$ is square integrable. Define the process

$$\phi(t) \triangleq E\left[-Y^{(\hat{y},\hat{v})}(T)\tilde{U}'(Y^{(\hat{y},\hat{v})}(T))\middle|\mathcal{F}_t\right],\ t \in [0,T].$$

By the martingale representation theorem, it is the unique solution to the BSDE

$$\phi(t) = -Y^{(\hat{y},\hat{v})}(T)\tilde{U}'(Y^{(\hat{y},\hat{v})}(T)) - \int_t^T \varphi^\intercal(s)dW(s),$$

where φ is a square integrable process with values in $\mathbb{R}^N$. Applying Ito's lemma to $\phi(t)/Y^{(\hat{y},\hat{v})}(t)$, we have

$$\begin{aligned} d\frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)} &= \left\{\frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)}\left[r(t)+\delta_K(\hat{v}(t)) + |\theta(t)+\sigma^{-1}(t)\hat{v}(t)|^2\right] + \frac{\varphi(t)}{Y^{(\hat{y},\hat{v})}(t)}[\theta(t)+\sigma(t)^{-1}\hat{v}(t)]\right\}dt \\ &\quad + \left\{\frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)}\left[\theta(t)+\sigma(t)^{-1}\hat{v}(t)\right]^\intercal + \frac{\varphi^\intercal(t)}{Y^{(\hat{y},\hat{v})}(t)}\right\}dW(t). \end{aligned}$$

Rearranging the above equation, we have

$$d\hat{p}_2(t) = \left\{[r(t)+\delta_K(\hat{v}(t))]\,\hat{p}_2(t) + \hat{q}_2^\intercal(t)\left[\theta(t)+\sigma(t)^{-1}\hat{v}(t)\right]\right\}dt + \hat{q}_2^\intercal(t)dW(t),$$

where $(\hat{p}_2, \hat{q}_2)$ are defined as

$$\hat{p}_2(t) \triangleq \frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)} \ \text{ and } \hat{q}_2(t) \triangleq \hat{p}_2(t)\left[\theta(t)+\sigma(t)^{-1}\hat{v}(t)\right] + \frac{\varphi(t)}{Y^{(\hat{y},\hat{v})}(t)}.$$

Hence, we conclude that $(\hat{p}_2, \hat{q}_2)$ solves the BSDE (20). □

Remark 10. *Note that if* $U(x) = \ln x$ *then* $\tilde{U}(y) = -\ln y - 1$. *We have* $Y^{(\hat{y},\hat{v})}(T)\tilde{U}'(Y^{(\hat{y},\hat{v})}(T)) \equiv -1$, *obviously square integrable. The conclusion of Lemma 9 holds. However, in this case,* $U(0) = -\infty$, *not finite. So the requirement of* $U(0)$ *being finite is only a sufficient condition for Lemma 9, not a necessary condition. We can apply all the results in the paper to log utility.*

We now state the necessary and sufficient optimality conditions for the dual problem.

Theorem 11. *(Dual problem and associated FBSDE) Let* $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$. *Then* $(\hat{y}, \hat{v})$ *is optimal for the dual problem if and only if the solution* $(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2)$ *of FBSDE*

$$\begin{cases} dY^{(\hat{y},\hat{v})}(t) = -Y^{(\hat{y},\hat{v})}(t) \left\{ [r(t) + \delta_K(\hat{v}(t))]dt + [\theta(t) + \sigma^{-1}(t)\hat{v}(t)]^\intercal dW(t) \right\}, \\ Y^{(\hat{y},\hat{v})}(0) = \hat{y}, \\ d\hat{p}_2(t) = \left\{ [r(t) + \delta_K(\hat{v}(t))]^\intercal \hat{p}_2(t) + \hat{q}_2^\intercal(t) \left[\theta(t) + \sigma^{-1}(t)\hat{v}(t)\right] \right\} dt + \hat{q}_2^\intercal(t) dW(t), \\ \hat{p}_2(T) = -\tilde{U}'(Y^{(\hat{y},\hat{v})}(T)) \end{cases} \tag{22}$$

satisfies the following conditions

$$\begin{cases} \hat{p}_2(0) = x_0, \\ \hat{p}_2(t)^{-1} \left[\sigma^\intercal(t)\right]^{-1} \hat{q}_2(t) \in K, \\ \hat{p}_2(t)\delta_K(\hat{v}(t)) + \hat{q}_2^\intercal(t)\sigma^{-1}(t)\hat{v}(t) = 0, \ \forall t \in [0,T] \ \mathbb{P} - a.s. \end{cases} \tag{23}$$

Proof. Let $(\hat{y}, \hat{v})$ be an optimal control of the dual problem and $Y^{(\hat{y},\hat{v})}$ be the corresponding state process. Define a function $h(\xi) \triangleq x_0\xi\hat{y} + E[\tilde{U}\left(\xi Y^{(\hat{y},\hat{v})}(T)\right)]$. Then $\inf_{\xi \in (0,\infty)} h(\xi) = h(1)$. Following the argument in [12, Lemma 11.7] by the convexity of $\tilde{U}$, the dominated convergence theorem and Lemma 9, we conclude that h is continuously differentiable at $\xi = 1$ and the derivative $h'(1) = x_0\hat{y} + E[Y^{(\hat{y},\hat{v})}(T)\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)]$ holds. Hence, we conclude that

$$\hat{p}_2(0) = -\frac{1}{\hat{y}} E\left[Y^{(\hat{y},\hat{v})}(T)\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)\right] = x_0.$$

Let $(\hat{y}, \tilde{v})$ be an admissible control and $\eta \triangleq \tilde{v} - \hat{v}$. Similar to the argument in [6, page 781-782], define the stopping time

$$\begin{aligned} \tau_n \ \triangleq \ & T \wedge \inf\{t \in [0,T]; \int_0^t \Big(|\delta_K(\eta(s))|^2 + |\theta^\intercal(s)\sigma^{-1}(s)\eta(s)|^2 + |\phi(s)\eta(s)|^2 \\ & + |\hat{v}^\intercal(s)[\sigma^{-1}(s)]^\intercal\sigma^{-1}(s)\eta(s)|^2 \Big) ds \geq n \ or \ \Big| \int_0^t \eta^\intercal(s)[\sigma^{-1}(s)]^\intercal dW(s) \Big| \geq n\}. \end{aligned}$$

