Ultrapatching

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Contents

In these notes we will develop the commutative algebra results needed for the Taylor–Wiles–Kisin patching method, as reformulated by Scholze in [Sch18].

Acknowledgments

These notes are an expanded version of the appendix to my thesis. Various portions of these notes appeared in my papers [Man19] and [MS19] (although some and definitions that appear in the current version of these notes are different from what was used in the these papers).

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Chapter I

Ultraproducts

I.1 General theory

Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the natural numbers. Recall that a *nonprincipal ultrafilter* on $\mathbb{N}$ is a collection, $\mathfrak{F}$, of subsets of $\mathbb{N}$ satisfying the following conditions:

1. $\mathfrak{F}$ does not contain any finite sets.
2. If $I, J \in \mathfrak{F}$ then $I \cap J \in \mathfrak{F}$
3. If $I \in \mathfrak{F}$ and $I \subseteq J \subseteq \mathbb{N}$, then $J \in \mathfrak{F}$ as well.
4. If $I \sqcup J = \mathbb{N}$ is a partition of $\mathbb{N}$, then either $I \in \mathfrak{F}$ or $J \in \mathfrak{F}$.

It is well known that such an $\mathfrak{F}$ must exist, if one assumes the axiom of choice.

Note that these conditions imply the following: If $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_a = \mathbb{N}$ is a partition of $\mathbb{N}$, then $I_i \in \mathfrak{F}$ for *exactly* one i.

For the remainder of this appendix, we will fix a nonprincipal ultrafilter $\mathfrak{F}$ on $\mathbb{N}$.

For convenience, we will say that a property $\mathcal{P}(i)$ holds for $\mathfrak{F}$*-many* i if there is some $I \in \mathfrak{F}$ such that $\mathcal{P}(i)$ is true for all $i \in I$. The four conditions above imply the following:

1. If $\mathcal{P}(i)$ holds for $\mathfrak{F}$-many i, then it holds for infinitely many i.
2. If $\mathcal{P}(i)$ and $\mathcal{Q}(i)$ each hold for $\mathfrak{F}$-many i, then $\mathcal{P}(i)$ and $\mathcal{Q}(i)$ are *simultaneously* true for $\mathfrak{F}$-many i.
3. $\mathcal{P}(i)$ holds for $\mathfrak{F}$-many i if and only if the set $\{i | \mathcal{P}(i) \text{ is true}\}$ is in $\mathfrak{F}$.
4. For any property $\mathcal{P}$, either $\mathcal{P}(i)$ is true for $\mathfrak{F}$-many i, or it is false for $\mathfrak{F}$-many i.

If $\mathscr{M} = \{M_n\}_{n \geq 1}$ is any sequence of sets, we define an equivalence relation $\sim$ on the set $\prod_{n \geq 1} M_n$ by $(m_1, m_2, \ldots) \sim (m'_1, m'_2, \ldots)$ if $m_i = m'_i$ for $\mathfrak{F}$-many i (the above properties of ultrafilters imply that this is an equivalence relation). We then define the *ultraproduct* of $\mathscr{M}$ to be

$$\mathcal{U}(\mathscr{M}) := \left(\prod_{n\geq 1} M_n\right) / \sim$$

For any $m = (m_1, m_2, \ldots) \in \prod_{n\geq 1} M_n$ we will denote the equivalence class of m in $\mathcal{U}(\mathscr{M})$ by $[m_i]_i = [m_1, m_2, \ldots]$. We will frequently define elements $m = [m_i]_i$ by only specifying m_i for $\mathfrak{F}$-many i. Doing so is unambiguous, as if m_i is specified for all $i \in I$ ($I \in \mathfrak{F}$) the choices of m_j for $j \in \mathbb{N} \smallsetminus I$ do not affect the equivalence class $[m_i]_i$.

If M is any set we will write $\underline{M} := \{M\}_{n\geq 1}$ for the constant sequence of sets, and define the *ultrapower* of M to be $M^{\mathcal{U}} := \mathcal{U}(\underline{M})$. Notice that we have a diagonal map $\Delta : M \to M^{\mathcal{U}}$ defined by $m \mapsto [m, m, \ldots]$. This map is clearly injective.

In our applications, we will generally consider the case where each M_n has a certain algebraic structure. Thus for the rest of this subsection we will fix a category, $\mathcal{C}$ of sets with algebraic structure, taken to be one of the following:

- The category of abelian groups;
- The category of (commutative) rings;
- The category of (continuous) R-modules;
- The category of (continuous) R-algebras,

for some fixed ring topological R (which we will often take to have the discrete topology, however the continuous version will be used in Lemma II.5.2). Using the language of universal algebra (or more generally, of model theory) it is possible phrase the results of this section for significantly more general categories of "sets with structure," however the specific cases we discuss here will be sufficient for our purposes.

We first show that if each M_n is in $\mathcal{C}$, then $\mathcal{U}(\mathscr{M})$ inherits a natural $\mathcal{C}$-object structure.

Proposition I.1.1. *Let $\mathscr{M} = \{M_n\}_{n\geq 1}$, and assume that each M_n is in $\mathcal{C}$. Then $\mathcal{U}(\mathscr{M})$ may be given the structure of object in $\mathcal{C}$ with the operations additions, multiplication and scalar multiplication (when appropriate) defined by:*

$$\begin{aligned}
[a_1, a_2, \ldots] + [b_1, b_2, \ldots] &= [a_1 + b_1, a_2 + b_2, \ldots] \\
[a_1, a_2, \ldots] \cdot [b_1, b_2, \ldots] &= [a_1 \cdot b_1, a_2 \cdot b_2, \ldots] \\
r[a_1, a_2, \ldots] &= [ra_1, ra_2, \ldots]
\end{aligned}$$

for $\alpha = [a_1, a_2, \ldots], \beta = [b_1, b_2, \ldots] \in \mathcal{U}(\mathscr{M})$, the elements $0, 1 \in \mathcal{U}(\mathscr{M})$ (again when appropriate) defined by:

$$0 = [0, 0, \ldots] \in \mathcal{U}(\mathscr{M}), \qquad 1 = [1, 1, \ldots] \in \mathcal{U}(\mathscr{M}),$$

and topology defined by the quotient map $\pi : \prod_{n\geq 1} M_n \twoheadrightarrow \mathcal{U}(\mathscr{M})$. Moreover:

1. *The natural surjection* $\pi : \prod_{n\geq 1} M_n \twoheadrightarrow \mathcal{U}(\mathscr{M})$, $(m_i)i \mapsto [m_i]_i$ *is a* $\mathcal{C}$*-morphism.*
2. *For* $M \in \mathcal{C}$*, the diagonal map* $\Delta : M \to M^{\mathcal{U}}$ *is a* $\mathcal{C}$*-morphism.*

Proof. We will prove this only in the case when $\mathcal{C}$ is taken to by the category of continuous R-algebras. The other cases are analogous.

First we check that the operations are well-defined. Take $\alpha = [a_i]_i, \alpha' = [a_i']_i, \beta = [b_i]_i, \beta' = [b_i'] \in \mathcal{U}(\mathscr{M})$ with $\alpha = \alpha'$ and $\beta = \beta'$. Then for $\mathfrak{F}$-many i we *simultaneously* have that $a_i = a_i'$ and $b_i = b_i'$. It follows that $a_i + b_i = a_i' + b_i'$, $a_i \cdot b_i = a_i' \cdot b_i'$ and $ra_i = ra_i'$ for $\mathfrak{F}$-many i, and so $\alpha + \beta = \alpha' + \beta'$, $\alpha \cdot \beta = \alpha' \cdot \beta'$ and $r\alpha = r\alpha'$.

Now as the operations are defined pointwise, they are clearly preserved by $\pi : \prod_{n\geq 1} M_n \twoheadrightarrow \mathcal{U}(\mathscr{M})$.Thus as $\prod_{n\geq 1} M_n$ is a *continuous* R-algebra, and π is continuous by definition, (1) will follow if we show that the operations make $\mathcal{U}(\mathscr{M})$ into a R-algebra (the operations will automatically be continuous as $\mathcal{U}(\mathscr{M})$ has the quotient topology).

Now let

$$\begin{aligned} K &= \left\{ (a_1, a_2, \ldots) \in \prod_{n\geq 1} M_n \,\middle|\, (a_1, a_2, \ldots) \sim (0, 0, \ldots) \right\} \\ &= \left\{ (a_1, a_2, \ldots) \in \prod_{n\geq 1} M_n \,\middle|\, a_i = 0 \text{ for } \mathfrak{F}\text{-many } i \right\} \subseteq \prod_{n\geq 1} M_n \end{aligned}$$

Now as the operations are well-defined, for any $a = (a_n)_n, b = (b_n)_n \in K$, any $m = (m_n) \in \prod_{n\geq 1} M_n$ and any $r \in R$ we get that:

$$\begin{aligned} (a_n + b_n)_n &= (a_n)_n + (b_n)_n \sim (0)_n + (0)_n = (0)_n \\ (m_n \cdot a_n)_n &= (m_n)_n \cdot (a_n)_n \sim (m_n)_n \cdot (0)_n = (0)_n \\ (ra_n)_n &= r(a_n)_n \sim r(0)_n = (0)_n, \end{aligned}$$

and so $a + b, ma, ra \in K$. It follows that $K \subseteq \prod_{n\geq 1} M_n$ is an *ideal.*

Also by definition, for $a = (a_n)_n, b = (b_n)_n \in \prod_{n\geq 1} M_n$, $a \sim b$ if and only if $a - b \in K$. It follows that $\pi : \prod_{n\geq 1} M_n \twoheadrightarrow \mathcal{U}(\mathscr{M})$ gives an identification $\overline{\pi} : \left(\prod_{n\geq 1} M_n\right)/K \xrightarrow{\sim} \mathcal{U}(\mathscr{M})$. As π, and thus $\overline{\pi}$, preserves the operations and $\left(\prod_{n\geq 1} M_n\right)/K$ is an R-algebra, it follows that $\mathcal{U}(\mathscr{M})$ is indeed an R-algebra, and π is an R-algebra homomorphism. This proves (1).

For (2), we simply note that $\Delta : M \to M^{\mathcal{U}}$ is the composition of the $\mathcal{C}$-morphisms $M \hookrightarrow \prod_{n\geq 1} M$, $m \mapsto (m, m, \dots)$ and $\pi : \prod_{n\geq 1} M \to \mathcal{U}(\underline{M}) = M^{\mathcal{U}}$. □

Given two sequences $\mathscr{M} = \{M_n\}_{n\geq 1}$ and $\mathscr{M}' = \{M'_n\}_{n\geq 1}$ in $\mathcal{C}$, we define an $\mathfrak{F}$-*morphism* $\varphi : \mathscr{M} \to \mathscr{M}'$ to be a collection of $\mathcal{C}$-morphisms $\varphi = \{\varphi_i : M_i \to M'_i\}_{i\in I}$ indexed by some $I \in \mathfrak{F}$. Then we have

Proposition I.1.2. *If $\varphi : \mathscr{M} \to \mathscr{M}'$ is an $\mathfrak{F}$-morphism, then the map $\varphi^{\mathcal{U}} : \mathcal{U}(\mathscr{M}) \to \mathcal{U}(\mathscr{M}')$ given by $\varphi^{\mathcal{U}}[a_i]_i = [\varphi_i(a_i)]_i$ is a well-defined $\mathcal{C}$-morphism. Moreover,*

1. *If $\varphi, \psi : \mathscr{M} \to \mathscr{M}'$ are two $\mathfrak{F}$-morphisms, and $\varphi_i = \psi_i$ for $\mathfrak{F}$-many i, then $\varphi^{\mathcal{U}} = \psi^{\mathcal{U}}$. In particular, if $\varphi : \mathscr{M} \to \mathscr{M}$ satisfies $\varphi_i = \mathrm{id}_{M_i} : M_i \to M_i$ for $\mathfrak{F}$-many i, then $\varphi^{\mathcal{U}} = \mathrm{id}_{\mathcal{U}(\mathscr{M})} : \mathcal{U}(\mathscr{M}) \to \mathcal{U}(\mathscr{M})$.*
2. *For two $\mathfrak{F}$-morphisms, $\varphi : \mathscr{M} \to \mathscr{M}'$ and $\psi : \mathscr{M}' \to \mathscr{M}''$, we have $\psi^{\mathcal{U}} \circ \varphi^{\mathcal{U}} = (\psi \circ \varphi)^{\mathcal{U}}$.*

Hence $\mathcal{U}(-)$ is a functor*.*

Proof. As in Proposition I.1.1, we will prove this only in the case where $\mathcal{C}$ is the category of continuous R-algebras.

Let $\varphi : \mathscr{M} \to \mathscr{M}'$ be an $\mathfrak{F}$-morphism. If we have $[a_i]_i = [a'_i]_i$ in $\mathcal{U}(\mathscr{M})$, then for $\mathfrak{F}$-many i we simultaneously have that φ_i exists and $a_i = a'_i$. Thus $\varphi^{\mathcal{U}}[a_i]_i = [\varphi_i(a_i)] = [\varphi(a'_i)] = \varphi^{\mathcal{U}}[a'_i]_i$, and so $\varphi^{\mathcal{U}}$ is well-defined. As each φ_i is continuous, it follows that $\varphi^{\mathcal{U}}$ is induced by a continuous map $\prod_{n\geq 1} M_n \to \prod_{n\geq 1} M'_n$, and thus is continuous.

Now for $\alpha = [a_i]_i, \beta = [b_i]_i \in \mathcal{U}(\mathscr{M})$ and $r \in R$, as φ_i is an R-algebra homomorphism for $\mathfrak{F}$-many i, we get

$$\begin{aligned} \varphi^{\mathcal{U}}(\alpha + \beta) &= \varphi^{\mathcal{U}}[a_i + b_i]_i = [\varphi_i(a_i + b_i)]_i = [\varphi_i(a_i) + \varphi(b_i)]_i = \varphi^{\mathcal{U}}(\alpha) + \varphi^{\mathcal{U}}(\beta) \\ \varphi^{\mathcal{U}}(\alpha \cdot \beta) &= \varphi^{\mathcal{U}}[a_i \cdot b_i]_i = [\varphi_i(a_i \cdot b_i)]_i = [\varphi_i(a_i) \cdot \varphi(b_i)]_i = \varphi^{\mathcal{U}}(\alpha) \cdot \varphi^{\mathcal{U}}(\beta) \\ \varphi^{\mathcal{U}}(r\alpha) &= \varphi^{\mathcal{U}}[ra_i]_i = [\varphi_i(ra_i)]_i = [r\varphi_i(a_i)]_i = r\varphi^{\mathcal{U}}(\alpha) \\ \varphi^{\mathcal{U}}(1) &= \varphi^{\mathcal{U}}[1]_i = [\varphi_i(1)]_i = [1]_i = 1, \end{aligned}$$

so indeed $\varphi^{\mathcal{U}}$ is an R-algebra homomorphism.

If $\varphi_i = \psi_i$ for $\mathfrak{F}$-many i, then by definition we have $\varphi^{\mathcal{U}}[a_i]_i = [\varphi_i(a_i)]_i = [\psi_i(a_i)]_i = \psi^{\mathcal{U}}[a_i]_i$, and if $\varphi_i = \mathrm{id}_{M_i}$ for $\mathfrak{F}$-many i, then $\varphi^{\mathcal{U}}[a_i]_i = [\varphi_i(a_i)]_i = [a_i]_i$. So (1) holds.

For (2), simply note that for $\mathfrak{F}$-many i, φ_i and ψ_i simultaneously exist, and so

$$\left(\psi^{\mathcal{U}} \circ \varphi^{\mathcal{U}}\right)[a_i]_i = \psi^{\mathcal{U}}\left(\varphi^{\mathcal{U}}\,[a_i]_i\right) = \psi^{\mathcal{U}}\,[\varphi_i(a_i)]_i = [\psi_i\left(\varphi_i(a_i)\right)]_i = (\psi \circ \varphi)^{\mathcal{U}}[a_i]_i.$$

□

In general, $\mathcal{U}(\mathscr{M})$ can be a quite complicated object. However in our setup, the M_n's will always be taken to be finite, of uniformly bounded cardinalities. In that case, we have the following:

Proposition I.1.3. *If $M \in \mathcal{C}$ has finite cardinality, the diagonal map $\Delta : M \to M^{\mathcal{U}}$ is an isomorphism.*

Now assume that $\mathcal{C}$ is the category of abelian groups or rings, or that the ring R is topologically finitely generated (in particular, if it is finite*). If $\mathscr{M} = \{M_n\}_{n\geq 1}$ where each $M_n \in \mathcal{C}$ is a* finite *set, and the cardinalities $\#M_n$ are bounded, then $\mathcal{U}(\mathscr{M})$ is also finite and we have $\mathcal{U}(\mathscr{M}) \cong M_i$ in $\mathcal{C}$ for $\mathfrak{F}$-many i.*

Proof. As $\Delta : M \to M^{\mathcal{U}}$ is already an injective $\mathcal{C}$-morphism, it suffices to show that it is surjective. Take any $\alpha = [a_i]_i \in M^{\mathcal{U}}$. As M is finite, $\displaystyle\bigsqcup_{a\in M} \{i | a_i = a\}$ is a finite partition of $\mathbb{N}$, and so for some $a \in M$, $a_i = a$ for $\mathfrak{F}$-many i. But then $\alpha = [a_i]_i = [a]_i = \Delta(a)$, so indeed Δ is surjective, and hence an isomorphism.

