Nonlinearity **16** (2003) 1823–1831 PII: S0951-7715(03)61100-X

Exact, infinite energy, blow-up solutions of the three-dimensional Euler equations

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Received 19 March 2003, in final form 30 June 2003
Published 28 July 2003
Online at stacks.iop.org/Non/16/1823

Recommended by K Ohkitani

Abstract

For the class of cylindrically symmetric velocity fields

$$\boldsymbol{U}(r, z, t) = \{u(r, t), v(r, t), z\gamma(r, t)\},$$

two infinite energy exact solutions of the three-dimensional incompressible Euler equations are exhibited that blow up at every point in space in finite time. The first solution is embedded within the second as a special case and in both cases $v = 0$. Both solutions represent three-dimensional vortices which take the form of hollow cylinders for which the vorticity vector is $\boldsymbol{\omega} = (0, \omega_\theta, 0)$. An analysis on characteristics shows how more general solutions can be constructed and analysed.

PACS numbers: 47.27.Ak, 03.40.Gc

1. Introduction

An important open question in inviscid fluid turbulence is whether the three-dimensional incompressible Euler equations

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{U} = -\nabla P, \qquad \operatorname{div}\boldsymbol{U} = 0, \tag{1}$$

develop a singularity in finite time. The pre-eminent theoretical result belongs to Beale *et al* [1] who showed that for the vorticity field $\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{U}$, the time integral of the maximum norm, $\int_0^t \|\boldsymbol{\omega}\|_\infty \,\mathrm{d}\tau$, must control any singularity that might develop in any variable, even in arbitrarily high derivatives. The reader is also referred to references in Majda and Bertozzi [2]. Kerr [3] has provided numerical support for the existence of a singularity by observing growth in the maximum norm (peak vorticity) like $\|\boldsymbol{\omega}\|_\infty \sim (t_0 - t)^{-1}$ from initial data consisting of a pair of perturbed parallel anti-parallel vortex tubes in a three-dimensional periodic domain. Pelz and Gulak [4] have pursued a different route by considering high symmetry flows; they used Kida's

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initial condition to follow the development of a real-time singularity using Padé methods of analysis. A further refinement of the BKM-criterion has been made by Constantin *et al* [5] who reduced the $\|\boldsymbol{\omega}\|_\infty$-norm to a finite L^p-norm ($1 \leqslant p \leqslant \infty$) at the price of needing to control the direction of vorticity. They showed that singularities are only possible if the misalignment between vortex lines is too great.

The above questions regarding the existence of singularities lie in the context of finite energy flows where a potential singularity would be localized in space. A different problem is that of singularities in infinite domains where blow-up might occur in some sections of the full three-dimensional domain, or even at every point. This latter class of singularities are infinite energy in nature; one of the tasks of this paper is to show that the three-dimensional Euler equations have exact solutions of this type. For the restricted class of velocity vectors where the z-coordinate appears only linearly in w and not elsewhere

$$\boldsymbol{U}(r,\,\theta,\,z,\,t) = \{u(r,\,\theta,\,t),\; v(r,\,\theta,\,t),\; w(r,\,\theta,\,z,\,t)\}, \tag{2}$$

$$w = z\gamma(r,\,\theta,\,t) \tag{3}$$

with u and v as radial and swirling components of velocity, respectively. The first and simplest cylindrically symmetric exact solution is

$$\gamma(r,\,t) = -\frac{\mathrm{e}^{-\alpha r^2}}{t_0 - t} \tag{4}$$

with corresponding velocity components

$$u(r,\,t) = \frac{1}{2\alpha r}\left(\frac{1 - \mathrm{e}^{-\alpha r^2}}{t_0 - t}\right), \qquad v = 0 \tag{5}$$

for any $\alpha \geqslant 0$. This solution becomes singular at a finite time $t_0 > 0$ and satisfies the boundary conditions $\gamma,\ u \to 0$ as $r \to \infty$ for $0 \leqslant t < t_0$. When $t_0 < 0$ solutions simply decay to zero. The second cylindrically symmetric exact solution, which has the first embedded within it, contains a further arbitrary parameter $\beta \geqslant 0$

$$\gamma(r,\,t) = \frac{1}{1+\beta t}\left\{\beta - \frac{\delta \mathrm{e}^{-\alpha(1+\beta t)r^2}}{1 - (\delta - \beta)t}\right\}. \tag{6}$$