Define $\eta_n(t) \triangleq \eta(t)1_{t \leq \tau_n}$ for $0 \leq t \leq T$ and

$$\tilde{\phi}_n(\varepsilon) \triangleq \tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) = \tilde{U}\left(\exp\left[\ln\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right)\right]\right)$$

for $0 \leq \varepsilon \leq 1$. According to Assumption 4, $g(z) \triangleq \tilde{U}(e^z)$ is a convex function that is nonincreasing. Moreover, since δ_K is convex, $f(\varepsilon) \triangleq \ln\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right)$ is a concave function of ε. Hence $\tilde{\phi}_n(\varepsilon) = g(f(\varepsilon))$ is a convex function and $\tilde{\Phi}_n(\varepsilon) \triangleq (\tilde{\phi}_n(\varepsilon) - \tilde{\phi}_n(0))/\varepsilon$

is an increasing function. Define $\tilde{H}^{\eta_n}_{\varepsilon}(t)$ and $\tilde{H}^{\eta_n}(t)$ as

$$
\begin{aligned}
\tilde{H}^{\eta_n}_{\varepsilon}(t) \triangleq & \int_0^t \delta_K(\hat{v}(s)+\varepsilon\eta_n(s)) - \delta_K(\hat{v}(s)) + \varepsilon\theta^{\intercal}(s)\sigma^{-1}(s)\eta_n(s) + \varepsilon\hat{v}^{\intercal}(s)[\sigma^{-1}(s)]^{\intercal}\sigma^{-1}(s)\eta_n(s) \\
& + \frac{1}{2}\varepsilon^2\eta_n^{\intercal}(s)[\sigma^{-1}(s)]^{\intercal}\sigma^{-1}(s)\eta_n(s)ds + \int_0^t \varepsilon\eta_n^{\intercal}(s)[\sigma^{-1}(s)]^{\intercal}dW(s), \\
\tilde{H}^{\eta_n}(t) \triangleq & \int_0^t \delta_K(\eta_n(s)) + \theta^{\intercal}(s)\sigma^{-1}(s)\eta_n(s) + \hat{v}^{\intercal}(s)[\sigma^{-1}(s)]^{\intercal}\sigma^{-1}(s)\eta_n(s)ds \\
& + \int_0^t \eta_n^{\intercal}(s)[\sigma^{-1}(s)]^{\intercal}dW(s).
\end{aligned}
$$

For $\varepsilon \in (0,1)$, we have

$$
\begin{aligned}
\tilde{\Phi}_n(\varepsilon) =& \frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{\varepsilon} \\
=& \frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T) - Y^{(\hat{y},\hat{v})}(T)} \frac{Y^{(\hat{y},\hat{v})}(T)}{\varepsilon} \left[\frac{Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)}{Y^{(\hat{y},\hat{v})}(T)} - 1\right] \\
=& \frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T) - Y^{(\hat{y},\hat{v})}(T)} \frac{Y^{(\hat{y},\hat{v})}(T)}{\varepsilon} \left[\exp\left(-\tilde{H}^{\eta_n}_{\varepsilon}(T)\right) - 1\right] \\
\leq& \frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T) - Y^{(\hat{y},\hat{v})}(T)} \frac{Y^{(\hat{y},\hat{v})}(T)}{\varepsilon} \\
& \Bigg\{-1 + \exp\Bigg[-\varepsilon\int_0^T \Bigg(\delta_K(\eta_n(t)) + \theta^{\intercal}(t)\sigma^{-1}(t)\eta_n(t) + \hat{v}^{\intercal}(t)[\sigma^{-1}(t)]^{\intercal}\sigma^{-1}(t)\eta_n(t) \\
& + \frac{1}{2}\varepsilon\eta_n^{\intercal}(t)[\sigma^{-1}(t)]^{\intercal}\sigma^{-1}(t)\eta_n(t)\Bigg)dt - \varepsilon\int_0^T \eta_n^{\intercal}(t)[\sigma^{-1}(t)]^{\intercal}dW(t)\Bigg]\Bigg\}.
\end{aligned}
$$

Hence, taking lim sup on both sides, we have

$$
\limsup_{\varepsilon\to 0} \tilde{\Phi}_n(\varepsilon) \leq -\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)Y^{(\hat{y},\hat{v})}(T)\tilde{H}^{\eta_n}(T)
$$

with

$$
E\left[\left|\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)Y^{(\hat{y},\hat{v})}(T)\tilde{H}^{\eta_n}(T)\right|\right] \leq E\left[\left(\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)Y^{(\hat{y},\hat{v})}(T)\right)^2\right]^{\frac{1}{2}} E\left[\tilde{H}^{\eta_n}(T)^2\right]^{\frac{1}{2}} < \infty.
$$

Moreover, note that as $\varepsilon \in (0,1)$ approaches zero, the sequence

$$
\left(\frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{\varepsilon}\right)_{\varepsilon\in(0,1)}
$$

is bounded from above by $|\tilde{\Phi}_n(1)|$ and $E[|\tilde{\Phi}_n(1)|] < \infty$. By the reverse Fatou lemma, we have

$$
0 \leq \limsup_{\varepsilon\to 0} E\left[\tilde{\Phi}_n(\varepsilon)\right] \leq E\left[\limsup_{\varepsilon\to 0}\tilde{\Phi}_n(\varepsilon)\right] \leq E\left[-\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)Y^{(\hat{y},\hat{v})}(T)\tilde{H}^{\eta_n}(T)\right].
$$