For the second statement, the assumption on $\mathcal{C}$ implies that there are only finitely many isomorphism classes of $\mathcal{C}$-objects of any fixed cardinality d. As the $\#M_n$'s are bounded, there are only finitely many distinct cardinalities $\{\#M_n\}_{n\geq 1}$. It thus follows that there are only finitely many isomorphism classes of $\mathcal{C}$-objects in $\mathscr{M}$.

Thus we may pick some $M \in \mathcal{C}$ (which is necessarily finite) for which $M \cong M_i$ for $\mathfrak{F}$-many i. Fix isomorphisms $\varphi_i : M \xrightarrow{\sim} M_i$ for $\mathfrak{F}$-many i, and define $\mathfrak{F}$-morphisms $\varphi : \underline{M} \to \mathscr{M}$ and $\psi : \mathscr{M} \to \underline{M}$ by $\varphi = \{\varphi_i\}$ and $\psi = \{\varphi_i^{-1}\}$. It follows from Proposition I.1.2 that $\psi^{\mathcal{U}} = (\varphi^{\mathcal{U}})^{-1}$ and so $\varphi^{\mathcal{U}} : M^{\mathcal{U}} = \mathcal{U}(\underline{M}) \to \mathcal{U}(\mathscr{M})$ is an isomorphism.

Combining this with the first claim, we indeed get $\mathcal{U}(\mathscr{M}) \cong M^{\mathcal{U}} \cong M \cong M_i$ for $\mathfrak{F}$-many i. □

I.2 A ring theoretic interpretation

In the case when $\mathcal{C}$ is taken to be the category of R-modules (or R-algebras), the construction of $\mathcal{U}(\mathscr{M})$ can be reformulated as a localization of modules, and is thus quite well behaved. We finish this section by discussing this situation.

For the reminder of this section, R will denote a *finite* local ring with maximal ideal m_R and residue field $\mathbb{F} = R/m_R$.

We will let $\mathcal{R} := \displaystyle\prod_{n\geq 1} R$, treated as an R-algebra via the diagonal embedding $r \mapsto (r, r, \ldots)$. Proposition I.1.1 implies that the natural map $\pi : \mathcal{R} \twoheadrightarrow R^{\mathcal{U}} = R$ is a surjective ring homomorphism.

Also for any $I \subseteq \mathbb{N}$, we will let $\mathcal{R}_I := \displaystyle\prod_{i\in I} R$, viewed as a quotient of $\mathcal{R}$ via the map $\pi_I : (r_n)_{n\geq 1} \mapsto (r_i)_{i\in I}$. Note that $\pi : \mathcal{R} \to R$ factors through π_I for each $I \in \mathfrak{F}$.

The key observation is that π may be viewed as a localization map:

Proposition I.2.1. *View R as a $\mathcal{R}$-algebra via the map $\pi : \mathcal{R} \twoheadrightarrow R$. There is a unique prime ideal $\mathfrak{Z}_R \in \operatorname{Spec} \mathcal{R}$ for which the $\mathcal{R}$-algebra localization map $R \to R_{\mathfrak{Z}_R}$ is an isomorphism. For this $\mathfrak{Z}_R$ we have:*

- *The map $\pi_{\mathfrak{Z}_R} : \mathcal{R}_{\mathfrak{Z}_R} \to R$ is an isomorphism.*
- *For all $I \in \mathfrak{F}$ the map $\pi_{I,\mathfrak{Z}_R} : \mathcal{R}_{\mathfrak{Z}_R} \twoheadrightarrow \mathcal{R}_{I,\mathfrak{Z}_R}$, induced by $\pi_I : \mathcal{R} \twoheadrightarrow \mathcal{R}_I$ is an isomorphism.*

We will call $\mathfrak{Z}_R$ the prime (of $\mathcal{R}$) associated to $\mathfrak{F}$*.*

Finally, if $\psi : R \to R'$ is a surjection of local rings, inducing the surjection $\Psi : \mathcal{R} \twoheadrightarrow \mathcal{R}' := \prod_{n \geq 1} R'$, and $\mathfrak{Z}_{R'} \in \operatorname{Spec} \mathcal{R}'$ is the prime associated to $\mathfrak{F}$, then $\mathfrak{Z}_R = \Psi^{-1}(\mathfrak{Z}_{R'})$.

Proof. Assume that there is some $\mathfrak{Z}_R \in \operatorname{Spec} \mathcal{R}$ which makes $R \to R_{\mathfrak{Z}_R}$ into an isomorphism. Clearly we must have $\ker(\pi : \mathcal{R} \to R) \subseteq \mathfrak{Z}_R$, or we would have $R_{\mathfrak{Z}_R} = 0$. Thus $\mathfrak{Z}_R = \pi^{-1}(P)$ for some $P \in \operatorname{Spec} R$ and $R_P \cong R_{\mathfrak{Z}_R}$. But now as R is a local ring, $R \to R_P$ is an isomorphism if and only if $P = m_R^{\mathcal{U}}$. Thus the unique prime $\mathfrak{Z}_R$ satisfying the condition is $\mathfrak{Z}_R = \pi^{-1}(m_R^{\mathcal{U}})$.

We now show that the map $\pi_{\mathfrak{Z}_R} : \mathcal{R}_{\mathfrak{Z}_R} \to R$ is an isomorphism. As localization is exact, it is surjective.

Take any $\frac{r}{s} \in \ker(\pi_{\mathfrak{Z}_R})$ where $r = (r_1, r_2, \ldots) \in \mathcal{R}$. Then $r \in \ker(\pi)$ so that $[r_i]_i = 0$ in R, and hence $r_i = 0$ for $\mathfrak{F}$-many i. Define $e = (\varepsilon_1, \varepsilon_2, \ldots) \in \mathcal{R}$ by $\varepsilon_i = 1$ if $r_i = 0$ and $e_i = 0$ if $r_i \neq 0$, and note that $er = 0$. But by definition $e_i = 1$ for $\mathfrak{F}$-many i, and so $\pi(e) = 1 \notin m_R^{\mathcal{U}}$. Hence $e \notin \mathfrak{Z}_R$, and so $\frac{e}{1}$ is a unit in $\mathcal{R}_{\mathfrak{Z}_R}$. As $\frac{e}{1}\frac{a}{s} = 0$, this implies that $\frac{a}{s} = 0$. Therefore $\ker(\pi_{\mathfrak{Z}_R}) = 0$ and so indeed, $\pi_{\mathfrak{Z}_R}$ is an isomorphism.

Now for any $I \in \mathfrak{F}$, $\pi : \mathcal{R} \twoheadrightarrow R$ is a composition of surjections $\pi_I : \mathcal{R} \twoheadrightarrow \mathcal{R}_I$ and $\mathcal{R}_I \twoheadrightarrow R$, and so $\pi_{\mathfrak{Z}_R} : \mathcal{R}_{\mathfrak{Z}_R} \to R$ is a composition of the surjections $\pi_{I,\mathfrak{Z}_R} : \mathcal{R}_{\mathfrak{Z}_R} \twoheadrightarrow \mathcal{R}_{I,\mathfrak{Z}_R}$ and $\mathcal{R}_{I,\mathfrak{Z}_R} \twoheadrightarrow R$. So as $\pi_{\mathfrak{Z}_R}$ is an isomorphism, the latter two maps are isomorphisms as well.

The last statement follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\pi} & R \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\psi^{\mathcal{U}}} \\ \mathcal{R}' & \xrightarrow[\pi']{} & R' \end{array}$$

□

From now on we will always use $\mathfrak{Z}_R$ to denote the prime of $\mathcal{R}$ associated to $\mathfrak{F}$, or just $\mathfrak{Z}$ if R is clear from context.

We will now investigate ultraproducts of R-modules (and R-algebras). Let $\mathscr{M} = \{M_n\}_{n\geq 1}$ be any sequence of R-modules, and write $\mathcal{M} = \prod_{n\geq 1} M_n$ with its natural $\mathcal{R}$-module structure. We claim that the natural surjection $\pi^{\mathscr{M}} : \mathcal{M} \twoheadrightarrow \mathcal{U}(\mathscr{M})$ is an $\mathcal{R}$-module homomorphism, where the $\mathcal{R}$-action on $\mathcal{U}(\mathscr{M})$ is given by $\pi : \mathcal{R} \twoheadrightarrow R$.

Indeed for any $r = (r_1, r_2, \ldots) \in \mathcal{R}$ and $m = (m_1, m_2, \ldots) \in \mathcal{M}$ we have $r_i = \pi(r)$ for $\mathfrak{F}$-many i, and so

$$\pi^{\mathscr{M}}(rm) = [r_i m_i]_i = [\pi(r) m_i]_i = \pi(r)[m_i]_i = \pi(r)\pi^{\mathscr{M}}(m).$$

If additionally the M_n's are A-*algebras*, then $\mathcal{U}(\mathscr{M})$ is an $\mathcal{R}$-algebra, and the above morphism is of $\mathcal{R}$-algebras.

Proposition I.2.1 now allows us to re-interpret $\pi^{\mathscr{M}}$ as a localization map of $\mathcal{R}$-modules:

Proposition I.2.2. *Let $\mathscr{M} = \{M_n\}_{n\geq 1}$ be a collection of R-modules and let $\mathcal{M}$ and $\pi^{\mathscr{M}} : \mathcal{M} \to \mathcal{U}(\mathscr{M})$ be as above. We have the following:*

1. *The map $\pi^{\mathscr{M}}_{\mathfrak{Z}} : \mathcal{M}_{\mathfrak{Z}} \to \mathcal{U}(\mathscr{M})_{\mathfrak{Z}} = \mathcal{U}(\mathscr{M})$ is an isomorphism of $\mathcal{R}_{\mathfrak{Z}} = R$-modules. If each M_n is an R-algebra then $\pi^{\mathscr{M}}_{\mathfrak{Z}}$ is an isomorphism of R-algebras.*
2. *If $\varphi = \{\varphi_i\}_{i\in I} : \mathscr{M} \to \mathscr{M}'$ (for $I \in \mathfrak{F}$) is a $\mathfrak{F}$-morphism of sequences of R-modules, then the map $\varphi^{\mathcal{U}} : \mathcal{U}(\mathscr{M}) \to \mathcal{U}(\mathscr{M}')$ from Proposition I.1.1 is the localization of the map*

$$\Phi_I := \prod_{i\in I} \varphi_i : \prod_{i\in I} M_i \to \prod_{i\in I} M'_i$$

   at $\mathfrak{Z}$.
3. *The functor $\mathscr{M} \mapsto \mathcal{U}(\mathscr{M})$ (from the category of sequences of R-modules, to the category of R-modules) is exact.*

Proof. As localization is exact, $\pi^{\mathscr{M}}_{\mathfrak{Z}}$ is surjective. Now arguing as in Proposition I.2.1, if $\frac{m}{s} \in \ker(\pi^{\mathscr{M}}_{\mathfrak{Z}})$ where $m = (m_1, m_2, \ldots) \in \mathcal{M}$, then $[m_i]_i = 0$ in $\mathcal{U}(\mathscr{M})$ and hence $m_i = 0$ for all $i \in I$ for some $I \in \mathfrak{F}$. But then $m \in \ker(\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_I)$ and so $\frac{m}{s} \in \ker(\mathcal{M}_{\mathfrak{Z}} \to \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_{I,\mathfrak{Z}} = \mathcal{M}_{\mathfrak{Z}}) = 0$. So indeed, $\ker(\pi^{\mathscr{M}}_{\mathfrak{Z}}) = 0$, and so $\pi^{\mathscr{M}}_{\mathfrak{Z}}$ is an isomorphism of R-modules. If each M_n is an R-algebra then $\pi^{\mathscr{M}}_{\mathfrak{Z}}$ is also a homomorphism of R-algebras, and thus is an isomorphism of R-algebras. This proves (1).

For (2), note that $\mathcal{M}_I := \prod_{i\in I} \varphi_i : \prod_{i\in I} M_i = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_I$, and so $\mathcal{M}_{I,\mathfrak{Z}} = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}_{I,\mathfrak{Z}} = \mathcal{M}_{\mathfrak{Z}}$, and similarly for $\mathcal{M}'_I := \prod_{i\in I} M'_i$. (2) then follows from localizing the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_I & \xrightarrow{\pi^{\mathscr{M}}} & \mathcal{U}(\mathscr{M}) \\ {\scriptstyle \Phi_I}\big\downarrow & & \big\downarrow{\scriptstyle \varphi^{\mathcal{U}}} \\ \mathcal{M}'_I & \xrightarrow[\pi^{\mathscr{M}'}]{} & \mathcal{U}(\mathscr{M}') \end{array}$$

Finally, (3) follows by noting that the functors $\{M_n\}_{n\geq 1} \mapsto \prod_{n\geq 1} M_n$ and $\mathcal{M} \mapsto \mathcal{M}_{\mathfrak{Z}}$ are both exact. $\square$

Chapter II

Ultrapatching

II.1 Patching systems

We are now ready to give the main patching construction. Fix a complete DVR $\mathcal{O}$ (which in practice will usually be the ring of integers in a finite extension of $\mathbb{Q}_\ell$) with uniformizer ϖ and *finite* residue field $\mathbb{F} = \mathcal{O}/\varpi$ of characteristic ℓ. Also fix some $d \geq 1$, and consider the ring:

$$S_\infty := \mathcal{O}[[t_1, \dots, t_d]].$$

And let $\mathfrak{n} = (t_1, \dots, t_d) \subseteq S_\infty$. Note that S_∞ is a compact topological ring, so that $S_\infty/\mathfrak{a}$ is finite for all open ideals $\mathfrak{a} \subseteq S_\infty$.

Fix a collection of ideals $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \dots \subseteq S_\infty$ with the following property:

$$\text{For all } n,\ \mathcal{I}_n \subseteq \mathfrak{n}, \text{ and for any open ideal } \mathfrak{a} \subseteq S_\infty,\ \mathcal{I}_n \subseteq \mathfrak{a} \text{ for all but finitely many } n. \qquad (\star)$$

It will often be important to work mod ϖ, so we will let $\overline{S}_\infty = S_\infty/\varpi = \mathbb{F}[[t_1, \dots, t_d]]$, and for each $n \geq 1$, let $\overline{\mathcal{I}}_n \subseteq \overline{S}_\infty$ be the image of $\mathcal{I}_n$.

In essentially all cases that arise in practice, the ideals $\mathcal{I}_n$ will have the following form:

Lemma II.1.1. *Pick a positive integer $d^\circ \leq d$ be an integer and assume that for each integer $n \geq 1$ and each $1 \leq j \leq d^\circ$ we are given an integer $e(n, j)$ with $e(n, j) \geq n$. Define ideals $\mathcal{I}_n \subseteq S_\infty$ by*

$$\mathcal{I}_n := \left((1 + t_1)^{\ell^{e(n,1)}} - 1, (1 + t_2)^{\ell^{e(n,2)}} - 1, \dots, (1 + t_{d^\circ})^{\ell^{e(n,d^\circ)}} - 1\right)$$

Then the collection of ideals $\{\mathcal{I}_n\}$ satisfies $(\star)$.

Proof. Clearly $(1 + t_j)^{\ell^{e(n,j)}} - 1 \subseteq (t_j)$ for all n and j, so it follows that $\mathcal{I}_n \subseteq \mathfrak{n}$ for all n.

Let $\mathfrak{a} \subseteq S_\infty$ be any open ideal. As $S_\infty/\mathfrak{a}$ is finite, and the group $1+\mathfrak{m}_{S_\infty}$ is pro-ℓ, the group $(1+\mathfrak{m}_{S_\infty})/\mathfrak{a} = \mathrm{im}(1+m_{S_\infty} \hookrightarrow S_\infty \twoheadrightarrow S_\infty/\mathfrak{a})$ is a finite ℓ-group. Since $1+t_i \in 1+\mathfrak{m}_{S_\infty}$ for all i, there is an integer $n_\mathfrak{a} \geq 0$ such that $(1+t_i)^{\ell^{n_\mathfrak{a}}} \equiv 1 \pmod{\mathfrak{a}}$ for all $i = 1, \dots, r$. Then for any $n \geq n_\mathfrak{a}$, $e(n,j) \geq n \geq k$ for all j, and so indeed $\mathcal{I}_n \subseteq \mathfrak{a}$. $\square$

The patching construction will take a sequence $\mathscr{M} = \{M_n\}_{n\geq 1}$ of finite type S_∞-modules satisfying certain properties, and produce a reasonably well behaved module $\mathscr{P}(\mathscr{M})$, which can be roughly though of as a "limit" of the M_n's.

We first make a precise definition of the sequences of S_∞-modules we will consider:

Definition II.1.2. Let $\mathscr{M} = \{M_n\}_{n\geq 1}$ be a sequence of finitely generated S_∞-modules.