Together with $v = 0$, the corresponding expression for u is

$$u(r,\,t) = \frac{1}{2(1+\beta t)}\left\{-\beta r + \frac{\delta(1 - \mathrm{e}^{-\alpha(1+\beta t)r^2})}{\alpha r(1+\beta t)[1 - (\delta - \beta)t]}\right\}. \tag{7}$$

These expressions for γ and u reduce to (4) and (5) when $\beta = 0$ and $t_0 = \delta^{-1}$. The singularity time t^* in this more general solution depends upon the relative signs and values of β and δ; the alternatives are listed in table 1. In general, neither of the expressions for γ and u decays to zero as $r \to \infty$ at some fixed pre-singular time. Nevertheless, they may represent an acceptable local cylindrically symmetric flow when embedded in a larger structure. Experience shows that exact solutions are always valuable for a variety of reasons.

Table 1. Singularity times t^* for the more general exact solution (6) and (7).

	$\beta < \delta$	$\beta > \delta$	$\beta = \delta$
$\beta > 0, \delta > 0$	$t^* = \dfrac{1}{\delta - \beta}$	No singularity	No singularity
$\beta < 0, \delta < 0$	$t^* = -\dfrac{1}{\beta}$	$t^* = -\dfrac{1}{\beta}$	$t^* = -\dfrac{1}{\beta}$
$\beta < 0,\ \delta > 0$	$t^* = \dfrac{1}{\delta - \beta}$		
$\beta > 0, \delta < 0$		No singularity	

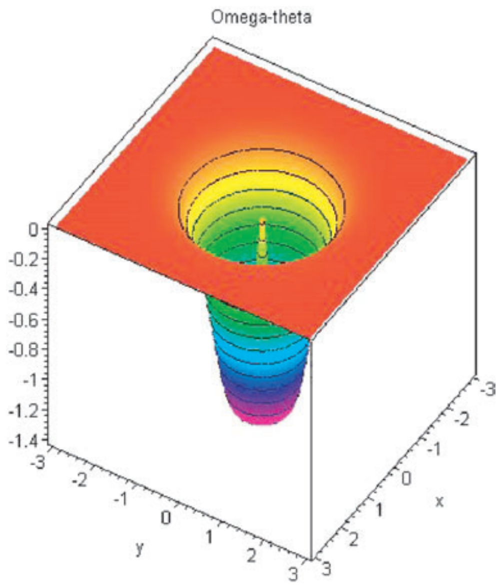

Figure 1. Plot of ω for the solution (6) and (7) at $z = 1$ with $\alpha = 1$ and $\beta = \frac{1}{2}$.

The full three-dimensional vorticity field corresponding to cylindrically symmetric velocity fields as in (2) and (3) is

$$\boldsymbol{\omega} = (0,\ \omega_\theta,\ 0), \qquad \omega_\theta = -z\frac{\partial \gamma}{\partial r}, \tag{8}$$

so both solutions represent stretched vortex rings that take the specific form of infinite hollow cylinders. It is also easy to calculate the pressure field and show that it is regular at $r = 0$. In section 2 a proof is given of the veracity of the two exact solutions, putting this in context with previous work (figure 1). The particle paths corresponding to these two solutions can be found from an analysis on characteristics which is performed in section 3. This also shows how simple the problem becomes when cast in the form

$$\gamma = \frac{\gamma_0}{1 + t\gamma_0}, \tag{9}$$

where γ_0 is a function of $\phi(t, r)$, a characteristic variable. Both exact solutions correspond to $v = 0$ but in section 4 it is shown how a class of corresponding non-zero solutions for v can be constructed that are regular at $r = 0$. These solutions have extra components of vorticity in addition to the azimuthal one in (8). Characteristic analysis also raises the question regarding the singularity process for initial data more general than the Gaussian-shaped profiles of γ in (5) and (6).