Let (ϕ, φ) be defined as in Lemma 9 and $(\hat{p}_2, \hat{q}_2)$ be the adjoint process corresponding to $(\hat{y}, \hat{v})$. Applying Ito's lemma to $\phi(t)\tilde{H}^{\eta}_n(t)$, we obtain

$$\begin{aligned}
& d\phi(t)\tilde{H}^{\eta_n}(t) \\
=& -\varphi^\intercal(t)\tilde{H}^{\eta_n}(t)dW(t) + \phi(t)\left(\delta_K(\eta_n(t)) + \theta^\intercal(t)\sigma^{-1}(t)\eta_n(t) + \hat{v}(t)[\sigma^{-1}(t)]^\intercal\sigma^{-1}(t)\eta_n(t)\right)dt \\
& + \phi(t)\eta_n^\intercal(t)[\sigma^{-1}(t)]^\intercal dW(t) + \eta_n^\intercal(t)[\sigma^{-1}(t)]^\intercal\varphi(t)dt \\
=& Y^{\hat{y},\hat{v}}(t)\left[\delta_K(\eta_n(t))\hat{p}_2(t) + \hat{q}_2(t)\sigma^{-1}(t)\eta_n(t))\right]dt + \left[\phi(t)\eta_n^\intercal(t)[\sigma^{-1}(t)]^\intercal - \varphi^\intercal(t)\tilde{H}^{\eta_n}(t)\right]dW(t).
\end{aligned}$$

Following a similar approach as in the proof of necessary condition for the primal problem, it can be shown that $\int_0^t[\phi(s)\eta_n^\intercal(s)[\sigma^{-1}(s)]^\intercal - \varphi^\intercal(s)\tilde{H}^{\eta_n}(s)]dW(s)$ is a true martingale. Taking expectation of the above equation, we obtain

$$E\left[\int_0^{\tau_n} Y^{(\hat{y},\hat{v})}(t)\left[\delta_K(\eta(t))\hat{p}_2(t) + \hat{q}_2(t)\sigma^{-1}(t)\eta(t))\right]dt\right] \geq 0. \tag{24}$$

Note that $\hat{p}_2(t) = \phi(t)/Y^{(\hat{y},\hat{v})}(t) > 0$, define the event $B \triangleq \{(\omega, t) : \hat{p}_2(t)^{-1}\sigma(t)^{-1}\hat{q}_2(t) \notin K\}$. According to [13, Lemma 5.4.2], there exists some $\mathbb{R}^N$-valued progressively measurable process η such that $|\eta(t)| \leq 1$ and $|\delta_K(\eta(t))| \leq 1$ a.e. and

$$\begin{aligned}
\delta_K(\eta(t)) + \hat{p}_2(t)^{-1}\hat{q}_2(t)'\sigma(t)^{-1} < 0 \ \textit{a.e. on } B, \\
\delta_K(\eta(t)) + \hat{p}_2(t)^{-1}\hat{q}_2(t)'\sigma(t)^{-1} = 0 \ \textit{a.e. on } B^c.
\end{aligned}$$

Let $\tilde{v} \triangleq \hat{v} + \eta$. We can easily verify that $\tilde{v}$ is progressively measurable and square integrable. Hence, we obtain that

$$E\left[\int_0^{\tau_n} Y^{(\hat{y},\hat{v})}(t)\left[\hat{p}_2(t)\left(\delta_K(\eta(t))\right) + \hat{q}_2(t)'\sigma(t)^{-1}\eta(t)\right]dt\right] < 0,$$

contradicting with (24). Hence, by the $\mathbb{P}$ strict positivity of $Y^{(\tilde{y},\tilde{v})}(t)\hat{p}_2(t)$, we conclude that $\hat{p}_2(t)^{-1}\sigma(t)^{-1}\hat{q}_2(t) \in K$ a.e. (this argument is essentially identical to the analysis in the proof of Proposition 4.17 in [16]). Take $\tilde{v} = 2\hat{v}$, and we have

$$E\left[\int_0^{\tau_n} Y^{(\tilde{y},\tilde{v})}(t)\left[\hat{p}_2(t)\left(\delta_K(\hat{v}(t))\right) + \hat{q}_2(t)'\sigma(t)^{-1}\hat{v}(t)\right]dt\right] \geq 0.$$

Lastly, to prove the third condition, simply take $\tilde{v} = 0$ and by the same analysis, we obtain

$$E\left[\int_0^{\tau_n} Y^{(\tilde{y},\tilde{v})}(t)\left[\hat{p}_2(t)\left(\delta_K(\hat{v}(t))\right) + \hat{q}_2(t)'\sigma(t)^{-1}\hat{v}(t)\right]dt\right] \leq 0.$$

On the other hand, by the definition of δ_K, we have $\delta_K(\hat{v}(t)) + \hat{p}_2(t)^{-1}\hat{q}_2^\intercal(t)\sigma^{-1}(t)\hat{v}(t) \geq 0$ a.e. Combining with the $\mathbb{P}$ strict positivity of $Y^{(\tilde{y},\tilde{v})}(t)\hat{p}_2(t)$ gives the last condition. We have proved the necessary condition.

Now we prove the sufficient condition. Let $\left(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2\right)$ be a solution to the FBSDE (22) and satisfy conditions (23). Let $(\tilde{y}, \tilde{v}) \in (0, \infty) \times \mathcal{D}$ be an admissible control such that $Y^{(\tilde{y},\tilde{v})}$ solves the SDE (7) and $E[\tilde{U}(Y^{(\tilde{y},\tilde{v})}(T))^2] < \infty$. By Lemma 9, there exists adjoint process $(\tilde{p}_2, \tilde{q}_2)$ that solves the BSDE with control $(\tilde{y}, \tilde{v})$. Applying Ito's lemma, we have