- We say that $\mathscr{M}$ is a *weak patching system* if $\mathcal{I}_n \subseteq \mathrm{Ann}_{S_\infty} M_n$ for all n and the S_∞-ranks of the M_n's are uniformly bounded. If we further have $\varpi M_n = 0$ for all n, we say that $\mathscr{M}$ is a *residual weak patching system.*
- We say that $\mathscr{M}$ is MCM (resp. MCM residual) if $\mathscr{M}$ is a *nonzero* weak patching system (resp. residual weak patching system) and M_n is free over $S_\infty/\mathcal{I}_n$ (resp. $\overline{S}_\infty/\overline{\mathcal{I}}_n$) for all n.
- We say that a *patching system* is a triple $(\mathscr{M}, M_0, \{\alpha_n\}_{n\geq 1})$ consisting of a weak patching system $\mathscr{M}$, a finite $\mathcal{O}$-module M_0 and a family of $\mathcal{O}$-module isomorphisms $\alpha_n : M_n/\mathfrak{n} \to M_0$.
- By slight abuse of notation, if $\mathscr{M}$ is a weak patching system (resp. residual weak patching system) we say that $\mathscr{M}$ is a *patching system* (resp. *residual patching system*) if there is a finite type $\mathcal{O}$-module M_0 and isomorphisms $\alpha_n : M_n/\mathfrak{n} \cong M_0$ making $(\mathscr{M}, M_0, \{\alpha_n\}_{n\geq 1})$ into a patching system. In this case, we say $\mathscr{M}$ is a *patching system over* M_0.

Furthermore, assume that $\mathscr{R} = \{R_n\}_{n\geq 1}$ is a sequence of finite local S_∞-*algebras.*

- We say that $\mathscr{R} = \{R_n\}_{n\geq 1}$ is a *weak (residual) patching algebra*, if it is a weak (residual) patching system.
- We say that a *patching algebra* is a triple $(\mathscr{R}, R_0, \{\alpha_n\}_{n\geq 1})$ consisting of a weak patching algebra $\mathscr{R}$, a finite $\mathcal{O}$-algebra R_0 and a family of $\mathcal{O}$-algebra isomorphisms $\alpha_n : R_n/\mathfrak{n} \to R_0$. Again by abuse of notation we also refer to $\mathscr{R}$ is a patching algebra over R_0.
- If M_n is an R_n-module (viewed as an S_∞-module via the S_∞-algebra structure on R_n) for all n we say that $\mathscr{M} = \{M_n\}_{n\geq 1}$ is a *(weak, residual) patching $\mathscr{R}$-module* if it is a (weak, residual) patching system.
- If $\mathscr{R}$ is a patching algebra over R_0 and M_0 is a finitely generated R_0-module, we say that $\mathscr{M}$ is a *patching $\mathscr{R}$-module over* M_0 if it is a patching system over M_0 and for each $n \geq 1$ the R_0-module structure on M_0 is induced by the R_n-module structure on M_n and the isomorphisms $R_n/\mathfrak{n} \cong R_0$ and $M_n/\mathfrak{n} \cong M_0$.

Let $\mathfrak{wP}$ be the category of weak patching systems, with the obvious notion of morphism. Similarly, let $\overline{\mathfrak{wP}}$ be the category of residual weak patching systems. Note that these are both abelian categories.

Also let $\mathbf{Alg}_{\mathfrak{wP}}$ be the category of weak patching algebras, and for any $\mathscr{R} \in \mathbf{Alg}_{\mathfrak{wP}}$, define $\mathfrak{wP}_\mathscr{R}$

to be the (abelian) category of weak patching $\mathscr{R}$-modules. Define $\overline{\mathbf{Alg}_{\mathfrak{wP}}}$ and $\overline{\mathfrak{wP}}_{\mathscr{R}}$ similarly

From now on, for any weak patching system $\mathscr{M}$ and any ideal $J \subseteq S_\infty$, we will write $\mathscr{M}/J := \{M_n/J\}_{n\geq 1}$.

If $\mathfrak{a} \subseteq S_\infty$ is open, note that each $M_n/\mathfrak{a}$ is a finite type $S_\infty/\mathfrak{a}$-module and the ranks of the $M_n/\mathfrak{a}$'s are bounded. As $S_\infty/\mathfrak{a}$ is finite, it follows that each $M_n/\mathfrak{a}$ is finite, and the cardinalities of the $M_n/\mathfrak{a}$'s are bounded. Proposition I.1.3 then implies that $\mathcal{U}(\mathscr{M}/\mathfrak{a}) \cong M_i/\mathfrak{a}$ as $S_\infty/\mathfrak{a}$-modules (and hence as S_∞-modules) for $\mathfrak{F}$-many i.

Now for any $\mathfrak{a}' \subseteq \mathfrak{a}$, the surjections $M_n/\mathfrak{a}' \twoheadrightarrow M_n/\mathfrak{a}$ induce a surjection $\mathcal{U}(\mathscr{M}/\mathfrak{a}') \twoheadrightarrow \mathcal{U}(\mathscr{M}/\mathfrak{a})$. In fact, by the exactness of $\mathcal{U}(-)$, this surjection induces an isomorphism $\mathcal{U}(\mathscr{M}/\mathfrak{a}')/\mathfrak{a} \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$ of S_∞-modules (or S_∞-algebras if $\mathscr{M}$ is a weak patching algebra).

Thus the $\mathcal{U}(\mathscr{M}/\mathfrak{a})$'s form an inverse system, and so we may make the following definition:

Definition II.1.3. For any weak patching system $\mathscr{M}$ define:

$$\mathscr{P}(\mathscr{M}) := \varprojlim_{\mathfrak{a}} \mathcal{U}(\mathscr{M}/\mathfrak{a}) \,.$$

As $\mathcal{U}(-)$ is functorial, it follows that $\mathscr{P}$ defines an additive functor $\mathscr{P} : \mathfrak{wP} \to \mathbf{Mod}_{S_\infty}$. For a morphism $f : \mathscr{M} \to \mathscr{N}$ of weak patching systems, let $f^{\mathscr{P}} : \mathscr{P}(\mathscr{M}) \to \mathscr{P}(\mathscr{N})$ denote the induced map.

Note that if $\mathscr{R}$ is a weak patching algebra then $\mathscr{P}(\mathscr{R})$ is an S_∞-algebra, and if $\mathscr{M}$ is a weak patching $\mathscr{R}$-module then $\mathscr{P}(\mathscr{M})$ is a $\mathscr{P}(\mathscr{R})$-module (with its S_∞-module structure induced from the S_∞-algebra structure on $\mathscr{P}(\mathscr{R})$). It follows that $\mathscr{P}$ induces functors $\mathbf{Alg}_{\mathfrak{wP}} \to \mathbf{Alg}_{S_\infty}$ and $\mathfrak{wP}_{\mathscr{R}} \to \mathbf{Mod}_{\mathscr{P}(\mathscr{R})}$, which we will also denote by $\mathscr{P}$.

II.2 Unframed patching systems

In Kisin's formulation of the patching method, the rings R_n and modules M_n must be modified with the addition of "framing variables" in order to make the patching argument work properly.[1] In this section, we briefly describe this modification.

Remark. As a small point about notation, typically the notation R_n and M_n are used for the unframed versions of these objects, and the notations $R_n^\square$ and $M_n^\square$ are used for the framed versions. Outside of this section, the unframed versions of these objects will rarely if ever appear explicitly, and so it makes more sense to allow R_n and M_n to represent the. For lack of a better notation, we will sometimes use a superscript of $^\circ$ to denote the unframed version of a framed object. If we

[1]The primary reason for this is that while the global Galois deformation functors are usually representable by the rings R_n, the local deformation functors usually are not representable, and so if local deformation rings are to be used explicitly in the patching argument, the local deformation functors must be modified, and hence the global deformation functors must be modified to account for this.

start with a unframed object, we will still use a superscript of $^\square$ to represent the framed version of that object. We apologize for not being able to think of a better choice of notation.

Fix an integer $d^\circ \le d$ and let $S_\infty^\circ = \mathcal{O}[[t_1, \dots, t_{d^\circ}]]$, treated as a subring of S_∞. Assume that the ideals $\mathcal{I}_n \subseteq S_\infty$ all have the form $\mathcal{I}_n = \mathcal{I}_n^\circ S_\infty$ for some ideals $\mathcal{I}_n^\circ \subseteq S_\infty^\circ$. Let $\mathfrak{n}^\circ = (t_1, \dots, t_{d^\circ})$.

We will define an *unframed* weak patching system $\mathscr{M}^\circ = \{M_n^\circ\}_{n\ge1}$ to be a sequence of finitely generated S_∞°-modules for which $\mathcal{I}_n^\circ \subseteq \operatorname{Ann}_{S_\infty^\circ} M_n^\circ$ and the S_∞°-ranks of the M_n°'s are uniformly bounded. We will define the obvious unframed analogues of all of the concepts listed in Definition II.1.2. Let $\mathfrak{wP}^\circ$ and $\mathbf{Alg}_{\mathfrak{wP}^\circ}$ be the categories of unframed weak patching systems, and unframed weak patching algebras, respectively.

We will again use the notation $\mathscr{P}$ to denote the functor $\mathscr{P} : \mathfrak{wP}^\circ \to \mathbf{Mod}_{S_\infty^\circ}$ given by

$$\mathscr{P}(\mathscr{M}^\circ) = \varprojlim_{\mathfrak{a}} \mathcal{U}(\mathscr{M}^\circ/\mathfrak{a}) .$$

Now treat S_∞ as an S_∞°-algebra via the inclusion $S_\infty^\circ \hookrightarrow S_\infty$, and treat S_∞° as an S_∞-algebra via the quotient map $S_\infty \twoheadrightarrow S_\infty/(t_{d^\circ+1}, \dots, t_d) = S_\infty^\circ$. We can then define functors $(-)^\square : \mathfrak{wP}^\circ \to \mathfrak{wP}$ and $(-)^\circ : \mathfrak{wP} \to \mathfrak{wP}^\circ$ via:

$$\begin{aligned}(\mathscr{M}^\circ)^\square &= \{M_n^\circ \otimes_{S_\infty^\circ} S_\infty\}_{n\ge1} = \{M_n^\circ \otimes_{\mathcal{O}} \mathcal{O}[[t_{d^\circ+1}, \dots, t_d]]\}_{n\ge1} \\ (\mathscr{M})^\circ &= \{M_n \otimes_{S_\infty} S_\infty^\circ\}_{n\ge1} = \{M_n/(t_{d^\circ+1}, \dots, t_d)\}_{n\ge1}.\end{aligned}$$

And note that the following basic properties are automatic from the definitions :

Proposition II.2.1. *Take any $\mathscr{M}^\circ = \{M_n^\circ\}_{n\ge1} \in \mathfrak{wP}^\circ$ and write $\mathscr{M}^\square := (\mathscr{M}^\circ)^\square \in wP$. Also let $\mathscr{R}^\circ = \{R_n^\circ\} \in \mathbf{Alg}_{\mathfrak{wP}^\circ}$ and $\mathscr{R}^\square = (\mathscr{R}^\circ)^\square \in \mathbf{Alg}_{\mathfrak{wP}}$. Then we have*

1. *If $\mathscr{M}^\circ$ is a unframed patching system over M_0, then $\mathscr{M}^\square$ is a patching system over M_0. The analogous statement holds for $\mathscr{R}^\circ$.*
2. *If $\mathscr{M}^\circ$ satisfies (the unframed version of) one of the additional properties listed in Definition II.1.2 (e.g. MCM, residual, patching algebra, etc.) then $\mathscr{M}^\square$ satisfies the corresponding property. The analogous statements holds for $\mathscr{R}^\circ$.*
3. *$\mathscr{P}(\mathscr{M}^\square) = \mathscr{P}(\mathscr{M}^\circ) \otimes_{S_\infty^\circ} S_\infty$ and $\mathscr{P}(\mathscr{M}^\circ) = \mathscr{P}(\mathscr{M}^\square) \otimes_{S_\infty} S_\infty^\circ$, and the same holds for $\mathscr{R}^\circ$.*
4. *If $\mathscr{M}^\circ$ is a unframed weak patching $\mathscr{R}^\circ$-module, then $\mathscr{M}^\square = \mathscr{M}^\circ \otimes_{\mathscr{R}^\circ} \mathscr{R}^\square = \{M_n^\circ \otimes_{R_n^\circ} R_n^\square\}_{n\ge1}$*

This allows us to translate statements about unframed objects to statements about framed objects, without losing any significant information.

Remark. When patching is used in practice, and R_n and $R_n^\square$ represent the unframed and unframed versions of a global Galois deformation ring, one typically has a canonical embedding $R_n \hookrightarrow R_n^\square$, but only a *noncanonical* choice of isomorphism $R_n^\square \cong R_n[[t_{d^\circ+1}, \dots, t_d]]$. In particular, this means that the maps $M_n^\square \twoheadrightarrow M_n$ and $R_n^\square \twoheadrightarrow R_n$ implied by the construction of the $(-)^\circ$ are noncanonical. This makes the treatment of framing we are using in this note somewhat nonstandard.

The approach taken in the note essentially amounts to fixing for all time isomorphisms $R_n^\square \cong R_n[[t_{d^\circ+1},\dots,t_d]]$ for each n, and hence fixing surjections $R_n^\square \twoheadrightarrow R_n$. While this choice is certainly noncanonical, making such a choice does not cause any issues with any of the standard patching arguments, so doing this is essentially harmless. (Also note that Proposition II.2.1(4) guarantees that our definition of $\mathscr{M}^\square$ lines up with the usual definition.)

The reason we are taking this approach is that part (1) of Proposition II.2.1 ensures that an unframed patching system over M_0 produces a framed patching system, still over M_0, as we have $M_n^\square/\mathfrak{n} \cong M_n^\circ/\mathfrak{n}^\circ = M_0$. In the standard approach, we would not be able to make such a statement, as we would not have a map $M_n^\square \twoheadrightarrow M_n^\circ$, so instead of being able to say that $\mathscr{M}^\square$ is a patching system over M_0, we would only be able to say (after some appropriate modification of our definition) that $\mathscr{M}^\square$ is a patching system over some framed object $M_0^\square$, which would only be finite over $\mathcal{O}[[t_{d^\circ+1},\dots,t_d]]$ not $\mathcal{O}$. One would then deduce results about $R_0^\square$ and $M_0^\square$ via patching, and then deduce the corresponding results about R_0 and M_0 from this. While setting up the theory in this way would not require any substantial changes to our arguments or results, it would introduce the extra baggage of having to worry about both the unframed objects M_0 and R_0 and their framed counterparts $M_0^\square$ and $R_0^\square$.

Ignoring the minor issue of fixing these noncanonical isomorphism, our approach is somewhat conceptually cleaner in that it is usually not necessary to remember that the objects R_n and M_n were originally constructed from unframed objects, and it is not necessary to distinguish the variables $t_1,\dots,t_{d^\circ}$ (coming from the "Taylor–Wiles primes") from the variables $t_{d^\circ+1},\dots,t_d$ (coming from the framing) in the definition of $S_\infty = \mathcal{O}[[t_1,\dots,t_d]]$.

II.3 A module theoretic interpretation

One could also give the following alternative construction of objects considered in Section II.1.

Define the S_∞-algebra

$$\mathfrak{S} = \prod_{n=1}^{\infty} S_\infty/\mathcal{I}_n$$

and note every weak patching system is naturally an $\mathfrak{S}$-module, and in fact is finitely generated (by the assumption that the S_∞-ranks of the M_n's were uniformly bounded). Moreover, it is not hard to see that every finitely generated $\mathfrak{S}$-module arises from a weak patching system in this way, and that in fact $\mathfrak{wP}$ is equivalent to the category of finitely generated $\mathfrak{S}$-modules.

Similarly if

$$\overline{\mathfrak{S}} = \mathfrak{S}/\varpi = \prod_{n=1}^{\infty} \overline{S}_\infty/\overline{\mathcal{I}}_n,$$

then $\overline{\mathfrak{wP}}$ is equivalent to the category of finitely generated $\overline{\mathfrak{S}}$-modules.

Now for any open ideal $\mathfrak{a}$ of S_∞, we have $(S_\infty/\mathcal{I}_n)/\mathfrak{a} \cong S_\infty/\mathfrak{a}$ for all but finitely many n, by $(\star)$.

It follows by Propositions I.2.1 and I.2.2 that there is a prime ideal $\mathfrak{Z}_\mathfrak{a} \subseteq \mathfrak{S}/\mathfrak{a} = \prod_{n=1}^{\infty}(\overline{S}_\infty/\overline{\mathcal{I}}_n)/\mathfrak{a}$ with the property that if $\mathscr{M}$ is any weak patching system (regarded as a $\mathfrak{S}$-module) then there is a (functorial) isomorphism $(\mathscr{M}/\mathfrak{a})_{\mathfrak{Z}_\mathfrak{a}} \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$.