2. Previous work and the proof of (6)

How can it be proved that equations (6) and (7) for γ and u are corresponding solutions of the Euler equations and what is the historical context? Stuart [6] considered solutions of the three-dimensional problem that had linear dependence in two variables x and z. The resulting differential equations in the remaining independent variables y and t displayed finite time singular behaviour. Childress *et al* [7] likewise analysed the two-dimensional problem in a similar manner. Gibbon *et al* [8] considered the three-dimensional problem as in (2) with an extension to w in which $w = z\gamma(r, \theta, t) + \sigma(r, \theta, t)$. The exact solutions stated above lie within this class with $\sigma = 0$. Their veracity can be confirmed by looking at the work in [8] in the context of cylindrical co-ordinates. The two-component velocity field $\boldsymbol{u}(r, \theta, t) = (u, v)$ allows a two-dimensional material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}, \tag{10}$$

to be defined, where $\boldsymbol{\nabla}$ is now a two-dimensional operator. The third component of the Euler equations, (1), upon integration with respect to z, gives an expression for the pressure P that is quadratic in z

$$P(r, \theta, z, t) = \frac{1}{2}z^2\left(\frac{D\gamma}{Dt} + \gamma^2\right) + z\left(\frac{D\sigma}{Dt} + \gamma\sigma\right) + p(r, \theta, t). \tag{11}$$

A contradiction can only be avoided if the coefficients of z^2 and z in (11) are uniform in space although they can be arbitrary functions of time. The first of these arbitrary functions of time is designated as $f(t)$ while the second will be put equal to zero; but see (43)–(45). The variables γ and $\boldsymbol{u}$ satisfy

$$\frac{D\gamma}{Dt} + \gamma^2 = f(t), \tag{12}$$

$$\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}p, \qquad \operatorname{div}\boldsymbol{u} = -\gamma. \tag{13}$$

In addition, σ and ω_3, the third component of the vorticity vector, satisfy

$$\frac{D\sigma}{Dt} = -\gamma\sigma, \qquad \frac{D\omega_3}{Dt} = \gamma\omega_3. \tag{14}$$

The result for ω_3 can be demonstrated by direct calculation using the vortex stretching vector $\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}$. Equations (12)–(14) are those derived in [8]. It has subsequently been pointed out in [9] that a time-independent version of these equations with $\sigma = 0$ can also be found in an appendix in Oseen [10].

Equations (12)–(14) have some interesting properties which have been discussed by Gibbon *et al* [8]. This was originally motivated by the work of Moffatt *et al* [11], who have performed an extensive analysis of the Burgers vortex. In regions where $\gamma > 0$, equation (14) shows that ω_3 stretches and σ compresses, thereby producing a dynamically stretched vortex with a rich internal structure. In contrast, in regions where $\gamma < 0$, the reverse process occurs and a ring-like object is produced. The dynamics of γ is therefore an important issue. If $f(t)$ is left undetermined, γ could be chosen as an arbitrary function of time and uniform in space.

Delbende *et al* [12] have exploited this by choosing γ such that it goes through a compressive phase ($0 \leqslant t \leqslant t^*$) and then a stretching phase ($t^* \geqslant t$) and found that the resulting Burgers vortices have a braided character.

The exact solution (6) corresponds to $f(t) = 0$ with $\sigma = 0$. Direct substitution shows that the three conditions in (12) and (13) are satisfied. $\omega_3 = 0$ so equation (14) is redundant. Once this step has been achieved, the first solution (5) is automatically true because it corresponds to the special case $\beta = 0$.

The appropriate domain for the two exact solutions (6) and (5) corresponding to $f(t) = 0$ is infinite but this choice of $f(t)$ needs to be reconsidered when a finite boundary is imposed. Ohkitani and Gibbon [9] considered a cross-sectional domain $\mathcal{A}$ with periodic boundary conditions imposed; in this case a circular boundary of radius L. The imposition of such a boundary changes the problem because the divergence theorem with periodic boundary conditions applied to the incompressibility condition $\operatorname{div} \boldsymbol{u} = -\gamma$ means that γ is a mean-zero function, namely

$$\int_{\mathcal{A}} \gamma \, \mathrm{d}A = 0. \tag{15}$$

In turn, this constraint applied to the equation for γ in (12) makes

$$f(t) = 2(\pi L^2)^{-1} \int_{\mathcal{A}} \gamma^2 \, \mathrm{d}A \tag{16}$$

and so γ satisfies

$$\frac{D\gamma}{Dt} + \gamma^2 = \frac{2}{\pi L^2} \int_{\mathcal{A}} \gamma^2 \, \mathrm{d}A. \tag{17}$$

Numerical integration of (17) by Ohkitani and Gibbon [9] suggested that $\gamma \to -\infty$ in finite time. Constantin [13] subsequently used integration along characteristics to prove that γ does indeed blow up to $-\infty$ but he also showed that the process is, in fact, two-sided in the sense that $\gamma \to +\infty$ simultaneously elsewhere in the domain. The positive blow-up is a process that begins only at a very late stage. Similar numerical behaviour has been observed by Gibbon and Ohkitani [14] in the equations for ideal magnetohydrodynamics but no analytical proof of blow-up has yet been found.