$$
\begin{aligned}
&\left(Y^{(\hat{y},\hat{v})}(t) - Y^{(\tilde{y},\tilde{v})}(t)\right)\hat{p}_2(t)\\
=&\hat{p}_2(0)y + \int_0^t \left\{Y^{(\tilde{y},\tilde{v})}(s)\left[r(s)+\delta_K(\tilde{v}(s))\right]^\intercal - Y^{(\hat{y},\hat{v})}(s)\left[r(s)+\delta_K(\hat{v}(s))\right]^\intercal\right\}\hat{p}_2(s)ds\\
&+\int_0^t \left\{Y^{(\tilde{y},\tilde{v})}(s)[\theta(s)+\sigma^{-1}(s)\tilde{v}(s)]^\intercal - Y^{(\hat{y},\hat{v})}(s)[\theta(s)+\sigma^{-1}(s)\hat{v}(s)]^\intercal\right\}\hat{p}_2(s)dW(s)\\
&+\int_0^t \left(Y^{(\hat{y},\hat{v})}(s) - Y^{(\tilde{y},\tilde{v})}(s)\right)\left\{\left[r(s)+\delta_K(\tilde{v}(s))\right]^\intercal\hat{p}_2(s) + \hat{q}_2^\intercal(s)\left[\theta(s)+\sigma^{-1}(s)\hat{v}(s)\right]\right\}ds\\
&+\int_0^t \left(Y^{(\hat{y},\hat{v})}(s) - Y^{(\tilde{y},\tilde{v})}(s)\right)\hat{q}_2^\intercal(s)dW(s)\\
&+\int_0^t \left\{Y^{(\tilde{y},\tilde{v})}(s)[\theta(s)+\sigma^{-1}(s)\tilde{v}(s)]^\intercal - Y^{(\hat{y},\hat{v})}(s)[\theta(s)+\sigma^{-1}(s)\hat{v}(s)]^\intercal\right\}\hat{q}_2(s)ds\\
=&\hat{p}_2(0)y + \int_0^t Y^{(\tilde{y},\tilde{v})}(s)\hat{p}_2(s)\left[\delta_K(\tilde{v}(s)) - \delta_K(\hat{v}(s)) + \hat{q}_2^\intercal(s)\sigma^{-1}(s)\left(\tilde{v}(s)-\hat{v}(s)\right)\right]ds\\
&+\int_0^t \left\{Y^{(\tilde{y},\tilde{v})}(s)[\theta(s)+\sigma^{-1}(s)\tilde{v}(s)]^\intercal - Y^{(\hat{y},\hat{v})}(s)[\theta(s)+\sigma^{-1}(s)\hat{v}(s)]^\intercal\right\}\hat{p}_2(s)dW(s)\\
&+\int_0^t \left(Y^{(\hat{y},\hat{v})}(s) - Y^{(\tilde{y},\tilde{v})}(s)\right)\hat{q}_2^\intercal(s)dW(s).
\end{aligned}
$$

By (23) and taking expectation, we have

$$E\left[\left(Y^{(\hat{y},\hat{v})}(T) - Y^{(\tilde{y},\tilde{v})}(T)\right)\hat{p}_2(T)\right] \geq y\hat{p}_2(0).$$

By the convexity of $\tilde{U}$ we obtain

$$x_0\tilde{y} + E\left[\tilde{U}(Y^{(\tilde{y},\tilde{v})}(T))\right] - x_0\hat{y} - E\left[\tilde{U}(Y^{(\hat{y},\hat{v})}(T))\right] \geq y(x_0 - \hat{p}_2(0)) = 0.$$

Hence, we conclude that $(\hat{y}, \hat{v})$ is indeed an optimal control of the dual problem. □

3.3 Dynamic relations of primal and dual problems

We can now state the dynamic relations of the optimal portfolio and wealth processes of the primal problem and the adjoint processes of the dual problem and vice versa.

Theorem 12. *(From dual problem to primal problem) Suppose that $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ is optimal for the dual problem. Let $\left(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2\right)$ be the associated process that solves the FBSDE* (22) *and satisfies condition* (23)*. Define*

$$\hat{\pi}(t) \triangleq \frac{\left[\sigma^\intercal(t)\right]^{-1}\hat{q}_2(t)}{\hat{p}_2(t)}, \;\; t \in [0, T]. \tag{25}$$

Then $\hat{\pi}$ is the optimal control for the primal problem with initial wealth x_0. The optimal wealth process and associated adjoint process are given by

$$\begin{cases} X^{\hat{\pi}}(t) = \hat{p}_2(t), \\ \hat{p}_1(t) = -Y^{(\hat{y},\hat{v})}(t), \\ \hat{q}_1(t) = Y^{(\hat{y},\hat{v})}(t)[\sigma^{-1}(t)\hat{v}(t) + \theta(t)]. \end{cases} \tag{26}$$

Proof. Suppose that $(\hat{y}, \hat{v}) \in (0.\infty) \times \mathcal{D}$ is optimal for the dual problem. By Theorem 11, the process $\big(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2\big)$ solves the dual FBSDE (22) and satisfies condition (23). Construct $\hat{\pi}$ and $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ as in (25) and (26), respectively. Substituting them back into (14), we conclude that $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ solves the FBSDE for the primal problem. By condition (23), it can be easily shown that $\hat{\pi} \in \mathcal{A}$. Moreover, we have

$$
\begin{aligned}
&X^{\hat{\pi}}(t)\sigma(t)\left[\hat{p}_1(t)\theta(t) + \hat{q}_1(t)\right] \\
&= \hat{p}_2(t)\sigma(t)\left\{-Y^{(\hat{y},\hat{v})}(t)\theta(t) + Y^{(\hat{y},\hat{v})}(t)\left[\sigma^{-1}(t)\hat{v}(t) + \theta(t)\right]\right\} \\
&= Y^{(\hat{y},\hat{v})}(t)\hat{p}_2(t)\hat{v}(t).
\end{aligned}
$$

Combining with the third statement of (23) and the almost sure positivity of $Y^{(\hat{y},\hat{v})}\hat{p}_2$, we claim that condition (15) holds. By Theorem 11 we conclude that $\hat{\pi}$ is indeed an optimal control to the primal problem. $\square$