Moreover, Proposition I.2.1 implies that the collection of ideals $\{\mathfrak{Z}_\mathfrak{a}\}_{\mathfrak{a}\subseteq S_\infty}$ is compatible with the transition maps $\mathfrak{S}/\mathfrak{a}' \to \mathfrak{S}/\mathfrak{a}$ (in the sense that $\mathfrak{Z}_{\mathfrak{a}'}$ is the preimage of $\mathfrak{Z}_\mathfrak{a}$) and so one can define a prime ideal $\mathfrak{Z} = \varprojlim_\mathfrak{a} \mathfrak{Z}_\mathfrak{a} \subseteq \mathfrak{S}$ with the property that $(\mathscr{M}/\mathfrak{a})_{\mathfrak{Z}_\mathfrak{a}} \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$ for any $\mathscr{M} \in \mathfrak{wP}$ and any open ideal $\mathfrak{a} \subseteq S_\infty$.

We may thus define $\mathscr{P} : \mathfrak{wP} \to \mathbf{Mod}_{S_\infty}$ by

$$\mathscr{P}(\mathscr{M}) = \varprojlim_\mathfrak{a}(\mathscr{M}/\mathfrak{a})_\mathfrak{Z},$$

which clearly agrees with our definition above.

We usually will use these two constructions interchangeably in the following discussions.

II.4 Basic properties of patching

In this section, we'll establish some basic properties of the functor $\mathscr{P}$.

First, for any finitely generated S_∞-module M, we will define the *constant* patching system $\underline{M}$ to be $\underline{M} = \{M/\mathcal{I}_n\}_{n\geq 1}$. The next lemma should justify this choice of terminology:

Lemma II.4.1. *For any finitely generated S_∞-module M, there is a natural isomorphism $\mathscr{P}(\underline{M}) \cong M$.*

Proof. Since $M/\mathfrak{a}$ is finite for all open $\mathfrak{a} \subseteq S_\infty$ Proposition I.1.3 gives a natural isomorphism $(M/\mathfrak{a})^\mathcal{U} \cong M/\mathfrak{a}$ of $S_\infty/\mathfrak{a}$-modules. Thus

$$\mathscr{P}(\underline{M}) = \varprojlim_\mathfrak{a} \mathcal{U}\left(\underline{M/\mathfrak{a}}\right) = \varprojlim_\mathfrak{a}(M/\mathfrak{a})^\mathcal{U} \cong \varprojlim_\mathfrak{a} M/\mathfrak{a} \cong M$$

as any finite type S_∞-module is complete. □

In turns out that $\mathscr{P}$ is a reasonably well behaved functor. In fact:

Proposition II.4.2. $\mathscr{P} : \mathfrak{wP} \to \mathbf{Mod}^{\mathrm{fg}}_{S_\infty}$ *is a right-exact functor.*

Proof. Let **Ab** be the category of abelian groups. For any directed index set I, let $\mathbf{finAb}^I$ be the category of inverse systems of *finite* abelian groups indexed by I. We claim that if I is countable, the functor $\varprojlim : \mathbf{finAb}^I \to \mathbf{Ab}$ is exact.

By [Sta19, Lemma 0598], it suffices to show that any $(A_i, f_{ji} : A_j \to A_i) \in \mathbf{finAb}^I$ satisfies the Mittag-Leffler condition: For any $i \in I$ there is a $j \geq i$ for which $\mathrm{im}(f_{ki}) = \mathrm{im}(f_{ji})$ for all $k \geq j$.

But as A_i is finite, it has only finitely many subgroups and so the collection $\{\mathrm{im}(f_{ji})\}_{j\geq i}$ of subgroups of A_i must have some minimal member, $\mathrm{im}(f_{ji})$, under inclusion. Then for any $k \geq j$, $\mathrm{im}(f_{ki}) = \mathrm{im}(f_{ji} \circ f_{kj}) \subseteq \mathrm{im}(f_{ji})$ and hence $\mathrm{im}(f_{ki}) = \mathrm{im}(f_{ji})$. So indeed every object of $\mathbf{finAb}^I$ satisfies the Mittag-Leffler condition, and so $\varprojlim$ is exact.

Now assume $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are weak patching systems, and we have an exact sequence

$$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$$

Then for any $\mathfrak{a} \subseteq S_\infty$, $\mathscr{A}/\mathfrak{a} \to \mathscr{B}/\mathfrak{a} \to \mathscr{C}/\mathfrak{a} \to 0$ is exact, so by the exactness of $\mathcal{U}(-)$ we get the exact sequence

$$\mathcal{U}(\mathscr{A}/\mathfrak{a}) \to \mathcal{U}(\mathscr{B}/\mathfrak{a}) \to \mathcal{U}(\mathscr{C}/\mathfrak{a}) \to 0.$$

Thus we have a exact sequence of complexes

$$\big(\mathcal{U}(\mathscr{A}/\mathfrak{a})\big)_{\mathfrak{a}} \to \big(\mathcal{U}(\mathscr{B}/\mathfrak{a})\big)_{\mathfrak{a}} \to \big(\mathcal{U}(\mathscr{C}/\mathfrak{a})\big)_{\mathfrak{a}} \to 0$$

But now as $\mathcal{U}(\mathscr{A}/\mathfrak{a})$, $\mathcal{U}(\mathscr{B}/\mathfrak{a})$ and $\mathcal{U}(\mathscr{C}/\mathfrak{a})$ are all finite, and S_∞ has only countably many open ideals, the above argument shows that taking inverse limits preserves exactness, and so indeed

$$\mathscr{P}(\mathscr{A}) \to \mathscr{P}(\mathscr{B}) \to \mathscr{P}(\mathscr{C}) \to 0$$

is exact. □

Remark. Note that in general, $\mathscr{P}$ is not left-exact. Indeed, assume that the ideals $\mathcal{I}_n$ are chosen so that $S_\infty/\mathcal{I}_n$ is ϖ-torsion free for all n (which is the case for the ideals considered in Lemma II.1.1). Then define $\varphi : \underline{S_\infty} \to \underline{S_\infty}$ by $\varphi_n(x) = \varpi^n x$. It is clear that φ is injective, but we can also see that $\varphi^{\mathscr{P}} : \mathscr{P}(\underline{S_\infty}) \to \mathscr{P}(\underline{S_\infty})$ is the zero map (since for any $\mathfrak{a}$, $\varphi_{n,\mathfrak{a}} : S_\infty/\mathfrak{a} \to S_\infty/\mathfrak{a}$ is the zero map for all but finitely many n). Thus $\mathscr{P}$ cannot be left-exact.

Proposition II.4.3. *For any $\mathscr{M} \in \mathfrak{wP}$, $\mathscr{P}(\mathscr{M})$ is a finitely generated S_∞-module. That is, $\mathscr{P}$ is a functor $\mathfrak{wP} \to \mathbf{Mod}^{\mathrm{fg}}_{S_\infty}$.*

Proof. Let $\mathscr{M} = \{M_n\}_{n\geq 1}$. As the S_∞-ranks of of the M_n's are bounded, there is some $N \geq 1$ such that there exist a family of surjections $\varphi_n : (S_\infty/\mathcal{I}_n)^N \twoheadrightarrow M_n$ for all $n \geq 1$. The φ_n's combine to form a surjection $\underline{S_\infty^N} \twoheadrightarrow \mathscr{M}$ in $\mathfrak{wP}$. By Proposition II.4.2 this gives a surjection $\varphi^{\mathscr{P}} : S_\infty^N \twoheadrightarrow \mathscr{P}(\mathscr{M})$ of S_∞-modules, and so $\mathscr{P}(\mathscr{M})$ is a finitely generated S_∞-module. □

Now Proposition II.4.2, and Definition II.1.2 easily imply the following basic properties:

Proposition II.4.4. *If $\mathscr{M} = \{M_n\}_{n\geq 1} \in \mathfrak{wP}$ then:*

1. *For any ideal $J \subseteq S_\infty$, $\mathscr{P}(\mathscr{M}/J) \cong \mathscr{P}(\mathscr{M})/J$*
2. *For any open ideal $\mathfrak{a} \subseteq S_\infty$, $\mathscr{P}(\mathscr{M})/\mathfrak{a} \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$.*

3. *If $\mathscr{M}$ is a weak patching system over M_0, then $\mathscr{P}(\mathscr{M})/\mathfrak{n} \cong M_0$.*
4. *If $\mathscr{M}$ is MCM, then $\mathscr{P}(\mathscr{M})$ is a finite free S_∞-module.*

Proof. Part (1) simply follows from Proposition II.4.2 applied to the exact sequence

$$0 \to J\mathscr{M} \to \mathscr{M} \to \mathscr{M}/J \to 0.$$

So now if $\mathfrak{a} \subseteq S_\infty$ is open, we have $(\mathscr{M}/\mathfrak{a})/\mathfrak{a}' \cong \mathscr{M}/\mathfrak{a}$ for all $\mathfrak{a}' \subseteq \mathfrak{a}$ and so $\mathcal{U}((\mathscr{M}/\mathfrak{a})/\mathfrak{a}') \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$. Thus we have

$$\mathscr{P}(\mathscr{M})/\mathfrak{a} \cong \mathscr{P}(\mathscr{M}/\mathfrak{a}) = \varprojlim_{\mathfrak{a}'} \mathcal{U}((\mathscr{M}/\mathfrak{a})/\mathfrak{a}') = \varprojlim_{\mathfrak{a}' \subseteq \mathfrak{a}} \mathcal{U}((\mathscr{M}/\mathfrak{a})/\mathfrak{a}') \cong \varprojlim_{\mathfrak{a}' \subseteq \mathfrak{a}} \mathcal{U}(\mathscr{M}/\mathfrak{a}) \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})\,,$$

proving (2).

Now assume that $\mathscr{M}$ is a weak patching system over M_0. Letting S_∞ act on M_0 via $S_\infty \twoheadrightarrow S_\infty/\mathfrak{n} = \mathcal{O}$ we see that for all $n \geq 1$, $M_0/\mathcal{I}_n = M_0 = M_n/\mathfrak{n}$ (as $\mathcal{I}_n \subseteq \mathfrak{n} \subseteq \mathrm{Ann}_{S_\infty}(M_0)$) and so $\mathscr{M}/\mathfrak{n} = \underline{M_0}$. Thus by part (1) and Lemma II.4.1

$$\mathscr{P}(\mathscr{M})/\mathfrak{n} \cong \mathscr{P}(\mathscr{M}/\mathfrak{n}) \cong \mathscr{P}\left(\underline{M_0}\right) \cong M_0,$$

proving (3).

Lastly, assume that $\mathscr{M}$ is MCM. Then for all $n \geq 1$, $M_n \cong (S_\infty/\mathcal{I}_n)^{r_n}$ for some r_n. As the r_n's are bounded, there is some r such that $r_i = r$, and hence $M_i \cong (S_\infty/\mathcal{I}_i)^r$, for $\mathfrak{F}$-many i.

Define an $\mathfrak{F}$-morphism $\varphi : \underline{S_\infty^r} \to \mathscr{M}$ by letting $\varphi_i : S_\infty^r \twoheadrightarrow (S_\infty/\mathcal{I}_i)^r \cong M_i$ for $\mathfrak{F}$-many i. Then for any open $\mathfrak{a} \subseteq S_\infty$, $\overline{\varphi_{i,\mathfrak{a}}} : S_\infty^r/\mathfrak{a} \to M_i/\mathfrak{a}$ is an isomorphism for $\mathfrak{F}$-many i, and so φ induces an isomorphism $\mathcal{U}\Big(\underline{(S_\infty/\mathfrak{a})^r}\Big) \cong \mathcal{U}(\mathscr{M}/\mathfrak{a})$ for all $\mathfrak{a}$, and thus an isomorphism $\varphi^{\mathscr{P}} : S_\infty^r = \mathscr{P}\left(\underline{S_\infty^r}\right) \to \mathscr{P}(\mathscr{M})$ is an isomorphism. So indeed, $\mathscr{P}(\mathscr{M})$ is a finite type, free S_∞-module, proving (4). $\square$

The following simple consequence of Proposition II.4.4 is central to most applications of this theory:

Corollary II.4.5. *If $\mathscr{R}$ is a weak patching algebra and $\mathscr{M}$ is an MCM weak patching $\mathscr{R}$-module, then:*

1. *The homomorphism $S_\infty \to \mathscr{P}(\mathscr{R})$ inducing the S_∞-algebra structure on $\mathscr{P}(\mathscr{R})$ is injective;*
2. *The Krull dimension of $\mathscr{P}(\mathscr{R})$ is $d+1$ $(= \dim S_\infty)$;*
3. *$\mathscr{P}(\mathscr{M})$ is a maximal Cohen–Macaulay module over $\mathscr{P}(\mathscr{R})$ and $(\varpi, t_1, \ldots, t_d) \subseteq S_\infty \subseteq \mathscr{P}(\mathscr{R})$ is a regular sequence for $\mathscr{P}(\mathscr{M})$.*

Proof. For (1), the map $\iota : S_\infty \to \mathscr{P}(\mathscr{R})$ induces the S_∞-module structure on $\mathscr{P}(\mathscr{M})$, which is faithful by Proposition II.4.4(4), and so ι must be injective.

It follows from (1) that $\dim \mathscr{P}(\mathscr{R}) \geq \dim S_\infty$. But as $\mathscr{P}(\mathscr{R})$ is finite over S_∞, it also follows that $\dim \mathscr{P}(\mathscr{R}) \leq \dim S_\infty$. This proves (2).

Now assume that $\mathscr{M}$ is MCM. By Proposition II.4.4(4), $\mathscr{P}(\mathscr{M})$ is finite free over $\iota(S_\infty) \cong S_\infty$ and so $\mathscr{P}(\mathscr{M})$ is indeed Cohen–Macaulay of dimension $d+1 = \dim S_\infty$. In particular $(t_1, \dots, t_d, \varpi) \subseteq S_\infty \subseteq \mathscr{P}(\mathscr{R})$ is a regular sequence. $\square$

II.5 Covers of Patching Algebras

In the classical setup of Taylor–Wiles–Kisin patching, one considers a patching algebra $\mathscr{R} = \{R_n\}_{n\ge1}$, where the R_n's are all taken to be quotients of a fixed ring R_∞. We thus make a the following definition:

Definition II.5.1. If $\mathscr{R} = \{R_n\}_{n\ge1}$ is a weak patching algebra we say that a *cover* $(R_\infty, \{\varphi_n\})$ of $\mathscr{R}$ is:

- A complete, local ring R_∞, which is topologically finitely generated as an $\mathcal{O}$-algebra of dimension $d+1$ $(= \dim S_\infty)$ together with:
- For each n, a continuous, surjective $\mathcal{O}$-algebra homomorphism $\varphi_n : R_\infty \twoheadrightarrow R_n$.

We say that $(R_\infty, \{\varphi_n\})$ is a *CM cover* if the ring R_∞ is Cohen–Macaulay, and we say that $(R_\infty, \{\varphi_n\})$ is a *regular cover* if the ring R_∞ is regular (for example, if $R_\infty \cong \mathcal{O}[[x_1, \dots, x_d]]$). We also say that $(R_\infty, \{\varphi_n\})$ is an *irreducible cover* if R_∞ is a domain.

We will often use R_∞ to denote the cover $(R_\infty, \{\varphi_n\})$.

Remark. We will often consider covers R_∞ of patching algebras $\mathscr{R} = \{R_n\}_{n\ge1}$ over a ring R_0. In such a situation one gets an infinite family of surjective morphisms $R_\infty \xrightarrow{\varphi_n} R_n \twoheadrightarrow R_n/\mathfrak{n} \xrightarrow{\sim} R_0$. In general, we will make no assumptions that these maps are in any way compatible with each other. Indeed, the lack of any compatibilities between the maps $\varphi_n : R_\infty \twoheadrightarrow R_n$ is part of reason why the pigeonhole principle argument (in the classical formulation) or the ultraproduct formalism (in the approach used here) is necessary for patching arguments.

Note that:

Lemma II.5.2. *If $(R_\infty, \{\varphi_n\})$ is a cover of a weak patching algebra $\mathscr{R}$, then the φ_n's induce a natural continuous surjection $\varphi_\infty : R_\infty \twoheadrightarrow \mathscr{P}(\mathscr{R})$.*

Proof. The φ_n's induce a continuous map $\Phi = \prod_{n\ge1} \varphi_n : R_\infty \to \prod_{n\ge1} R_n$, and thus induce continuous maps

$$\Phi_\mathfrak{a} : R_\infty \xrightarrow{\Phi} \prod_{n\ge1} R_n \twoheadrightarrow \prod_{n\ge1} (R_n/\mathfrak{a}) \twoheadrightarrow \mathcal{U}(\mathscr{R}/\mathfrak{a})$$

for all open $\mathfrak{a} \subseteq S_\infty$, and thus they indeed induce a continuous map

$$\varphi_\infty = (\Phi_\mathfrak{a})_\mathfrak{a} : R_\infty \to \varprojlim_{\mathfrak{a}} \mathcal{U}(\mathscr{R}/\mathfrak{a}) = \mathscr{P}(\mathscr{R}).$$

We now claim that each $\Phi_\mathfrak{a}$ is surjective. As each map $R_\infty \xrightarrow{\varphi_n} R_n \twoheadrightarrow R_n/\mathfrak{a}$ is continuous, we may give each $R_n/\mathfrak{a}$ the structure of a continuous R_∞-algebra. Then the map $\Phi_\mathfrak{a} : R_\infty \to \mathcal{U}(\mathscr{R}/\mathfrak{a})$ defines the continuous R_∞-algebra structure on $\mathcal{U}(\mathscr{R}/\mathfrak{a})$ from Proposition I.1.1. By Proposition I.1.3, $\mathcal{U}(\mathscr{R}/\mathfrak{a}) \cong R_i/\mathfrak{a}$ as R-algebras for $\mathfrak{F}$-many i. But for any such i, the map $R_\infty \xrightarrow{\varphi_i} R_i \twoheadrightarrow R_i/\mathfrak{a}$ defining the R_∞-algebra structure is surjective, and so $\Phi_\mathfrak{a} : R_\infty \to \mathcal{U}(\mathscr{R}/\mathfrak{a})$ must indeed be surjective.