3. Characteristics

In order to expose the nature of solutions of the incompressible Euler equations (1), at least for cylindrically symmetric cases of the form (2) with θ omitted, it is helpful to study the relevant scalar equations afresh.

We consider the velocity field to have the form (2), with no dependence on θ and with the pressure given by

$$p = p(r, t). \tag{18}$$

An extension to the case where $p \to \frac{1}{2} z^2 f(t) + p(r, t)$ as in (11) is readily achievable, but here we restrict attention to (18). In cylindrical co-ordinates, equations (13) and (12), with $f(t) = 0$, yield

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} = -\frac{\partial p}{\partial r}, \tag{19}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} = 0, \tag{20}$$

$$\frac{\partial \gamma}{\partial t} + u\frac{\partial \gamma}{\partial r} + \gamma^2 = 0, \tag{21}$$

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \gamma = 0. \tag{22}$$

It is seen that (21) and (22) decouple; if u and γ were known, v would readily follow from (20) and p from (19). If we eliminate γ by use of (22) in (21), we obtain

$$\frac{\partial^2 u}{\partial t \partial r} + u\frac{\partial^2 u}{\partial r^2} + \text{lower derivatives} = 0, \tag{23}$$

an equation which has a hyperbolic operator. The characteristics are given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u, \qquad \mathrm{d}t = 0, \tag{24}$$

the former of which gives the particle paths. Although the flow is incompressible, it is of hyperbolic character, a consequence of the presence of vorticity. Indeed the vorticity field is given by

$$\boldsymbol{\omega} = (0,\, \omega_\theta,\, \omega_z)\,, \qquad \omega_\theta = -z\frac{\partial \gamma}{\partial r}, \quad \omega_z = \frac{1}{r}\frac{\partial (rv)}{\partial r}, \tag{25}$$

which is a generalization of (8) to include the azimuthal flow v.

We proceed now to solve (21) and (22) in a general manner. Let us use the transformation

$$\gamma = \frac{\gamma_0(t, r)}{1 + t\gamma_0(t, r)} \tag{26}$$

and substitute in (21), which becomes

$$\frac{\partial \gamma_0}{\partial t} + u\frac{\partial \gamma_0}{\partial r} = 0. \tag{27}$$

This has the characteristic (24), which can be used to effect a solution of (27). Suppose for the moment that u is known; then a solution of (24) is

$$\phi(t, r) = \text{const.}, \tag{28}$$

where ϕ is a Lagrangian variable associated with the particle paths. It follows that

$$\gamma = \frac{\gamma_0(\phi)}{1 + t\gamma_0(\phi)}. \tag{29}$$

Moreover, if, at $t = 0$, $\gamma(0,\, \phi) = \gamma_0(\phi)$ and $\phi(0,\, r) = r$, then it is seen that $\gamma_0(\phi)$ plays the role of an initial condition. It is clear from this formula for γ that a singularity (a pole) could occur if the denominator were zero. Since an initial condition is specified at $t = 0$, a necessary requirement is that $\gamma_0(\phi)$ must have a negative region. But we do not yet know ϕ, the Lagrangian (particle) co-ordinate in terms of r and t. Can ϕ achieve a value at some time, say t_0, in the region where $\gamma_0(\phi)$ is negative?

We now discuss this question, and in so doing relate our discussion to the solutions given in section 1. The continuity equation (22) is the focus of our attention. We note that the velocity component u is given by

$$u = \frac{\partial}{\partial t} r(t,\, \phi), \tag{30}$$

where the radial co-ordinate r is replaced as a function of time, t, and the Lagrangian variable, ϕ; so $r(\gamma,\, \phi)$ comes from the inversion of (28). Then

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\frac{\partial}{\partial t} r(t,\, \phi) = \left[\frac{\partial}{\partial \phi}\frac{\partial}{\partial t} r(t,\, \phi)\right]\left(\frac{\partial r}{\partial \phi}\right)^{-1} = \frac{\partial}{\partial t}\ln\left(\frac{\partial r}{\partial \phi}\right). \tag{31}$$