Theorem 13. *(From primal problem to dual problem) Suppose that $\hat{\pi} \in \mathcal{A}$ is optimal for the primal problem with initial wealth x_0. Let $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ be the associated process that satisfies the FBSDE* (14) *and conditions* (15)*. Define*

$$
\begin{cases}
\hat{y} \triangleq -\hat{p}_1(0), \\
\hat{v}(t) \triangleq -\sigma(t)\left[\dfrac{\hat{q}_1(t)}{\hat{p}_1(t)} + \theta(t)\right], \ \forall t \in [0, T].
\end{cases} \tag{27}
$$

Then $(\hat{y}, \hat{v})$ is an optimal control for the dual problem. The optimal dual process and associated adjoint process are given by

$$
\begin{cases}
Y^{(\hat{y},\hat{v})}(t) = -\hat{p}_1(t), \\
\hat{p}_2(t) = X^{\hat{\pi}}(t), \\
\hat{q}_2(t) = \sigma^\intercal(t)\hat{\pi}(t)X^{\hat{\pi}}(t).
\end{cases} \tag{28}
$$

Proof. Suppose that $\hat{\pi} \in \mathcal{A}$ is an optimal control for the primal problem. By Theorem 7, the process $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ solves the FBSDE (14) and satisfies conditions (15). Define $(\hat{y}, \hat{v})$ and $(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2)$ as in (27) and (28), respectively. Substituting them back into (22), we obtain that $(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2)$ solves the FBSDE for the dual problem. Moreover, by the construction in (27) and (28), we have $\hat{p}_2(0) = x_0$ and $[\sigma^\intercal(t)]^{-1}\hat{q}_2(t) = \hat{\pi}(t)X^{\hat{\pi}}(t)^{-1} \in K$. Substituting $\hat{v}$ into (23), we can easily show that the third statement in (23) holds. Hence, by Theorem 11, we conclude that $(\hat{y}, \hat{v})$ is indeed an optimal control to the dual problem. $\square$

4 Examples

In this section, we use the main results in the previous section to address several classical constrained utility maximization problems.

4.1 Constrained power utility maximization

In this subsection, we assume U is a power utility function defined by $U(x) \triangleq \frac{1}{\beta}x^\beta, \ x \in (0, \infty)$, where $\beta \in (0, 1)$ is a constant. In addition, we assume that coefficients $r(t), b(t), \sigma(t)$

in (1) and (2) are deterministic and $K \subseteq \mathbb{R}^N$ is a closed convex cone. In this case, the dual problem can be written as

$$Minimize\ x_0 y + E\left[\tilde{U}\left(Y^{(y,v)}(T)\right)\right]$$

over $(y,v) \in (0,\infty) \times \mathcal{D}$, where $\tilde{U}(y) = \frac{1-\beta}{\beta} y^{\frac{\beta}{\beta-1}}$, $y \in (0,\infty)$, is the dual function of U. We solve the above problem in two steps: first, fix y and solve a stochastic optimal control problem

$$\tilde{u}(y) \triangleq \inf_v E\left[\tilde{U}\left(Y^{(y,v)}(T)\right)\right],$$

where $Y^{(y,v)}$ satisfies the SDE (8) and, second, solve a convex minimization problem

$$\inf_y (x_0 y + \tilde{u}(y)).$$

Step 1: Define the dual value function

$$v(t,y) = \inf_v E\left[\tilde{U}\left(Y^{(y,v)}(T)\right) \Big| Y^{(y,v)}(t) = y\right],$$

where the expectation above is the conditional expectation given $Y(t) = y$. Using the dynamic programming principle (DPP) (see [20]), we have v satisfies the following HJB (Hamilton-Jacobi-Bellman) equation:

$$\begin{cases} \frac{\partial}{\partial t} v(t,y) - r(t) y v_y(t,y) + \frac{1}{2} \inf_{v \in \tilde{K}} |\theta(t) + \sigma^{-1}(t) v|^2 y^2 v_{yy}(t,y) = 0, \\ v(T,y) = \tilde{U}(y), \end{cases} \tag{29}$$

for $(t,y) \in [0,T] \times \mathbb{R}$. Let $\hat{v}(t) \in \tilde{K}$ be the minimizer of $\inf_{v \in \tilde{K}} |\theta(t) + \sigma^{-1}(t) v|^2$ and $\hat{\theta}(t) \triangleq \sigma^{-1}(t)\hat{v}(t) + \theta(t)$. Note that the dual optimal control $\hat{v}$ is independent of y. Then the HJB equation (29) becomes

$$\begin{cases} \frac{\partial}{\partial t} v(t,y) - r(t) y v_y(t,y) + \frac{1}{2} |\hat{\theta}(t)|^2 y^2 v_{yy}(t,y) = 0 \\ v(T,y) = \tilde{U}(y), \end{cases}$$

for $(t,y) \in [0,T] \times \mathbb{R}$. Assume $|\hat{\theta}(t)| \geq c > 0$ for $t \in [0,T]$. According to the Feynman-Kac formula, we can represent v as

$$v(t,y) \quad = \quad E\left[\tilde{U}(Y(T)) \Big| Y(t) = y\right],$$

where the stochastic process Y follows a geometric Brownian motion

$$dY(s) = -Y(s)\{r(s)ds + \hat{\theta}^\intercal(s) dW(s)\},\ t \leq s \leq T$$

with $Y(t) = y$. Simple calculation gives

$$v(t,y) = \tilde{U}(y) \exp\left(\int_t^T \left[\frac{1}{2} \frac{\beta}{(\beta-1)^2} |\hat{\theta}(s)|^2 - \frac{\beta}{\beta-1} r(s) \right] ds \right).$$

Step 2: Since $\tilde{u}(y) = v(0,y)$, solving the following convex minimization problem

$$\inf_{y>0} (x_0 y + v(0,y)),$$

we find the minimum is achieved at point

$$\hat{y} = x_0^{\beta-1} \exp\left((1-\beta) \int_0^T \left[\frac{\beta}{2(\beta-1)^2} |\hat{\theta}(s)|^2 - \frac{\beta}{\beta-1} r(s) \right] ds \right).$$