It follows that $\varphi_\infty(R_\infty) \subseteq \mathscr{P}(\mathscr{R})$ is *dense*. But now as R_∞ is topologically finitely generated over $\mathcal{O}$, it is compact, and so $\varphi_\infty(R_\infty)$ is also closed in $\mathscr{P}(\mathscr{R})$. Therefore φ_∞ is indeed surjective. □

We will say that the cover R_∞ is *minimal* if φ_∞ is an isomorphism.

From now on, if $\mathscr{M}$ is weak patching $\mathscr{R}$ module and R_∞ is a cover of $\mathscr{R}$, we will treat $\mathscr{P}(\mathscr{M})$ as a R_∞-module via the map $\varphi_\infty : R_\infty \twoheadrightarrow \mathscr{P}(\mathscr{R})$ from Lemma II.5.2.

Lemma II.5.2 and Corollary II.4.5 give the following useful result:

Corollary II.5.3. *If $\mathscr{R}$ is a weak patching algebra with a cover R_∞, and $\mathscr{M}$ is any* MCM *weak patching $\mathscr{R}$-module, then $\mathscr{P}(\mathscr{M})$ is maximal Cohen–Macaulay over R_∞.*

Proof. By Lemma II.5.2, $\mathscr{P}(\mathscr{R})$ may be thought of as a quotient of R_∞. From the definition of Cohen–Macaulay modules, if $f : A \twoheadrightarrow B$ is any surjective map of rings, and M is a B-module, then M is Cohen–Macaulay over B if and only if it is Cohen–Macaulay over A. Thus by Corollary II.4.5, $\mathscr{P}(\mathscr{M})$ is Cohen–Macaulay over R_∞.

Furthermore, by Corollary II.4.5 and Definition II.5.1, we have $\dim R_\infty = d + 1 = \dim \mathscr{P}(\mathscr{R}) = \dim \mathscr{P}(\mathscr{M})$, so $\mathscr{P}(\mathscr{M})$ is *maximal* Cohen–Macaulay over R_∞. □

For the remainder of this section we will consider a finite $\mathcal{O}$-algebra R_0 and a nonzero R_0-module R_0, and we will assume that M_0 be a nonzero R_0-module, which is finite type and free over $\mathcal{O}$. One of the primary goals of patching is to deduce information about the R_0-module structure of M_0 by considering patching systems over $\mathscr{R}$ and $\mathscr{M}$.

With this in mind assume that we are given a triple $(R_\infty, \mathscr{R}, \mathscr{M})$ where:

- $\mathscr{R} = \{R_n\}_{n\geq 1}$ is a patching algebra over R_0;
- $\mathscr{M} = \{M_n\}_{n\geq 1}$ is a *MCM* patching $\mathscr{R}$-module over M_0;
- R_∞ is a cover of $\mathscr{R}$.

We will be primarily interested in triples $(R_\infty, \mathscr{R}, \mathscr{M})$ satisfying the following property:

$$R_\infty \text{ acts faithfully on the module } \mathscr{P}(\mathscr{M}) \tag{Supp}$$

In general, it can be quite difficult to check if (Supp) if we do not have much direct information about R_0 and M_0, and this presents one of the major challenges in the study of automorphy lifting. For now, we will not attempt to give general strategies for testing (Supp) and instead only note the following special case:

Lemma II.5.4. *If R_∞ is a domain, then $(R_\infty, \mathscr{R}, \mathscr{M})$ always satisfies (Supp).*

Proof. This follows immediately from standard properties of maximal Cohen–Macaulay modules, as $\mathscr{P}(\mathscr{M})$ is maximal Cohen–Macaulay over R_∞.

Alternatively, assume that (Supp) fails. Then $\operatorname{Ann}_{R_\infty} \mathscr{P}(\mathscr{M}) \neq (0)$, so as R_∞ is a domain, we get that $\dim R_\infty / \operatorname{Ann}_{R_\infty} \mathscr{P}(\mathscr{M}) < \dim R_\infty = \dim S_\infty$. But now lifting the structure homomorphism $S_\infty \to \mathscr{P}(\mathscr{R})$ to $S_\infty \to R_\infty$, we see that the action of S_∞ on $\mathscr{P}(\mathscr{M})$ factors through the composition $S_\infty \to R_\infty \to R_\infty / \operatorname{Ann}_{R_\infty} \mathscr{P}(\mathscr{M})$. Since $\dim S_\infty > \dim R_\infty / \operatorname{Ann}_{R_\infty} \mathscr{P}(\mathscr{M})$, this map cannot be surjective, and so S_∞ cannot act faithfully on $\mathscr{P}(\mathscr{M})$. But this contradicts the fact that $\mathscr{P}(\mathscr{M})$ is free over S_∞. Hence (Supp) must hold. □

We can now prove the main result of this section:

Theorem II.5.5. *Let R_0 be a finite $\mathcal{O}$-algebra and let M_0 be a nonzero R_0-module, which is finite and free over $\mathcal{O}$. Assume that we are given:*

- *A patching algebra $\mathscr{R} = \{R_n\}_{n \geq 1}$ over R_0;*
- *A* MCM *patching $\mathscr{R}$-module $\mathscr{M} = \{M_n\}_{n \geq 1}$ over M_0;*
- *A cover R_∞ of $\mathscr{R}$.*

such that $(R_\infty, \mathscr{R}, \mathscr{M})$ satisfies (Supp). Then we have the following:

1. *R_∞ is a minimal cover, i.e. $R_\infty \cong \mathscr{P}(\mathscr{R})$, and $R_0 = R_\infty/\mathfrak{n}$ (where R_∞ is given the structure of a S_∞-algebra via the isomorphism $R_\infty \cong \mathscr{P}(\mathscr{R})$).*
2. $\operatorname{Supp}_{R_0} M_0 = \operatorname{Spec} R_0$. *In particular, for any generic point η of* $\operatorname{Spec} R_0$ *with function field $K(\eta)$ (i.e. $K(\eta)$ is the field of fractions of R_0/η), $M_0 \otimes_{R_0} K(\eta) \neq 0$.*
3. *If η is any generic point of* $\operatorname{Spec} R_0$*, and $\widetilde{\eta}$ is a generic point of* $\operatorname{Spec} R_\infty$ *with $\eta \in \overline{\widetilde{\eta}}$ (i.e. $\widetilde{\eta} \subseteq \eta$ treating both as ideals of R_∞), then*
$$\dim_{K(\eta)} M_0 \otimes_{R_0} K(\eta) \geq \dim_{K(\widetilde{\eta})} \mathscr{P}(\mathscr{M}) \otimes_{R_\infty} K(\widetilde{\eta}) \geq 1,$$
where $K(\eta)$ and $K(\widetilde{\eta})$ are the function fields of η and $\widetilde{\eta}$, respectively (i.e. the field of fractions of R_0/η and $R_\infty/\widetilde{\eta}$).
4. *If R_∞ is a CM cover, then $(t_1, \dots, t_d, \varpi) \subseteq R_\infty$ is a regular sequence for R_∞, and R_0 is ϖ-torsion free (and hence Cohen–Macaulay).*
5. *If R_∞ is a regular cover, then M_0 is free over R_0.*

Proof. By definition, the action of R_∞ on $\mathscr{P}(\mathscr{M})$ factors through the map $\varphi_\infty : R_\infty \twoheadrightarrow \mathscr{P}(\mathscr{R})$ from Lemma II.5.2. By (Supp), this map must be injective, and thus an isomorphism. In particular Proposition II.4.4(3) implies that $R_\infty/\mathfrak{n} \cong \mathscr{P}(\mathscr{R})/\mathfrak{n} \cong R_0$. This proves (1).

For (2), note that $\operatorname{Ann}_{R_\infty} \mathscr{P}(\mathscr{M}) = (0)$ by (Supp), which implies that $\operatorname{Supp}_{R_\infty} \mathscr{P}(\mathscr{M}) = \operatorname{Spec} R_\infty$, as $\mathscr{P}(\mathscr{M})$ is a finitely generated R_∞-module. This now implies that

$$\operatorname{Supp}_{R_0} M_0 = \operatorname{Supp}_{R_\infty/\mathfrak{n}} \mathscr{P}(\mathscr{M})/\mathfrak{n} = \operatorname{Supp}_{R_\infty} \mathscr{P}(\mathscr{M})/\mathfrak{n} = V(\mathfrak{n}) = \operatorname{Spec} R_\infty/\mathfrak{n} = \operatorname{Spec} R_0.$$

Now for any $P \in \operatorname{Spec} R_\infty$, let $K(P)$ be the residue field of P (that is, the field of fractions of R_∞/P). As $\mathscr{P}(\mathscr{M})$ is a finite type R_∞ algebra, the map $P \mapsto \dim_{K(P)} \mathscr{P}(M) \otimes_{R_\infty} K(P)$ is upper semi-continuous on $\operatorname{Spec} R_\infty$. In particular, if η is a generic point of $\operatorname{Spec} R_0$ and $\widetilde{\eta}$ is a generic point of $\operatorname{Spec} R_\infty$ contained in η,

$$\dim_{K(\eta)} \mathscr{P}(M) \otimes_{R_\infty} K(\eta) \geq \dim_{K(\widetilde{\eta})} \mathscr{P}(\mathscr{M}) \otimes_{R_\infty} K(\widetilde{\eta}) \geq 1$$

(where the second inequality is just from the fact that $\widetilde{\eta} \in \operatorname{Supp}_{R_\infty} \mathscr{P}(\mathscr{M}) = \operatorname{Spec} R_\infty$). As

$$\mathscr{P}(\mathscr{M}) \otimes_{R_\infty} K(\eta) \cong (\mathscr{P}(\mathscr{M}) \otimes_{R_\infty} R_0) \otimes_{R_0} K(\eta) \cong M_0 \otimes K(\eta),$$

this gives (3).

Now as $\mathscr{M}$ is MCM, Corollary II.4.5(3) implies that $(t_1, \dots, t_d, \varpi) \subseteq \mathscr{P}(\mathscr{R}) = R_\infty$ is a regular sequence for $\mathscr{P}(\mathscr{M})$, and hence is a system of parameters for R_∞. Now assume that R_∞ is Cohen–Macaulay. This implies that any system of parameters for R_∞ is also a regular sequence, so indeed $(t_1, \dots, t_d, \varpi)$ is a regular sequence for R_∞.

But by the definition of regular sequences, it follows that $R_0 = R_\infty/(t_1, \dots, t_d)$ is Cohen–Macaulay and (ϖ) is an R_0-regular sequence, which implies that R_0 is ϖ-torsion free. This proves (4).

Finally, if R_∞ is a regular local ring, then as $\mathscr{P}(\mathscr{M})$ is maximal Cohen–Macaulay over R_∞, the Auslander–Buchsbaum formula formula implies that $\mathscr{P}(\mathscr{M})$ is free over R_∞ (see [Dia97, Theorem 2.1] for more details). Modding out by $\mathfrak{n}$ this now implies that $\mathscr{P}(\mathscr{M})/\mathfrak{n} \cong M_0$ is free over $\mathscr{P}(\mathscr{R})/\mathfrak{n} \cong R_0$, proving (5). □

II.6 Generically smooth covers

Theorem II.5.5 gives a significantly stronger result in the case when the cover is regular. The covers that arise in practice are typically only regular in the simplest cases, however they do sometimes satisfy a weaker condition, which we summarize in the following definition:

Definition II.6.1. We say that a cover $(R_\infty, \{\varphi_n\}_{n\geq 1})$ is *generically smooth* if R_∞ is a domain and $\operatorname{Spec} R_\infty[1/\varpi]$ is formally smooth over E.

In the case of a generically smooth cover, we get the following stronger version of Theorem II.5.5

Theorem II.6.2. *Let R_0 be a finite $\mathcal{O}$-algebra and let M_0 be a nonzero R_0-module, which is finite and free over $\mathcal{O}$. Assume that we are given:*

- *A patching algebra $\mathscr{R} = \{R_n\}_{n\geq 1}$ over R_0;*
- *A* MCM *patching $\mathscr{R}$-module $\mathscr{M} = \{M_n\}_{n\geq 1}$ over M_0;*
- *A* generically smooth *cover R_∞ of $\mathscr{R}$.*

Then $\mathscr{P}(\mathscr{M}) \otimes_{\mathcal{O}} E$ is a projective $R_\infty[1/\varpi]$-module, and $M_0 \otimes_{\mathcal{O}} E$ is a free $R_0[1/\varpi]$ module.

Proof. By assumption, R_∞ is a domain, and hence $(R_\infty, \mathscr{R}, \mathscr{M})$ satisfies (Supp), so the results of Theorem II.5.5 are applicable. In particular, $R_\infty \cong \mathscr{P}(\mathscr{R})$ (making R_∞ into an S_∞-algebra) and $R_\infty/\mathfrak{n} \cong R_0$.

Now as $\mathscr{M}$ is MCM, $\mathscr{P}(\mathscr{M})$ is free over S_∞, and so $\mathscr{P}(\mathscr{M})_E := \mathscr{P}(\mathscr{M}) \otimes_{\mathcal{O}} E$ is free over $S_\infty[1/\varpi] = S_\infty \otimes_{\mathcal{O}} E$. To show that $\mathscr{P}(\mathscr{M}) \otimes_{\mathcal{O}} E$ is a projective $R_\infty[1/\varpi]$-module, it suffices to prove that for any prime $\mathfrak{p} \subseteq R_\infty[1/\varpi]$ the completion $\mathscr{P}(\mathscr{M})^\wedge_{E,\mathfrak{p}} := \mathscr{P}(\mathscr{M})_E \otimes_{R_\infty[1/\varpi]} R_\infty[1/\varpi]^\wedge_{\mathfrak{p}}$ is free over $R_\infty[1/\varpi]^\wedge_{\mathfrak{p}}$.

Let $\mathfrak{q} = (S_\infty \otimes_{\mathcal{O}} E) \cap \mathfrak{p}$ be the prime ideal of $S_\infty \otimes_{\mathcal{O}} E$ lying under $\mathfrak{p}$. Then $\mathscr{P}(\mathscr{M})_E \otimes_{S_\infty[1/\varpi]} S_\infty[1/\varpi]^\wedge_{\mathfrak{q}}$ is free over $S_\infty[1/\varpi]^\wedge_{\mathfrak{q}}$. Let $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_k$ be the primes of $R_\infty[1/\varpi]$ lying over $\mathfrak{q}$. By [Sta19, Lemma 07N9],

$$R_\infty[1/\varpi] \otimes_{S_\infty[1/\varpi]} S_\infty[1/\varpi]^\wedge_{\mathfrak{q}} \cong \prod_{i=1}^{k} R_\infty[1/\varpi]^\wedge_{\mathfrak{p}_i}$$

and so

$$\mathscr{P}(\mathscr{M})_E \otimes S_\infty[1/\varpi]^\wedge_{\mathfrak{q}} = \bigoplus_{i=1}^{k} \mathscr{P}(\mathscr{M})_E \otimes R_\infty[1/\varpi]^\wedge_{\mathfrak{p}_i} = \bigoplus_{i=1}^{k} \mathscr{P}(\mathscr{M})^\wedge_{E,\mathfrak{p}_i}.$$

It follows that $\mathscr{P}(\mathscr{M})^\wedge_{E,\mathfrak{p}} = \mathscr{P}(\mathscr{M})^\wedge_{E,\mathfrak{p}_1}$ is a direct summand of a free $S_\infty[1/\varpi]^\wedge_{\mathfrak{q}}$-module, and is thus a projective $S_\infty[1/\varpi]^\wedge_{\mathfrak{q}}$-module. As $S_\infty[1/\varpi]^\wedge_{\mathfrak{q}}$ is local, $\mathscr{P}(\mathscr{M})^\wedge_{E,\mathfrak{p}}$ is a free $S_\infty[1/\varpi]^\wedge_{\mathfrak{q}}$-module.