Also

$$\frac{u}{r} = \frac{1}{r}\frac{\partial r}{\partial t} = \frac{\partial}{\partial t}\ln r, \tag{32}$$

so that (22) becomes

$$\frac{\partial}{\partial t}\ln\left(r\frac{\partial r}{\partial \phi}\right) + \gamma = 0. \tag{33}$$

This can be integrated with respect to t to yield

$$r\frac{\partial r}{\partial \phi} = \frac{G(\phi)}{1 + t\gamma_0(\phi)}, \tag{34}$$

where $G(\phi)$ is a 'constant' of the time integration. An integration with respect to ϕ yields

$$r^2(t,\ \phi) = \int_{\phi_0}^{\phi} \frac{2G(\phi)\,\mathrm{d}\phi}{1 + t\gamma_0(\phi)}, \tag{35}$$

imposition of $\phi = r$ at $t = 0$ requires $G(\phi) \equiv \phi$ and $\phi_0 \equiv 0$, so that

$$r^2(t,\ \phi) = \int_{0}^{\phi} \frac{2\phi\,\mathrm{d}\phi}{1 + t\gamma_0(\phi)}. \tag{36}$$

This formula implies ϕ in terms of t and r by inversion.

It may be noted here that (29) may be extended to the non-axisymmetric case, when there are two Lagrangian variables, ϕ and ψ. However, a result corresponding to (34) does not follow straightforwardly. It is also the case that (29) applies for a compressible fluid, particularly for barotropic (including adiabatic) cases in which the pressure and density are related. For a discussion of a related compressible problem, see [15].

Let us now turn to the two examples given by equations (5) and (6). If we set

$$\gamma_0(\phi) = -\frac{\mathrm{e}^{-\alpha\phi^2}}{t_0}, \qquad \alpha > 0, \tag{37}$$

we find that (36) can be integrated exactly to yield

$$t_0\mathrm{e}^{\alpha\phi^2} = t + (t_0 - t)\mathrm{e}^{\alpha r^2} \tag{38}$$

and then it follows from (29), (37) and (38) that

$$\gamma = -\frac{\mathrm{e}^{-\alpha r^2}}{t_0 - t}, \tag{39}$$

which is (4). Utilization of (22) gives exactly the expression for u in (5)

$$u(r,\ t) = \frac{1}{2\alpha r}\left(\frac{1 - \mathrm{e}^{-\alpha r^2}}{t_0 - t}\right), \tag{40}$$

while (20) can be used to calculate v. A singularity in both γ and u is clear for $t \to t_0$ and at all values of r.

The more general example in (6) and (7) is obtained by choosing

$$\gamma_0(\phi) = \beta - \delta\mathrm{e}^{-\alpha\phi^2}, \qquad \alpha > 0. \tag{41}$$

We find that (36) yields

$$\mathrm{e}^{\alpha\phi^2} = 1 + \frac{1 + (\beta - \delta)t}{1 + \beta t}(\mathrm{e}^{\alpha(1+\beta t)r^2} - 1). \tag{42}$$

From (29), (41) and (42) we obtain (6) and (7).

We pursue equation (14) further and note that if (3) is replaced (in the axisymmetric case) with

$$w = z\gamma(r,t) + \sigma(r,t), \tag{43}$$

then σ satisfies

$$\frac{\partial\sigma}{\partial t} + u\frac{\partial\sigma}{\partial r} + \gamma\sigma = 0, \tag{44}$$

the solution being

$$\sigma = \frac{\sigma_0(\phi)}{1 + t\gamma_0(\phi)}. \tag{45}$$

Here $\sigma_0(\phi) = \sigma(r,0)$ and $\phi(0,r) = r$. Since σ_0 may be different from γ_0, the solution (43) does not represent a mere shift of the z-coordinate. However, no important change in the singular structure emerges.

4. Swirling component of velocity

By analytical and computational methods, and by use of characteristics, we have found singular solutions of a certain type. Thus the solution formulae for $\gamma(r,t)$ and $u(r,t)$ are given by (6) and (7), while the characteristic variable $\phi(r,t)$ is given by (42); these formulae apply to the more general case ($\beta \neq 0$), but the simpler example can be obtained by setting $\beta = 0$ and setting $t_0 = \delta^{-1}$. Let us now turn to the calculation of the swirl velocity, $v(r,t)$, and the pressure $p(r,t)$, as given by (20) and (19).