Solving the adjoint BSDE (20), we have

$$\begin{aligned} \hat{p}_2(t) &= x_0 \exp\left(\int_0^t \left[r(s) + \frac{(1-2\beta)}{2(1-\beta)^2} |\hat{\theta}(s)|^2 \right] ds + \frac{1}{1-\beta} \int_0^t \hat{\theta}^\intercal(s) dW(s) \right), \\ \hat{q}_2(t) &= \frac{1}{1-\beta} \hat{p}_2(t) \hat{\theta}(t). \end{aligned}$$

Applying Theorem 12, we can construct the optimal solution to the primal problem using the optimal solutions of the dual problem and hence arrive at the following closed form solutions:

$$\begin{cases} \hat{\pi}(t) = \dfrac{1}{1-\beta} [\sigma(t)^\intercal]^{-1} \hat{\theta}(t), \\ X^{\hat{\pi}}(t) = x_0 \exp\left(\displaystyle\int_0^t \left[r(s) + \frac{(1-2\beta)}{2(1-\beta)^2} |\hat{\theta}(s)|^2 \right] ds + \frac{1}{1-\beta} \int_0^t \hat{\theta}^\intercal(s) dW(s) \right). \end{cases}$$

4.2 Constrained log utility maximization with random coefficients

In this section, we assume that U is a log utility defined by $U(x) = \log x$ for $x > 0$. The dual function of U is given by $\tilde{U}(y) = -(1 + \log y),\ y \geq 0$. Assume that $K \subseteq \mathbb{R}^N$ is a closed convex set and r, b, σ are uniformly bounded $\{\mathcal{F}_t\}$ progressively measurable processes on $\Omega \times [0, T]$.

Step 1: We fix y and solve the dual optimal control problem. Note that the DPP is not appropriate in this case due to the non-Markov nature of the problem. However, following the approach in [6, Section 11] the problem can be solved explicitly due to the special property of the logarithmic function.

Let $v \in \mathcal{D}$ be a given dual admissible control and the dual objective function becomes

$$E\left[\tilde{U}\left(Y^{(y,v)}(T) \right) \right] = -1 - \log y - E\left[\int_0^T r(t) + \delta_K(v(t)) + \frac{1}{2} |\theta(t) + \sigma^{-1}(t) v(t)|^2 dt \right],$$

where Y satisfies the SDE (7). The dual optimization boils down to the following problem of pointwise minimization of a convex function $\delta_K(v) + \frac{1}{2}|\theta(t) + \sigma^{-1}(t)v|^2$ over $v \in \mathbb{R}^N$ for all $t \in [0, T]$. Applying the classical measurable selection theorem (see [23, 24]), we conclude that the process defined by

$$\hat{v}(t) \triangleq \operatorname*{arg\,min}_{v \in \mathbb{R}^N} \left[\delta_K(v) + \frac{1}{2} |\theta(t) + \sigma(t)^{-1} v|^2 \right] \tag{30}$$

is $\{\mathcal{F}_t\}$ progressively measurable and therefore is the optimal control given y and is independent of y.

Step 2: Solve the following static optimization problem

$$\inf_{y \in \mathbb{R}} x_0 y - 1 - \log y - E\left[\int_0^T r(t) + \delta_K(\hat{v}(t)) + \frac{1}{2} |\theta(t) + \sigma^{-1}(t) \hat{v}(t)|^2 dt \right].$$

We obtain $\hat{y} = 1/x_0$. Hence, the optimal state process for the dual problem is the exponential process satisfying (7). Solving the adjoint BSDE (20), we have

$$\hat{p}_2(t)Y^{(\hat{y},\hat{v})}(t) = E\left[-\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)Y^{(\hat{y},\hat{v})}(T)\middle|\mathcal{F}_t\right] = 1.$$

Hence, we have $\hat{p}_2(t) = Y^{(\hat{y},\hat{v})}(t)^{-1}$. Applying Ito's lemma on $\hat{p}_2$, we get

$$\hat{q}_2(t) = Y^{(\hat{y},\hat{v})}(t)^{-1}[\theta(t) + \sigma(t)^{-1}\hat{v}(t)],\ \forall t \in [0,T],\ a.e.$$

Finally, according to Theorem 12, we construct the optimal control to the primal problem explicitly form the optimal solution of the dual problem as

$$\hat{\pi}(t) = [\sigma(t)\sigma^\intercal(t)]^{-1}[\hat{v}(t) + b(t) - r(t)\mathbf{1}],\ \forall t \in [0,T],\ a.e. \tag{31}$$

Remark 14. *In the case where K is a closed convex cone, it is trivial to see that $\delta_K(\hat{v}(t)) = 0$ and $\hat{v}(t) \in \tilde{K}$ a.s for a.e. $t \in [0,T]$. The pointwise minimization problem* (30) *becomes a simple constrained quadratic minimization problem*

$$\hat{v}(t) \triangleq \underset{v\in\tilde{K}}{\arg\min}\,|\theta(t) + \sigma(t)^{-1}v|^2,\ \forall t \in [0,T].$$

Furthermore, in the case where $K = \mathbb{R}^N$, the complete market case, then $\tilde{K} = \{0\}$ and $\hat{v} = 0$, the optimal control (31) *reduces to $\hat{\pi}(t) = [\sigma(t)\sigma^\intercal(t)]^{-1}[b(t) - r(t)\mathbf{1}]$ for all $t \in [0,T]$, and we recover the unconstrained log utility maximization problem discussed in [10].*

Remark 15. *From the above two examples, we contrast our method to the approach in [6, 12, 13], which rely on the introduction of a family of auxiliary unconstrained problems formulated in auxiliary markets parametrized by money market and stock mean return rates [6, Section 8]. The existence of a solution to the original problem is then equivalent to finding the fictitious market that provides the correct optimal solution to the primal problem. On the other hand, we explicitly write out the dual problem to the original constrained problem only relying on the elementary convex analysis results and characterize its solution in terms of FBSDEs. The dynamic relationship between the primal and dual FBSDEs then allows us to explicitly construct the optimal solution to the primal problem from that to the dual problem.*