But now as S_∞ is regular, so is $S_\infty[1/\varpi]_{\mathfrak{q}}$, and hence $S_\infty[1/\varpi]^\wedge_{\mathfrak{q}}$ is a complete regular local ring, which contains a field E. By the Cohen structure theorem, $S_\infty[1/\varpi]_{\mathfrak{q}} \cong K[[y_1, \dots, y_a]]$ for some field K. As $\mathscr{P}(\mathscr{M})^\wedge_{E,\mathfrak{p}}$ is free over $S_\infty[1/\varpi]_{\mathfrak{q}}$ and, $R_\infty[1/\varpi]^\wedge_{\mathfrak{p}}$ is finite over $S_\infty[1/\varpi]_{\mathfrak{q}}$ we get that $\mathscr{P}(\mathscr{M})^\wedge_{E,\mathfrak{p}}$ is maximal Cohen–Macaulay over $R_\infty[1/\varpi]^\wedge_{\mathfrak{p}}$ (with regular sequence $(y_1, \dots, y_a)$).

But now as $R_\infty[1/\varpi]$ is formally smooth, $R_\infty[1/\varpi]^\wedge_{\mathfrak{p}}$ is regular. Thus the Auslander–Buchsbaum formula (just as in the proof of Theorem II.5.5) implies that $\mathscr{P}(\mathscr{M})^\wedge_{E,\mathfrak{p}}$ is free over $R_\infty[1/\varpi]^\wedge_{\mathfrak{p}}$.

So indeed $\mathscr{P}(\mathscr{M})_E$ is projective over $R_\infty[1/\varpi]$. Since $R_\infty[1/\varpi]$ is a domain, this implies that $\mathscr{P}(\mathscr{M})_E$ is locally free of some rank, say m. It follows that $\mathscr{P}(\mathscr{M})_E/\mathfrak{n} \cong (\mathscr{P}(\mathscr{M})/\mathfrak{n}) \otimes_{\mathcal{O}} E \cong M_0 \otimes_{\mathcal{O}} E$ is locally free of rank m over $R_0[1/\varpi]$.

But now as R_0 is finite over $\mathcal{O}$, $R_0[1/\varpi]$ is finite over E and so it is a direct sum of finitely many local E-algebras. It follows that a locally free rank m $R_0[1/\varpi]$-module is actually free of rank m, completing the proof. □

Remark. The conditions we imposed on R_∞ in this section were fairly restrictive, namely we required the entire scheme $\operatorname{Spec} R_\infty[1/\varpi]$ to be formally smooth. A weaker condition, and one which is satisfied far more often in practice, would be to only require that $\operatorname{Spec} R_\infty[1/\varpi]$ is formally smooth at each point of $\operatorname{Spec} R_0[1/\varpi]$, for all of the embeddings $\iota_n : \operatorname{Spec} R_0[1/\varpi] \hookrightarrow \operatorname{Spec} R_\infty[1/\varpi]$ given by the maps $R_\infty \twoheadrightarrow R_n \twoheadrightarrow R_0$.

In such a situation we would expect to be able to give a similar result to Theorem II.6.2[2] *except* that there is no way to guarantee that the embedding $\iota_\infty : \operatorname{Spec} R_0[1/\varpi] \hookrightarrow \operatorname{Spec} R_\infty[1/\varpi]$ induced by $R_\infty \twoheadrightarrow \mathscr{P}(\mathscr{R}) \twoheadrightarrow R_0$ also only hits the formally smooth points of $\operatorname{Spec} R_\infty[1/\varpi]$. The reason for this is that we haven't imposed any compatibility between the different embeddings ι_n, and so all we can say about ι_∞ is that it's a limit of some subsequence of the ι_n's, which can have a non formally smooth point in it's image as the formally smooth locus of $\operatorname{Spec} R_\infty[1/\varpi]$ is not typically closed.

To get around this issue, we will need to impose extra restrictions on the maps $R_\infty \twoheadrightarrow R_n$ to make the embeddings ι_n somewhat more compatible. We will return to this idea later.

II.7 Quasi-Patching Algebras

In the situation described in Theorem II.5.5 it was very convenient that the rings $\mathscr{R} = \{R_n\}_{n\geq 1}$ formed a patching system over R_0, in particular, that they were finitely generated as S_∞-modules of bounded rank.

When the patching argument is used in practice, one typically considers a sequence of rings $\mathscr{R} = \{R_n\}_{n\geq 1}$ and a sequence of modules R_n-modules $\mathscr{M} = \{M_n\}_{n\geq 1}$, over a ring R_0 and an R_0-module M_0. In almost all situations, $\mathscr{M}$ is known to form a patching system (and usually a MCM patching system, in fact). However the sequence of rings $\mathscr{R}$ is not always known to form a patching algebra. In particular the rings R_n, or even the ring R_0, are not known to be finitely generated S_∞-modules. In fact, one common application of the patching argument is to prove that the ring R_0 is actually finite over $\mathcal{O}$.

So to use the full strength of the patching arguments, we will sometimes need to consider the following slightly more general situation:

Definition II.7.1. We say that a *quasi-patching algebra* is a triple $(\mathscr{R} = \{R_n\}_{n\geq 1}, R_0, \{\alpha_n\}_{n\geq 1})$ where

- For each $n \geq 0$, R_n is a, topologically finitely generated $\mathcal{O}$-algebra (not necessarily finite over $\mathcal{O}$).
- For each $n \geq 1$, R_n has the structure of a $S_\infty/\mathcal{I}_n$-algebra.
- For each $n \geq 1$, α_n is an isomorphism $\alpha_n : R_n/\mathfrak{n} \xrightarrow{\sim} R_0$ of $\mathcal{O}$-algebras.
- There is some $g \geq 0$ such that for each $n \geq 0$, R_n is topologically generated an an $\mathcal{O}$-algebra by at most g elements (equivalently, the set $\{\dim_{\mathbb{F}}(R_n/\mathfrak{m}_{R_n})\}_{n\geq 0}$ is bounded).

Again, in this situation we will also refer to $\mathscr{R}$ as a quasi-patching algebra over R_0.

We first observe the following:

[2] Possibly at the cost of only proving that $M_0 \otimes_{\mathcal{O}} E$ is locally free of constant rank over each component of $\operatorname{Spec} R_\infty$ in the case when $\operatorname{Spec} R_\infty$ is not irreducible.

Lemma II.7.2. *Let R be a complete, local S_∞-algebra. Assume that R is topologically finitely generated over $\mathcal{O}$ by g elements and $\dim_{\mathbb{F}} R/\mathfrak{m}_{S_\infty}R = r < \infty$. Then R is finitely generated as an S_∞-module by at most r^g elements.*

Proof. Let $J = \mathfrak{m}_{S_\infty}R \subseteq \mathfrak{m}_R$. By assumption, R/J is a finite, and hence Artinian, local ring with length at most r. It follows that $\mathfrak{m}_{R/J}^r = 0$ in R/J and so $\mathfrak{m}_R^r \subseteq J$ in R. In particular, $\mathfrak{m}_R^{rk} \subseteq J^k \subseteq \mathfrak{m}_R^k$ for all $k \geq 1$, and so $\{\mathfrak{m}_R^k\}$ is cofinal with $\{J^k\}$ and so $R \cong \varprojlim R/J^k$ as topological rings.

Now let $x_1, \dots, x_g \in \mathfrak{m}_R$ be a set of topological generators for R over $\mathcal{O}$, and let

$$\mathcal{A} = \left\{ x_1^{e_1} \cdots x_g^{e_g} \middle| 0 \leq e_i \leq r-1 \right\} \subseteq R,$$

so that $\#\mathcal{A} \leq r^g$. We claim that R is generated by $\mathcal{A}$ as an S_∞-module. Let $B \subseteq R$ be the S_∞-submodule of R generated by $\mathcal{A}$. As R/J is topologically generated by $x_1, \dots, x_g$ as a S_∞-algebra and $x_i^{e_i} \equiv 0 \pmod{J}$ whenever $e_i \geq r$, it clearly follows that R/J is generated by $\mathcal{A} \pmod{J}$ as a S_∞-module, and so $B + J = R$. Now for any $k \geq 1$ it follows that

$$B + J^k = B + J^k R = B + J^k(B + J) = (B + J^k B) + J^{k+1} = B + J^{k+1},$$

and so $B + J^k = B + J = R$ for all $k \geq 1$ (i.e. that R/J^k is generated $\mathcal{A} \pmod{J^k}$). Since B is clearly closed in the profinite topology on R (as the structure map $f : S_\infty \to R$ satisfies $f^{-1}(\mathfrak{m}_R) = \mathfrak{m}_{S_\infty}$, and is thus continuous) we now have

$$B = \bigcap_{k \geq 1} (B + J^k) = \bigcap_{k \geq 1} R = R,$$

as desired. □

Lemma II.7.3. *If $\mathscr{R}$ is a quasi-patching algebra over R_0. Then the following are equivalent:*

1. *$\mathscr{R}$ is a patching algebra;*
2. *R_0 is finite over $\mathcal{O}$;*
3. *R_0/ϖ is a finite dimensional $\mathbb{F}$-vector space.*

Proof. Let $\mathscr{R} = \{R_n\}_{n \geq 1}$.

By definition, if $\mathscr{R}$ is a patching algebra over R_0, then R_0 is finite over $\mathcal{O}$ (as each R_n is for $n \geq 1$, and R_0 is a quotient of R_n), so (1) ⇒ (2). (2) ⇒ (3) is trivial.

So now assume (3). Let $r = \dim_{\mathbb{F}} R_0/\varpi$. Then for each $n \geq 1$,

$$R_n/\mathfrak{m}_{S_\infty}R_n \cong (R_n/\mathfrak{n}R_n)/\varpi \cong R_0/\varpi$$

and so $\dim_{\mathbb{F}} R_n/\mathfrak{m}_{S_\infty}R_n = r < \infty$. By the definition of quasi-patching algebra, there is a $g \geq 1$ such that each R_n is topologically generated by at most g elements as an $\mathcal{O}$-algebra. Lemma II.7.2 now implies that each R_n is finitely generated as a S_∞-module of rank at most r^g. Thus $\mathscr{R}$ is indeed a patching algebra, so (3) ⇒ (1). □

At this point, one could define $\mathscr{P}(\mathscr{R})$ for a quasi-patching algebra $\mathscr{R}$ by the formula $\mathscr{P}(\mathscr{R}) = \varprojlim_{\mathfrak{a}} \mathcal{U}(\mathscr{R}/\mathfrak{a})$ just as for weak patching algebras. However we will refrain from making such a definition as when $\mathscr{R}$ is only a quasi-patching algebra, the rings $R_n/\mathfrak{a}$ do not necessarily have cardinalities bounded independently of n (and in fact, are not necessarily even finite) and so $\mathcal{U}(\mathscr{R}/\mathfrak{a})$ may be a complicated, and rather poorly behaved object. Instead, we will define a different object $\widetilde{\mathscr{P}}(\mathscr{R})$ which will be better behaved and will agree with $\mathscr{P}(\mathscr{R})$ in the case when $\mathscr{R}$ is a patching algebra.

Lemma II.7.4. *Let $\mathscr{R} = \{R_n\}_{n\geq 1}$ be a quasi-patching algebra over R_0. For each $k \geq 1$, let $\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k = \{R_n/\mathfrak{m}_{R_n}^k\}_{k\geq 1}$. Then $\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k$ is a patching algebra over $R_0/\mathfrak{m}_{R_0}^k$. Moreover for any $k \geq 1$, we have $\mathscr{P}(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k) = \mathcal{U}\left(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k\right)$, and $\mathcal{U}\left(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k\right)$ is a finite ring.*

Proof. If each R_n is topologically generated by g elements, the same is true of $R_n/\mathfrak{m}_{R_n}^k$. Now for any n, let $\alpha_n : R_n \to R_n/\mathfrak{n} \xrightarrow{\sim} R_0$ denote the surjection, and note that $\alpha_n(\mathfrak{m}_{R_n}) = \mathfrak{m}_{R_0}$, and hence $\alpha_n(\mathfrak{m}_{R_n}^k) = \alpha_n(\mathfrak{m}_{R_n})^k = \mathfrak{m}_{R_0}^k$. But then

$$\frac{R_n/\mathfrak{m}_{R_n}^k}{\mathfrak{n}(R_n/\mathfrak{m}_{R_n}^k)} \cong \frac{R_n}{\mathfrak{n}R_n + \mathfrak{m}_{R_n}^k} \cong \frac{R_n/\mathfrak{n}R_n}{\mathfrak{m}_{R_n}^k(R_n/\mathfrak{n}R_n)} \cong \frac{R_0}{\alpha_n(\mathfrak{m}_{R_n}^k)} \cong R_0/\mathfrak{m}_{R_0}^k,$$

and so $\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k$ is a quasi-patching algebra over $R_0/\mathfrak{m}_{R_0}^k$. But now R_0 is topologically finitely generated over $\mathcal{O}$, so $R_0/\mathfrak{m}_{R_0}^k$ is finite, and hence finite over $\mathcal{O}$. Thus Lemma II.7.3 implies that $R_n/\mathfrak{m}_{R_n}^k$ is actually a patching algebra over $R_0/\mathfrak{m}_{R_0}^k$.

For the last statement, note that for any fixed k we have $\mathfrak{m}_{S_\infty} \subseteq \mathfrak{m}_{R_n}$ and so $\mathfrak{m}_{S_\infty}^k \subseteq \mathfrak{m}_{R_n}^k$ for all n. Then each $R_n/\mathfrak{m}_{R_n}^k$ is a finitely generated $S_\infty/\mathfrak{m}_{S_\infty}^k$-module of bounded rank. Since $S_\infty/\mathfrak{m}_{S_\infty}^k$ is finite, it follows that each $R_n/\mathfrak{m}_{R_n}^k$ is a finite ring, of bounded rank. It follows that $\mathcal{U}\left(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k\right)$ is also a finite ring. Finally,

$$\mathscr{P}(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k) = \varprojlim_{\mathfrak{a}} \mathcal{U}\left(\frac{R_n/\mathfrak{m}_{R_n}^k}{\mathfrak{a}}\right) = \varprojlim_{\mathfrak{a}\subseteq\mathfrak{m}_{S_\infty}^k} \mathcal{U}\left(\frac{R_n/\mathfrak{m}_{R_n}^k}{\mathfrak{a}}\right) = \varprojlim_{\mathfrak{a}\subseteq\mathfrak{m}_{S_\infty}^k} \mathcal{U}\left(R_n/\mathfrak{m}_{R_n}^k\right) = \mathcal{U}\left(R_n/\mathfrak{m}_{R_n}^k\right).$$

□

Thus we may define

Definition II.7.5. For any quasi-patching algebra $\mathscr{R}$, define

$$\widetilde{\mathscr{P}}(\mathscr{R}) = \varprojlim_{k} \mathscr{P}(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k) = \varprojlim_{k} \mathcal{U}\left(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k\right),$$

given the structure of a local S_∞ algebra.

We first establish some basic properties of $\widetilde{\mathscr{P}}(\mathscr{R})$:

Lemma II.7.6. *Let $\mathscr{R} = \{R_n\}_{n\geq 1}$ be a quasi-patching algebra over R_0. Then the following hold:*

1. *$\widetilde{\mathscr{P}}(\mathscr{R})$ is complete and topologically finitely generated over $\mathcal{O}$.*
2. *For any ideal $J \subseteq S_\infty$, $\widetilde{\mathscr{P}}(\mathscr{R})/J \cong \widetilde{\mathscr{P}}(\mathscr{R}/J)$.*
3. *$\widetilde{\mathscr{P}}(\mathscr{R})/\mathfrak{n} \cong R_0$.*
4. *For any integer $k \geq 1$, $\widetilde{\mathscr{P}}(\mathscr{R})/\mathfrak{m}^k_{\widetilde{\mathscr{P}}(\mathscr{R})} \cong \mathscr{P}(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}})$.*

Proof. By Lemma II.7.4 each $\mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big)$ is a finite ring, and so $\widetilde{\mathscr{P}}(\mathscr{R})$ is profinite, and hence complete. Now by assumption, each R_n is topologically generated by g elements, for some fixed $g \geq 0$. Thus there exist surjective maps $f_{n,k} : \mathcal{O}[[x_1, \dots, x_g]] \twoheadrightarrow R_n \twoheadrightarrow R_n/\mathfrak{m}^k_{R_n}$. By the same argument as in Lemma II.5.2 this induces a compatible system of surjective maps $f_k : \mathcal{O}[[x_1, \dots, x_g]] \twoheadrightarrow \mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big)$ and hence a continuous surjective map $\mathcal{O}[[x_1, \dots, x_g]] \twoheadrightarrow \varprojlim_k \mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big) = \widetilde{\mathscr{P}}(\mathscr{R})$. This proves (1).

Now fix any ideal $J \subseteq S_\infty$. For any fixed k we have a natural isomorphism $\mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big)/J \cong \mathcal{U}\Big((\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}})/J\Big) \cong \mathcal{U}\Big((\mathscr{R}/J)/\mathfrak{m}^k_{\mathscr{R}}\Big)$. Taking inverse limits gives

$$\widetilde{\mathscr{P}}(\mathscr{R}/J) = \varprojlim_k \mathcal{U}\Big((\mathscr{R}/J)/\mathfrak{m}^k_{\mathscr{R}}\Big) \cong \varprojlim_k \Big(\mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big)/J\Big).$$

Now noting that the rings $\mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big)$ are all finite, and the S_∞-module S_∞/J is finitely presented, exactness of $\varprojlim$ (as in Proposition II.4.2) gives that

$$\widetilde{\mathscr{P}}(\mathscr{R}/J) \cong \varprojlim_k \Big(\mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big)/J\Big) \cong \varprojlim_k \Big(\mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big)\Big)/J = \widetilde{\mathscr{P}}(\mathscr{R})/J,$$

proving (2).