We note that equation (20) can be written

$$\frac{\partial(rv)}{\partial t} + u\frac{\partial(rv)}{\partial r} = 0, \tag{46}$$

which has the same characteristic as equation (27). Thus

$$rv = G(\phi), \tag{47}$$

where G is an arbitrary function. We note, however, that it can be determined by the given initial condition

$$v = v_0(r), \qquad t = 0 \tag{48}$$

and by $\phi(0,r) = r$. So we have

$$v = \frac{\phi^2}{r}v_0(\phi), \tag{49}$$

where $\phi(t,r)$ is given by (42) in the case under consideration.

By way of example, we consider a particular condition (48), say

$$v_0(r) = r\mathrm{e}^{-\mu r^2}, \qquad \mu > 0. \tag{50}$$

Then (49) yields

$$v(r,t) = \frac{\phi^2}{r}\mathrm{e}^{-\mu\phi^2}, \tag{51}$$

where ϕ is given by (42). It can be shown that when $r \to \infty$ with t fixed,

$$v \sim (1+\beta t)r\mathrm{e}^{-\mu(1+\beta t)r^2}, \tag{52}$$

where $1 + \beta t > 0$. On the other hand, when $r \to 0$ with t fixed

$$v \sim r\left[1 + (\beta - \delta)t\right], \tag{53}$$

so that v remains finite on the axis. The pressure can be determined from (19) by integration, with u given by (7) and v by (51) with (42).

Acknowledgments

The authors would like to thank Professor Steve Cowley of Imperial College's Physics Department for his help and suggestions.

References

[1] Beale J T, Kato T and Majda A 1984 Remarks on the breakdown of smooth solutions for the 3D Euler equations *Commun. Math. Phys.* **94** 61–6
[2] Majda A J and Bertozzi A 2002 *Vorticity and Incompressible Flow* (Cambridge: Cambridge University Press)
[3] Kerr R 1993 Evidence for a singularity of the 3-dimensional, incompressible Euler equations *Phys. Fluids* A **5** 1725–46
[4] Pelz R B and Gulak Y 1997 Evidence for a real-time singularity in hydrodynamics from time series analysis *Phys. Rev. Lett.* **79** 4998
[5] Constantin P, Fefferman Ch and Majda A 1996 Geometric constraints on potentially singular solutions for the 3D Euler equations *Commun. Partial. Diff. Eqns* **21** 559–71
[6] Stuart J T 1987 Nonlinear Euler partial differential equations: singularities in their solution *Proc. Symp. Honor of C C Lin* ed D J Benney *et al* (Singapore: World Scientific) pp 81–95
[7] Childress S, Ierley S, Spiegel E and Young W 1989 Blow-up of unsteady two-dimensional Euler and Navier–Stokes solutions having stagnation point form *J. Fluid Mech.* **203** 1–22
[8] Gibbon J D, Fokas A and Doering C R 1999 Dynamically stretched vortices as solutions of the 3D Navier–Stokes equations *Physica* D **132** 497–510
[9] Ohkitani K and Gibbon J D 2000 Numerical study of singularity formation in a class of Euler and Navier–Stokes flows *Phys. Fluids* **12** 3181–94
[10] Oseen C W 1927 Exakte Lösungen der hydrodynamischen Differentialgleichungen. II *Arkiv für Matematik, Astronomi och Fysik* A **20** 1
[11] Moffatt H K, Kida S and Ohkitani K 1994 Stretched vortices—the sinews of turbulence; large-Reynolds-number asymptotics *J. Fluid Mech.* **259** 241
[12] Delbende I, Rossi M and le Dizès S 2002 Stretching effects on the three-dimensional stability of vortices with axial flow *J. Fluid Mech.* **454** 419–42
[13] Constantin P 2000 The Euler equations and nonlocal conservative Riccati equations *Internat. Math. Res. Notices* **9** 455–65
[14] Gibbon J D and Ohkitani K 2001 Singularity formation in a class of stretched solutions of the equations for ideal MHD *Nonlinearity* **14** 1239–64
[15] Stuart J T 1998 Singularities in three-dimensional compressible Euler flows with vorticity *Theoret. Comput. Fluid Dyn.* **10** 385–91