4.3 Constrained non-HARA utility maximization

In this subsection, we assume U is a non-HARA utility function defined by

$$U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$$

for $x > 0$, where $H(x) = \sqrt{2}\left(-1 + \sqrt{1+4x}\right)^{-\frac{1}{2}}$. The dual function of U is given by

$$\tilde{U}(y) = \frac{1}{3}y^{-3} + y^{-1}$$

for $y > 0$. Assume that $K \subseteq \mathbb{R}^N$ is a closed convex cone and r, b, σ are constants. The dual problem becomes

$$\text{Minimize } x_0 y + E\left[\frac{1}{3}\left(Y^{(y,v)}(T)\right)^{-3} + \left(Y^{(y,v)}(T)\right)^{-1}\right] \text{ over } (y,v) \in (0,\infty) \times \mathcal{D}.$$

We solve the above problem in two steps: first, fix y and find the optimal control $\tilde{v}(y)$; second, find the optimal $\hat{y}$. We can then construct the optimal solution explicitly.

Step 1: The DPP implies that the dual value function $v(t,y)$ satisfies the following HJB equation

$$\begin{cases} \frac{\partial}{\partial t} v(t,y) - ryv_y(t,y) + \frac{1}{2}\inf_{v \in \tilde{K}} |\theta + \sigma^{-1}v|^2 y^2 v_{yy}(t,y) = 0, \\ v(T,y) = \frac{1}{3}y^{-3} + y^{-1}, \end{cases}$$

for $(t,y) \in [0,T] \times [0,\infty)$. Let $\hat{v}$ be the minimizer of $\inf_{v \in \tilde{K}} |\theta + \sigma^{-1}v|^2$ and $\hat{\theta} \triangleq \theta + \sigma^{-1}\hat{v}$. Assume $|\hat{\theta}| > 0$. Using the Feynman-Kac formula, we can easily find

$$v(t,y) = \frac{1}{3}y^{-3}e^{(3r+6|\hat{\theta}|^2)(T-t)} + \frac{1}{y}e^{(r+|\hat{\theta}|^2)(T-t)}.$$

Step 2: Consider the following scalar convex minimization problem

$$\inf_{y>0}(x_0 y + v(0,y)).$$

The minimizer $\hat{y}$ satisfies the equation

$$x_0 - \hat{y}^{-4}e^{(3r+6|\hat{\theta}|^2)T} - \hat{y}^{-2}e^{(r+|\hat{\theta}|^2)T} = 0.$$

Hence, we have

$$\hat{y} = \frac{1}{\sqrt{2x_0}}\left[e^{(r+|\hat{\theta}|^2)T} + \sqrt{e^{2(r+|\hat{\theta}|^2)T} + 4x_0 e^{3(r+2|\hat{\theta}|^2)T}}\right]^{\frac{1}{2}}$$

and the optimal state process for the dual problem is given by

$$\hat{Y}(t) = \hat{y}e^{-(r+\frac{1}{2}|\hat{\theta}|^2)t - \hat{\theta}^\intercal W(t)}. \tag{32}$$

Using the martingale property of $\hat{p}_2\hat{Y}$, see (21), we have

$$\begin{aligned} \hat{p}_2(t)\hat{Y}(t) &= E\left[\hat{Y}(T)^{-3} + \hat{Y}(T)^{-1} | \mathcal{F}_t\right] \\ &= \hat{y}^{-3}e^{3(r+\frac{1}{2}|\hat{\theta}|^2)T}e^{3\hat{\theta}^\intercal W(t)}e^{\frac{9}{2}|\hat{\theta}|^2(T-t)} + \hat{y}^{-1}e^{(r+\frac{1}{2}|\hat{\theta}|^2)T}e^{\hat{\theta}^\intercal W(t)}e^{\frac{1}{2}|\hat{\theta}|^2(T-t)}. \end{aligned}$$

Substituting (32) back into the above equation and rearranging, we have

$$\hat{p}_2(t) = a_1 S_1(t) + a_2 S_2(t),$$

where $a_1 = \hat{y}^{-4}e^{3(r+2|\hat{\theta}|^2)T}$, $a_2 = \hat{y}^{-2}e^{(r+|\hat{\theta}|^2)T}$, $S_1(t) = e^{(r-4|\hat{\theta}|^2)t + 4\hat{\theta}^\intercal W(t)}$, and $S_2(t) = e^{rt + 2\hat{\theta}^\intercal W(t)}$. Applying Ito's lemma, we have

$$d\hat{p}_2(t) = [r\hat{p}_2(t) + \hat{q}_2^\intercal(t)\hat{\theta}]dt + \hat{q}_2^\intercal(t)dW(t),$$

where

$$\hat{q}_2(t) = (4a_1 S_1(t) + 2a_2 S_2(t))\hat{\theta},\ t \in [0, T].$$

Finally, according to Theorem 12, we can construct the optimal solution of the primal problem explicitly as

$$\begin{cases} \hat{\pi}(t) = [\sigma^\intercal]^{-1}\hat{q}_2(t)\hat{p}_2^{-1}(t), \\ X^{\hat{\pi}}(t) = \hat{p}_2(t) = \hat{y}^{-4}e^{3(r+2|\hat{\theta}|^2)T}e^{(r-4|\hat{\theta}|^2)t+4\hat{\theta}^\intercal W(t)} + \hat{y}^{-2}e^{(r+|\hat{\theta}|^2)T}e^{rt+2\hat{\theta}^\intercal W(t)}. \end{cases}$$

Remark 16. *Suppose that after attaining the dual value function v, we try to recover the optimal solution to the primal problem directly. By the duality relation between the primal and dual value functions, see [2, Theorem 2.6], the primal value function is given by*

$$u(t,x) = v(t, \hat{y}(t,x)) + v_y(t, \hat{y}(t,x))\hat{y}(t,x) = \frac{2}{3}\left(\hat{y}(t,x)^{-1}e^{(r+|\hat{\theta}|^2)(T-t)} + 2x\hat{y}(t,x)\right),$$

where $\hat{y}(t,x)$ is the solution of equation $v_y(t,y) + x = 0$ and is given by

$$\hat{y}(t,x) = \frac{1}{\sqrt{2x}}\left[e^{(r+|\hat{\theta}|^2)(T-t)} + \sqrt{e^{2(r+|\hat{\theta}|^2)(T-t)} + 4xe^{3(r+2|\hat{\theta}|^2)(T-t)}}\right]^{\frac{1}{2}}$$