Setting $J = \mathfrak{n}$, this gives $\widetilde{\mathscr{P}}(\mathscr{R})/\mathfrak{n} \cong \widetilde{\mathscr{P}}(\mathscr{R}/\mathfrak{n})$ and by assumption $\mathscr{R}/\mathfrak{n} = \{R_n/\mathfrak{n}\}_{n\geq 1} = \{R_0\}_{n\geq 1}$. Hence

$$\widetilde{\mathscr{P}}(\mathscr{R})/\mathfrak{n} \cong \widetilde{\mathscr{P}}(\mathscr{R}/\mathfrak{n}) = \varprojlim_k \mathcal{U}\Big((\mathscr{R}/\mathfrak{n})/\mathfrak{m}^k_{\mathscr{R}}\Big) = \varprojlim_k \mathcal{U}\Big(R_0/\mathfrak{m}^k_{R_0}\Big) = \varprojlim_k R_0/\mathfrak{m}^k_{R_0} = R_0,$$

proving (3).

Now similarly to part (2), for any fixed k we have

$$\begin{aligned}\widetilde{\mathscr{P}}(\mathscr{R})/\mathfrak{m}^k_{\widetilde{\mathscr{P}}(\mathscr{R})} &\cong \varprojlim_m \left(\mathcal{U}(\mathscr{R}/\mathfrak{m}^m_{\mathscr{R}})\right)/\mathfrak{m}^k_{\widetilde{\mathscr{P}}(\mathscr{R})} \cong \varprojlim_m \Big(\mathcal{U}(\mathscr{R}/\mathfrak{m}^m_{\mathscr{R}})/\mathfrak{m}^k_{\widetilde{\mathscr{P}}(\mathscr{R})}\Big) \cong \varprojlim_{m\geq k} \mathcal{U}\Big((\mathscr{R}/\mathfrak{m}^m_{\mathscr{R}})/\mathfrak{m}^k_{\widetilde{\mathscr{P}}(\mathscr{R})}\Big)\\ &\cong \varprojlim_{m\geq k} \mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big) \cong \mathcal{U}\Big(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\Big) \cong \mathscr{P}(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}),\end{aligned}$$

proving (4). □

Lemma II.7.7. *Let $\mathscr{R} = \{R_n\}$ be a quasi-patching algebra over R_0, and $\mathscr{M} = \{M_n\}$ be a patching $\mathscr{R}$-module over M_0. For any k, define $\mathfrak{m}^k_{\mathscr{R}}\mathscr{M} = \{\mathfrak{m}^k_{R_n} M_n\}_{n\geq 1}$. We have the following:*

1. *There is a natural isomorphism $\mathscr{P}(\mathscr{M}) \cong \varprojlim_k \mathcal{U}\left(\mathscr{M}/\mathfrak{m}^k_{\mathscr{R}}\mathscr{M}\right)$.*
2. *$\mathscr{P}(\mathscr{M})$ can be given a natural $\widetilde{\mathscr{P}}(\mathscr{R})$-module structure, inducing the S_∞-module structure on $\mathscr{P}(\mathscr{M})$ via the structure map $S_\infty \to \widetilde{\mathscr{P}}(\mathscr{R})$ and the R_0-module structure on M_0 via the isomorphisms $\widetilde{\mathscr{P}}(\mathscr{R})/\mathfrak{n} \cong R_0$ and $\mathscr{P}(\mathscr{M})/\mathfrak{n} \cong M_0$.*

Proof. Write $\mathscr{M} = \prod_{n=1}^{\infty} M_n$. We claim that the two inverse systems:

$$\left\{\mathfrak{m}^k_{\mathscr{R}}\mathscr{M} = \prod_{n=1}^{\infty} \mathfrak{m}^k_{R_n} M_n \,\middle|\, k \geq 1\right\} \qquad \text{and} \qquad \left\{\mathfrak{a}\mathscr{M} = \prod_{n=1}^{\infty} \mathfrak{a} M_n \,\middle|\, \mathfrak{a} \subseteq S_\infty\right\}$$

of submodules of $\mathscr{M}$ are cofinal. First, for any k and n we have $\mathfrak{m}^k_{S_\infty} M_n \subseteq \mathfrak{m}^k_{R_n} M_n$. Now as $\mathscr{M}$ is a patching system, there is some $N \geq 1$ such that each M_n is generated as a S_∞-module by N elements. Then it follows that for any open $\mathfrak{a} \subseteq S_\infty$ and any n that $\operatorname{length}(M_n/\mathfrak{a}) \leq N \operatorname{length}(S_\infty/\mathfrak{a})$ and so for any $k \geq N \operatorname{length}(S_\infty/\mathfrak{a})$ we have $\mathfrak{m}^k_{R_n} M_n \leq \mathfrak{a} M_n$. So the above systems are indeed cofinal systems of submodules of $\mathscr{M}$, and hence of the localization $\mathscr{M}_{\mathfrak{Z}}$ (where $\mathfrak{Z}$ is as in Section II.3). By standard properties of inverse limits it follows that

$$\mathscr{P}(\mathscr{M}) = \varprojlim_{\mathfrak{a}} \mathcal{U}(\mathscr{M}/\mathfrak{a}) = \varprojlim_{\mathfrak{a}} (\mathscr{M}/\mathfrak{a}\mathscr{M})_{\mathfrak{Z}} = \varprojlim_k (\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}})_{\mathfrak{Z}} = \varprojlim_k \mathcal{U}\left(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\right),$$

proving (1).

Now for each k and n, $R_n/\mathfrak{m}^k_{R_n}$ acts naturally on $M_n/\mathfrak{m}^k_{R_n} M_n$. This implies that $\mathcal{U}\left(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\right)$ acts naturally on $\mathcal{U}\left(\mathscr{M}/\mathfrak{m}^k_{\mathscr{R}}\mathscr{M}\right)$ and so taking inverse limits gives a natural action of $\widetilde{\mathscr{P}}(\mathscr{R}) = \varprojlim_k \mathcal{U}\left(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\right)$ on $\mathscr{P}(\mathscr{M}) = \varprojlim_k \mathcal{U}\left(\mathscr{M}/\mathfrak{m}^k_{\mathscr{R}}\mathscr{M}\right)$. The listed properties of this action now follow automatically. □

Corollary II.7.8. *If $\mathscr{R}$ is a patching algebra, then there is a natural isomorphism $\widetilde{\mathscr{P}}(\mathscr{R}) \cong \mathscr{P}(\mathscr{R})$.*

Proof. Applying Lemma II.7.7(1) with $\mathscr{R} = \mathscr{M}$ gives $\mathscr{P}(\mathscr{R}) = \varprojlim_k \mathcal{U}\left(\mathscr{R}/\mathfrak{m}^k_{\mathscr{R}}\right) = \widetilde{\mathscr{P}}(\mathscr{R})$. □

Quasi-patching algebras, $\mathscr{R}$, that arise in practice usually do so in a context similar to Theorem II.5.5 (so together with a "cover" R_∞ and an MCM patching $\mathscr{R}$-module $\mathscr{M}$). We would like to find a criterion to ensure that such quasi-patching algebras are actually a patching algebra, or equivalently (by Lemma II.7.3) that the ring R_0 is finite over $\mathcal{O}$.

We can define a *cover* $(R_\infty, \{\varphi_n\})$ of a quasi-patching algebra $\mathscr{R}$ just as in definition II.5.1 (recalling that this requires $\dim R_\infty = \dim S_\infty$). Note that if $\mathscr{R} = \{R_n\}_{n\geq 1}$ is covered by a finitely generated

$\mathcal{O}$-algebra then the last condition of Definition II.7.1 is automatically satisfied — simply let g be the cardinality of a topological generating set for R_∞ over $\mathcal{O}$. We then have the natural analogue of Lemma II.5.2:

Lemma II.7.9. *Let $\mathscr{R}$ be a quasi-patching algebra over a ring R_0, and let R_∞ be a cover of R_∞. Then there exists a continuous, surjective $\mathcal{O}$-algebra homomorphism $\varphi_\infty : R_\infty \twoheadrightarrow \widetilde{\mathscr{P}}(\mathscr{R})$. In particular,* $\dim \widetilde{\mathscr{P}}(\mathscr{R}) \le \dim R_\infty = d+1$.

Proof. For any k we have a continuous map

$$\Phi_k : R_\infty \to \prod_{n=1}^{\infty} R_n \twoheadrightarrow \prod_{n=1}^{\infty} (R_n/\mathfrak{m}_{R_n}^k) \twoheadrightarrow \mathcal{U}\Big(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k\Big)$$

which induces a continuous map

$$\varphi_\infty = (\Phi_k)_k : R_\infty \to \varprojlim_k \mathcal{U}\Big(\mathscr{R}/\mathfrak{m}_{\mathscr{R}}^k\Big) = \widetilde{\mathscr{P}}(\mathscr{R}).$$

The proof that Φ_k and φ_∞ are surjective is identical to the proof of Lemma II.5.2. □

Thus if $\mathscr{R}$ is a quasi-patching algebra covered by R_∞ and $\mathscr{M}$ is a weak patching $\mathscr{R}$-module, then R_∞ acts on $\mathscr{P}(\mathscr{M})$, and if $\mathscr{M}$ is MCM, then $\mathscr{P}(\mathscr{M})$ is maximal Cohen–Macaulay over R_∞. From now on we consider a triple $(R_\infty, \mathscr{R}, \mathscr{M})$ where:

- $\mathscr{R} = \{R_n\}_{n\ge1}$ is a quasi-patching algebra over R_0;
- $\mathscr{M} = \{M_n\}_{n\ge1}$ is a *MCM* patching $\mathscr{R}$-module over M_0;
- R_∞ is a cover of $\mathscr{R}$.

We again say that such a triple satisfies (Supp) if R_∞ acts faithfully on $\mathscr{P}(\mathscr{M})$, and note that this is still automatically satisfied if R_∞ is a domain.

We can now prove the main theorem of this section:

Theorem II.7.10. *Let $\mathscr{R}$ be a quasi-patching system over R_0. Assume that we are given a cover R_∞ of $\mathscr{R}$ and an MCM patching $\mathscr{R}$-module $\mathscr{M}$ over M_0. Then if $(R_\infty, \mathscr{R}, \mathscr{M})$ satisfies (Supp) then $\mathscr{R}$ is a patching system, and so in particular R_0 is finite over $\mathcal{O}$.*

Proof. Since $\mathscr{M}$ is MCM, $\mathscr{P}(\mathscr{M})$ is free of finite rank over S_∞ and so $\operatorname{depth}_{S_\infty} \mathscr{P}(\mathscr{M}) = \dim S_\infty = \dim R_\infty$. By Lemma II.7.7 the S_∞-module structure on $\mathscr{P}(\mathscr{M})$ is induced by its $\widetilde{\mathscr{P}}(\mathscr{R})$-module structure. It follows that

$$\operatorname{depth}_{R_\infty} \mathscr{P}(\mathscr{M}) = \operatorname{depth}_{\widetilde{\mathscr{P}}(\mathscr{R})} \mathscr{P}(\mathscr{M}) \ge \operatorname{depth}_{S_\infty} \mathscr{P}(\mathscr{M}) = \dim R_\infty \ge \dim \widetilde{\mathscr{P}}(\mathscr{R})$$

and so $\operatorname{depth}_{R_\infty} \mathscr{P}(\mathscr{M}) = \dim R_\infty = \dim \widetilde{\mathscr{P}}(\mathscr{R})$. Hence $\mathscr{P}(\mathscr{M})$ is maximal Cohen–Macaulay over R_∞. It now follows that if R_∞ is a domain that (Supp) is satisfied.

Now assume that (Supp) holds. As the action of R_∞ on $\mathscr{P}(\mathscr{M})$ factors through $\varphi_\infty : R_\infty \twoheadrightarrow \widetilde{\mathscr{P}}(\mathscr{R})$, φ_∞ must be an isomorphism. Give R_∞ the structure of an S_∞-algebra via φ_∞. Then by Lemma II.7.6(3), $R_\infty/\mathfrak{n} \cong \widetilde{\mathscr{P}}(\mathscr{R})/\mathfrak{n} \cong R_0$ and so $R_\infty/\mathfrak{m}_{S_\infty} \cong R_0/\varpi$. Let $\overline{R}_0 = R_0/\varpi$. By Lemma II.7.3 it suffices to show that $\overline{R}_0$ is finite dimensional over $\mathbb{F}$. As $\overline{R}_0$ is topologically finitely generated over $\mathbb{F}$, this is equivalent so saying that $\dim \overline{R}_0 = 0$.

So let $\overline{P} \subseteq \overline{R}_0$ be any prime. Lift $\overline{P}$ to a prime ideal $P \subseteq R_\infty$ via the isomorphism $R_\infty/\mathfrak{m}_{S_\infty} \cong \overline{R}_0$, so that $\mathfrak{m}_{S_\infty} R_\infty \subseteq P$. Since $\mathscr{P}(\mathscr{M})$ has full support over R_∞ and $P \subseteq R_\infty$ is prime, it follows that R_∞/P acts faithfully on

$$\mathscr{P}(\mathscr{M})/P = (\mathscr{P}(\mathscr{M})/\mathfrak{m}_{S_\infty}\mathscr{P}(\mathscr{M}))/P = (M_0/\varpi M_0)/P,$$

which is finite, as M_0 is finite over $\mathcal{O}$. It follows that $R_\infty/P \cong \overline{R}_0/\overline{P}$ is finite. As $\overline{R}_0$ is a local ring, this implies that $\overline{P} = \mathfrak{m}_{\overline{R}_0}$. Thus $\mathfrak{m}_{\overline{R}_0}$ is the only prime ideal of $\overline{R}_0$ and so indeed $\dim \overline{R}_0 = 0$, completing the proof. □

This theorem means that the main results of the previous sections (in particular Theorems II.5.5 and II.6.2) can be applied in the case when $\mathscr{R}$ is merely assumed to be a quasi-patching algebra, instead of a patching algebra.

II.8 $R = \mathbb{T}$ theorems

When the theory of patching is applied in practice, in addition to a patching algebra $\mathscr{R} = \{R_n\}_{n\geq 1}$ over R_0 and a (usualy MCM) patching $\mathscr{R}$-module $\mathscr{M} = \{M_n\}_{n\geq 1}$ over M_0, one generally also has another collection of rings $\{\mathbb{T}_n\}_{n\geq 0}$ (which arise as the completions of various Hecke algebras), such that each $\mathbb{T}_n$ is naturally a quotient of R_n, and R_n acts on M_n via the quotient map $R_n \to \mathbb{T}_n$. The rings $\mathbb{T}_n$ carry a great deal of number theoretic significance, so it is often quite important to understand their structure and their relation to the rings R_n.

To incorporate these rings $\mathbb{T}_n$ into the picture, we make the following definition:

Definition II.8.1. If $\mathscr{R} = \{R_n\}_{n\geq 1}$ is a quasi-patching algebra over a ring R_0 and $\mathscr{M} = \{M_n\}_{n\geq 1}$ is a patching $\mathscr{R}$-module over an R_0-module M_0, then for each $n \geq 0$, let $\mathbb{T}_n^{\mathscr{R}}(\mathscr{M})$ be the image of R_n in $\operatorname{End}_{S_\infty}(M_n)$. Let $\mathscr{T}^{\mathscr{R}}(\mathscr{M}) = \{\mathbb{T}_n^{\mathscr{R}}(\mathscr{M})\}_{n\geq 1}$.

Lemma II.8.2. *If $\mathscr{R}$ be a quasi-patching algebra and $\mathscr{M} = \{M_n\}_{n\geq 1}$ is a weak patching $\mathscr{R}$-module, then $\mathscr{T}^{\mathscr{R}}(\mathscr{M})$ is a weak patching algebra.*

Proof. Since $\mathscr{M}$ is a weak patching system, there must exist an integer N for which $\operatorname{rank}_{S_\infty} M_n \leq N$ for all $n \geq 1$. It follows that for any $n \geq 1$, $\operatorname{rank}_{S_\infty} \mathbb{T}_n^{\mathscr{R}}(\mathscr{M}) \leq \operatorname{rank}_{S_\infty} \operatorname{End}_{S_\infty}(M_n) \leq N^2$, and so $\mathscr{T}^{\mathscr{R}}(\mathscr{M})$ is a weak patching algebra. □

Now the results of the previous sections can be used to deduce information about the quotient map $\pi_0 : R_0 \twoheadrightarrow \mathbb{T}_0^{\mathscr{R}}(\mathscr{M})$:

Theorem II.8.3. *Assume we are given the following:*

- *A quasi-patching algebra $\mathscr{R}$ over a ring R_0.*
- *An MCM patching $\mathscr{R}$-module $\mathscr{M}$ over an R_0-module M_0.*
- *A cover $(R_\infty, \{\varphi_n\})$ of $\mathscr{R}$, where R_∞ is a domain.*

and assume that $(R_\infty, \mathscr{R}, \mathscr{M})$ satisfies (Supp). Then letting $\mathbb{T}_0 = \mathbb{T}_0^{\mathscr{R}}(\mathscr{M})$ we have:

1. *The map $\pi_0 : R_0 \twoheadrightarrow \mathbb{T}_0$ induces an isomorphism $R_0^{red} \xrightarrow{\sim} \mathbb{T}_0^{red}$.*
2. *If R_∞ is a generically smooth cover of $\mathscr{R}$, then π_0 induces an isomorphism $R_0[1/\varpi] \xrightarrow{\sim} \mathbb{T}_0[1/\varpi]$.*
3. *If R_∞ is a generically smooth cover of $\mathscr{R}$ and R_∞ is Cohen–Macaulay, then π_0 is an isomorphism $R_0 \xrightarrow{\sim} \mathbb{T}_0$.*

Proof. First note that Theorem II.7.10 implies that R_0 is finite over $\mathcal{O}$ and $\mathscr{R}$ is a patching algebra over R_0. By definition R_0 acts on M_0 via $\pi_0 : R_0 \twoheadrightarrow \mathbb{T}_0$, and $\mathbb{T}_0$ acts faithfully on M_0.