Hence, to find the optimal control $\hat{\pi}$, we would need to solve the following optimization

$$\sup_{\pi \in K}\left(x\left(r + \pi^\intercal\sigma\theta\right)u_x(t,x) + \frac{1}{2}x^2\pi^\intercal\sigma\sigma^\intercal\pi u_{xx}(t,x)\right)$$

to get an optimal feedback control $\hat{\pi}(t,x)$. To find the optimal wealth process $X^{\hat{\pi}}$, we would need to substitute $\hat{\pi}(t,x)$ into (3) *and solve a highly complicated nonlinear SDE. However, in the approach we proposed, the optimal adjoint process of the dual problem can be written out explicitly as conditional expectations of the dual process. The optimal solution to the primal problem can be constructed explicitly thanks to their dynamic relation as stated in Theorem 12.*

5 Conclusions

In this paper, we study the constrained utility maximization problem following the convex duality approach. After formulating the primal and dual problems, we construct the necessary and sufficient optimality conditions for both the primal and dual problems in terms of FBSDEs plus some additional conditions. Such formulation allows us to establish the explicit connection between the primal and dual optimal solutions in a dynamic fashion. Finally we solve three constrained utility maximization problems to illustrate our dynamic convex duality FBSDE approach.

Acknowledgements. The authors are very grateful to the reviewer and the associate editor for their constructive comments and suggestions which have helped to improve the paper of the previous two versions.

References

[1] J. M. Bismut, *Conjugate convex functions in optimal stochastic control*, J. Math. Anal. Appl., **44** (1973), pp. 384–404.

[2] B. Bian and H. Zheng, *Turnpike Property and Convergence Rate for an Investment Model with General Utility Functions*, J. Econom. Dynam. Control., **51** (2015), pp. 28–49.

[3] A. Cadenillas and I. Karatzas, *The stochastic maximum principle for linear convex optimal control with random coefficients*, SIAM J. Control Optim., **33** (1995), pp. 590–624.

[4] J.C. Cox and C.F. Huang *Optimal consumption and portfolio policies when asset prices follow a diffusion process*, J. Econom. Theory, **49** (1989), pp. 33–83.

[5] J.C. Cox and C.F. Huang *A variational problem arising in financial economics*, J. Math. Econom., **20** (1991), pp. 465–487.

[6] J. Cvitanić and I. Karatzas, *Convex duality in constrained portfolio optimization*, Ann. Appl. Probab, **2** (1992), pp. 767–818.

[7] H. He and N.D. Pearson, *Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite-dimensional case*, J. Econom. Theory, **54** (1991), pp. 259–304.

[8] H. He and N.D. Pearson, *Consumption and portfolio policies with incomplete markets and short-sale constraints: the finite-dimensional case*, Math. Finance, **3** (1991), pp. 1–10.

[9] U. Horst, Y. Hu, P. Imkeller, A. Reveillac and J. Zhang, *Forward-backward systems for expected utility maximization*, Stochastic Process. Appl., **124** (2014), pp. 1813–1848.

[10] I. Karatzas, *Optimization problems in the theory of continuous trading*, SIAM J. Control Optim., **27** (1989), pp. 1221–1259.

[11] I. Karatzas, J.P. Lehoczky and S.E. Shreve, *Optimal portfolio and consumption decisions for a "small" investor on a finite horizon*, SIAM J. Control Optim., **25** (1987), pp. 1557–1586.

[12] I. Karatzas, J.P. Lehoczky, S.E. Shreve and G.L. Xu, *Martingale and duality methods for utility maximization in an incomplete market*, SIAM J. Control Optim., **29** (1991), pp. 702–731.

[13] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, Springer, 2001.

[14] D. Kramkov and W. Schachermayer, *The asymptotic elasticity of utility functions and optimal investment in incomplete markets*, Ann. Appl. Probab., **9** (1999), pp. 904–950.

[15] C. Labbé and A. J. Heunis, *Convex duality in constrained mean-variance portfolio optimization*, Adv. in Appl. Probab., **39** (2007), pp. 77–104.

[16] C. Labbé and A. J. Heunis, *Convex duality in problems of constrained utility maximization*, Stochastics., **81** (2009), pp. 545–565.

[17] R.C. Merton, *Lifetime portfolio selection under uncertainty: the continuous-time case*, Rev. Econom. Statist., **51** (1969), pp. 247–257.

[18] R.C. Merton, *Optimal consumption and portfolio rules in a continuous-time model*, J. Econom. Theory, **3** (1971), pp. 373–413.

[19] S. Peng, *A General Stochastic Maximum Principle for Optimal Control Problems*, SIAM J. Control Optim., **28** (1990), pp. 966–979.

[20] H. Pham, *Continuous-Time Stochastic Control and Optimization with Financial Applications*, Springer, 2009.

[21] S.R. Pliska, *A stochastic calculus model of continuous-time trading: optimal portfolios*, Math. Oper. Res., **11** (1986), pp. 371–382.

[22] M. Santacroce and B. Trivellato, *Forward backward semimartingale systems for utility maximization*, SIAM J. Control Optim., **52** (2014), pp. 3517–3537.

[23] M. Schäl, *A selection theorem for optimization problems*, Arch. Math., **25** (1974), pp. 219–224.

[24] M. Schäl, *Conditions for optimality in dynamic programming, and for the limit of n-state optimal policies to be optimal*, Z. Wahrsch. Verw. Geviete, **32** (1975), pp. 179–196.

[25] G. L. Xu and S. E. Shreve, *A duality method for optimal consumption and investment under short-selling prohibition. I. General market coefficients*, Ann. Appl. Probab., **2** (1992), pp. 87–112.