By Theorem II.5.5(2) we have $\operatorname{Supp}_{R_0} M_0 = R_0$. Thus for any prime ideal $\mathfrak{p} \subseteq R_0$, if $K(R_0/\mathfrak{p})$ is the fraction field of $R_0/\mathfrak{p}$ then $M_0 \otimes_{R_0} K(R_0/\mathfrak{p}) \neq 0$ and so as $K(R_0/\mathfrak{p})$ is a field, it acts faithfully on $M_0 \otimes_{R_0} K(R_0/\mathfrak{p})$. Hence $R_0/\mathfrak{p} \subseteq K(R_0/\mathfrak{p})$ acts faithfully on $M_0 \otimes_{R_0} K(R_0/\mathfrak{p})$ and thus on $M_0/\mathfrak{p}M_0$.

But now the action of $R_0/\mathfrak{p}$ on $M_0/\mathfrak{p}M_0$ still factors through a surjection $R_0/\mathfrak{p} \twoheadrightarrow \mathbb{T}_0/\mathfrak{p}T_0$ and so we see that $R_0/\mathfrak{p} \cong \mathbb{T}_0/\mathfrak{p}\mathbb{T}_0$ for all primes $\mathfrak{p} \subseteq R_0$. This implies that the kernel of $\pi_0 : R_0 \to \mathbb{T}_0$ is in the intersection of all prime ideals of R_0, and thus is nilpotent, so we indeed get $R_0^{\text{red}} \xrightarrow{\sim} \mathbb{T}_0^{\text{red}}$, proving (1).

Now assume that R_∞ is a generically smooth cover of $\mathscr{R}$. Then by Theorem II.6.2, $M_0 \otimes_{\mathcal{O}} E$ is free over $R_0[1/\varpi]$, and so $R_0[1/\varpi]$ certainly acts faithfully on $M_0 \otimes_{\mathcal{O}} E$. As the action of $R_0[1/\varpi]$ on $M_0 \otimes_{\mathcal{O}} E$ still factors through the map $R_0[1/\varpi] \twoheadrightarrow \mathbb{T}_0[1/\varpi]$ induced by π_0, this map must be an isomorphism, proving (2).

In particular, if R_∞ is a generically smooth cover, then $\ker \pi_0 \subseteq R_0$ must be ϖ^N-torsion for some N. But now if we further assume that R_∞ is Cohen–Macaulay, Theorem II.5.5(4) implies that R_0 is ϖ-torsion free, and hence $\ker \pi_0 = 0$. So indeed if R_∞ is generically smooth and CM, then π_0 is an isomorphism $R_0 \xrightarrow{\sim} \mathbb{T}_0$, proving (3). □

II.9 Duality

For applications of patching beyond automorphy lifting (e.g. computing multiplicities) it is often necessary to precisely determine the R_∞-module structure of a patched module $\mathscr{P}(\mathscr{M})$. Theorem II.5.5 gives this structure in the case when R_∞ is regular, but it does not give enough information to determine this structure in general.

In many cases that arise in practice, the modules M_n satisfy some form of self-duality, which can be used to impose extra restrictions on the module $\mathscr{P}(\mathscr{M})$, and even precisely determine it in many cases (see [Man19]).

To study duality we make the following definitions:

Definition II.9.1. Let $\mathscr{M} = \{M_n\}_{n\geq 1}$ be an MCM weak patching system. We define

$$\mathscr{M}^* = \{M_n^*\}_{n\geq 1} = \big\{\operatorname{Hom}_{S_\infty}(M_n, S_\infty/\mathcal{I}_n)\big\}_{n\geq 1},$$

and note that this is clearly also an MCM weak patching module. If $\mathscr{M}$ is an MCM weak patching $\mathscr{R}$-module, for some weak patching algebra $\mathscr{R} = \{R_n\}_{n\geq 1}$, we will treat $\mathscr{M}^*$ as an MCM weak patching $\mathscr{R}$-module, by letting R_n act on $M_n^* = \operatorname{Hom}_{S_\infty}(M_n, S_\infty/\mathcal{I}_n)$ by $(rf)(x) = f(rx)$.

If $\mathscr{M}$ is an MCM weak patching MCM $\mathscr{R}$-module we say that $\mathscr{M}$ is *self-dual* if $\mathscr{M} \cong \mathscr{M}^*$ as weak patching $\mathscr{R}$-modules.

Note that for any weak MCM patching module $\mathscr{M}$, we clearly have a natural isomorphism $\mathscr{M}^{**} \cong \mathscr{M}$, compatible with the $\mathscr{R}$-module structure in the case when $\mathscr{M}$ is a weak MCM patching $\mathscr{R}$-algebra.

From now on, if A is any local Cohen–Macaulay $\mathcal{O}$-algebra, we will let ω_A denote its dualizing module (which will always exist if A is complete and topologically finitely generated over $\mathcal{O}$). We will need the following easy lemma in our discussion below:

Lemma II.9.2. *If A is a local Cohen–Macaulay ring and B is an A-algebra which is also Cohen–Macaulay with $\dim A = \dim B$, then for any B-module M,*

$$\operatorname{Hom}_A(M, \omega_A) \cong \operatorname{Hom}_B(M, \omega_B)$$

as left $\operatorname{End}_B(M)$-modules.

Proof. By [Sta19, Tag 08YP] there is an isomorphism

$$\operatorname{Hom}_A(M, \omega_A) \cong \operatorname{Hom}_B(M, \operatorname{Hom}_A(B, \omega_A))$$

sending $\alpha : M \to \omega_A$ to $\alpha' : m \mapsto (b \mapsto \alpha(bm))$, which clearly preserves the action of $\operatorname{End}_B(M)$ (as $(\alpha\circ\psi)(bm) = \alpha(b\psi(m))$ for any $\psi \in \operatorname{End}_B(M)$). It remains to show that $\operatorname{Hom}_A(B, \omega_A) \cong \omega_B$, which is just Theorem 21.15 from [Eis95] in the case $\dim A = \dim B$. $\square$

Before going on, we should say how this notion of duality arises in practice, as our definition above takes a somewhat different form. Consider the setup and notation from Section II.2. In particular, assume that the ideals $\mathcal{I}_n \subseteq S_\infty$ all take the form $\mathcal{I}_n = \mathcal{I}_n^\circ S_\infty$ and moreover that the ideals $\mathcal{I}_n^\circ \subseteq S_\infty^\circ$ satisfy the condition:

For all $n \geq 1$, $S_\infty^\circ/\mathcal{I}_n^\circ$ is a finite free $\mathcal{O}$-module and $\mathcal{I}_n^\circ$ is generated by d° elements. $(\star\star)$

and note that the system of ideals constructed in Lemma II.1.1 clearly satisfy $(\star\star)$. Now we have

Proposition II.9.3. *Assume that we are given:*

- *An unframed weak patching algebra* $\mathscr{R}^\circ = \{R_n^\circ\}_{n\geq 1}$.
- *Unframed MCM weak patching* $\mathscr{R}^\circ$*-modules* $\mathscr{M}^\circ = \{M_n^\circ\}_{n\geq 1}$ *and* $\mathscr{N}^\circ = \{N_n^\circ\}_{n\geq 1}$
- *For each* $n \geq 1$ *an* R_n*-equivariant perfect pairing* $\langle\ ,\ \rangle_n : M_n \times N_n \to \mathcal{O}$

Let $\mathscr{R}^\square = (\mathscr{R}^\circ)^\square$, $\mathscr{M}^\square = (\mathscr{M}^\circ)^\square$ *and* $\mathscr{N}^\square = (\mathscr{N}^\circ)^\square$.

If the ideals $\mathcal{I}_n^\circ \subseteq S_\infty^\circ$ *satisfy* ($\star\star$) *then* $(\mathscr{M}^\square)^* \cong \mathscr{N}^\square$ *as weak patching* $\mathscr{R}^\square$*-modules.*

Proof. The R_n-equivariant perfect $\langle,\rangle_n : M_n \times N_n \to \mathcal{O}$ implies that $N_n^\circ \cong \mathrm{Hom}_\mathcal{O}(M_n^\circ, \mathcal{O})$ as R_n-modules. Tensoring with $\mathcal{O}[[t_{d^\circ+1}, \dots, t_d]]$ over $\mathcal{O}$ implies that we have isomorphisms

$$\begin{aligned} N_n^\square = N_n^\circ \otimes_\mathcal{O} \mathcal{O}[[t_{d^\circ+1}, \dots, t_d]] &\cong \mathrm{Hom}_{\mathcal{O}[[t_{d'+1},\dots,t_d]]}\left(M_n^\circ \otimes_\mathcal{O} \mathcal{O}[[t_{d^\circ+1}, \dots, t_d]], \mathcal{O}[[t_{d'+1}, \dots, t_d]]\right) \\ &\cong \mathrm{Hom}_{\mathcal{O}[[t_{d'+1},\dots,t_d]]}\left(M_n^\square, \mathcal{O}[[t_{d'+1}, \dots, t_d]]\right) \end{aligned}$$

$R_n^\square = R_n^\circ \otimes_\mathcal{O} \mathcal{O}[[t_{d'+1}, \dots, t_d]]$-modules.

Now we have $S_\infty/\mathcal{I}_n = S_\infty/\mathcal{I}_n^\circ S_\infty^\circ \cong (S_\infty^\circ/\mathcal{I}_n^\circ) \otimes_\mathcal{O} \mathcal{O}[[t_{d'+1}, \dots, t_d]]$. It follows from ($\star\star$) that $S_\infty^\circ/\mathcal{I}_n^\circ$ is a complete intersection of relative dimension 0 over $\mathcal{O}$, and so $S_\infty/\mathcal{I}_n$ is also a complete intersection with $\dim S_\infty/\mathcal{I}_n = d - d^\circ + 1 = \dim \mathcal{O}[[t_{d'+1}, \dots, t_d]]$. In particular, $S_\infty/\mathcal{I}_n$ is Gorenstein, and so $\omega_{S_\infty/\mathcal{I}_n} \cong S_\infty/\mathcal{I}_n$. Lemma II.9.2 now implies that

$$N_n^\square \cong \mathrm{Hom}_{\mathcal{O}[[t_{d'+1},\dots,t_d]]}\left(M_n^\square, \mathcal{O}[[t_{d'+1}, \dots, t_d]]\right) \cong \mathrm{Hom}_{S_\infty/\mathcal{I}_n}\left(M_n^\square, S_\infty/\mathcal{I}_n\right)$$

for all $n \geq 1$. So indeed $(\mathscr{M}^\square)^* \cong \mathscr{N}^\square$ as weak patching $\mathscr{R}^\square$-modules. □

We are now ready to show that patching preserves duality:

Theorem II.9.4. *Let* $\mathscr{R}$ *be a weak patching algebra and let* $\mathscr{M}$ *be an MCM patching* $\mathscr{R}$*-module. Then we have* $\mathscr{P}(\mathscr{M}^*) \cong \mathrm{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)$ *as* $\mathscr{P}(\mathscr{R})$*-modules.*

Furthermore, if R_∞ *is a CM cover of* $\mathscr{R}$ *then* $\mathscr{P}(\mathscr{M}^*) \cong \mathrm{Hom}_{R_\infty}(\mathscr{P}(\mathscr{M}), \omega_{R_\infty})$ *as* R_∞*-modules.*

Proof. We shall first compute $\mathcal{U}(\mathscr{M}^*/\mathfrak{a})$ for any open ideal $\mathfrak{a} \subseteq S_\infty$. For any such $\mathfrak{a}$, we have $\mathcal{I}_n \subseteq \mathfrak{a}$ for all but finitely many n, and so $S_\infty/\mathfrak{a}$ is a $S_\infty/\mathcal{I}_n$-module for all but finitely many n.

But now for all n, M_n is finite free over $S_\infty/\mathcal{I}_n$ by assumption, and so it is projective. Thus the functor $\mathrm{Hom}_{S_\infty}(M_n, -) = \mathrm{Hom}_{S_\infty/\mathcal{I}_n}(M_n, -)$ is exact and so if $\mathcal{I}_n \subseteq \mathfrak{a}$ then

$$M_n^*/\mathfrak{a} = \mathrm{Hom}_{S_\infty}(M_n, S_\infty/\mathcal{I}_n)/\mathfrak{a} \cong \mathrm{Hom}_{S_\infty}(M_n, S_\infty/\mathfrak{a}) = \mathrm{Hom}_{S_\infty/\mathfrak{a}}(M_n/\mathfrak{a}, S_\infty/\mathfrak{a})$$

as $R_n/\mathfrak{a}$-modules.

Now by Proposition I.1.3, for $\mathfrak{F}$-many i we have that $\mathcal{U}(\mathscr{R}/\mathfrak{a}) \cong R_i/\mathfrak{a}$ and $\mathcal{U}(\mathscr{M}/\mathfrak{a}) \cong M_i/\mathfrak{a}$ and $\mathcal{U}(\mathscr{M}^*/\mathfrak{a}) \cong M_i^*/\mathfrak{a}$ as $R_i/\mathfrak{a}$-modules. Taking any such i, the above computation gives that

$$\mathcal{U}(\mathscr{M}^*/\mathfrak{a}) \cong \mathrm{Hom}_{S_\infty/\mathfrak{a}}\left(\mathcal{U}(\mathscr{M}/\mathfrak{a}), S_\infty/\mathfrak{a}\right)$$

as $\mathcal{U}(\mathscr{R}/\mathfrak{a})$-modules. Taking inverse limits, it now follows that

$$\mathscr{P}(\mathscr{M}^*) \cong \varprojlim_{\mathfrak{a}} \operatorname{Hom}_{S_\infty/\mathfrak{a}}\left(\mathcal{U}(\mathscr{M}/\mathfrak{a}), S_\infty/\mathfrak{a}\right)$$

as $\mathscr{P}(\mathscr{R})$-modules. It remains to show that the right hand side is just $\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)$. But using the fact that $\mathscr{P}(\mathscr{M})$, and thus $\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)$ is a finite free S_∞-module (and thus is m_{S_∞}-adically complete) we get that

$$\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty) \cong \varprojlim_{\mathfrak{a}} \operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)/\mathfrak{a}$$

as $\mathscr{P}(\mathscr{R}) = \varprojlim_{\mathfrak{a}} \mathscr{P}(\mathscr{R})/\mathfrak{a}$-modules. But now for any $\mathfrak{a}$, as $\mathscr{P}(\mathscr{M})$ is a finite free, and hence projective, S_∞-module

$$\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty)/\mathfrak{a} \cong \operatorname{Hom}_{S_\infty/\mathfrak{a}}(\mathscr{P}(\mathscr{M})/\mathfrak{a}, S_\infty/\mathfrak{a}) \cong \operatorname{Hom}_{S_\infty/\mathfrak{a}}\left(\mathcal{U}(\mathscr{M}/\mathfrak{a}), S_\infty/\mathfrak{a}\right)$$

as $\mathscr{P}(\mathscr{R})/\mathfrak{a} = \mathcal{U}(\mathscr{R}/\mathfrak{a})$-modules. So indeed

$$\operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty) \cong \varprojlim_{\mathfrak{a}} \operatorname{Hom}_{S_\infty/\mathfrak{a}}\left(\mathcal{U}(\mathscr{M}/\mathfrak{a}), S_\infty/\mathfrak{a}\right) \cong \mathscr{P}(\mathscr{M}^*)$$

as $\mathscr{P}(\mathscr{R})$-modules.

Now assume that R_∞ is a CM cover of $\mathscr{R}$. Then $\dim R_\infty = d+1 = \dim S_\infty$, so Lemma II.9.2 implies that

$$\mathscr{P}(\mathscr{M}^*) \cong \operatorname{Hom}_{S_\infty}(\mathscr{P}(\mathscr{M}), S_\infty) \cong \operatorname{Hom}_{R_\infty}(\mathscr{P}(\mathscr{M}), \omega_{R_\infty})$$

as R_∞-modules (where we have used the fact that $\omega_{S_\infty} = S_\infty$). □

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