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A quaternionic structure in the three-dimensional Euler and ideal magneto-hydrodynamics equations

J.D. Gibbon*

Department of Mathematics, Imperial College of Science, Technology and Medicine, London SW7 2BZ, UK

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Abstract

By considering the three-dimensional incompressible Euler equations, a 4-vector $\boldsymbol{\zeta}$ is constructed out of a combination of scalar and vector products of the vorticity $\boldsymbol{\omega}$ and the vortex stretching vector $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = S\boldsymbol{\omega}$. The evolution equation for $\boldsymbol{\zeta}$ can then be cast naturally into a quaternionic Riccati equation. This is easily transformed into a quaternionic zero-eigenvalue Schrödinger equation whose potential depends on the Hessian matrix of the pressure. With minor modifications, this system can alternatively be written in complex notation. An infinite set of solutions of scalar zero-eigenvalue Schrödinger equations has been found by Adler and Moser, which are discussed in the context of the present problem. Similarly, when the equations for ideal magneto-hydrodynamics (MHD) are written in Elsasser variables, a pair of 4-vectors $\boldsymbol{\zeta}^{\pm}$ are governed by coupled quaternionic Riccati equations.

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1. Quaternions and the three-dimensional Euler equations

1.1. Introduction

Quaternions are 4-vectors whose multiplication rules are governed by a simple non-commutative division algebra. The concept was originally invented by Hamilton to generalize complex numbers to $\mathbb{R}^4$. The purpose of this paper is to demonstrate that the three-dimensional incompressible Euler equations have a natural quaternionic Riccati structure in the dependent variable. To convince the reader that this structure is robust and no accident, it is shown in Section 2 that the more complicated equations for ideal magneto-hydrodynamics (MHD) can also be written in an equivalent quaternionic form. Roubtsov and Roulstone [1,2] have recently shown that a quaternionic formulation, different from the one in this paper, can be made for two-dimensional nearly geostrophic flows whose origin lies in balanced models of the atmosphere. They have also shown that a Kähler structure can be associated with this.

* Tel.: +44-207-594-8504; fax: +44-207-594-8517.
E-mail address: j.d.gibbon@ic.ac.uk (J.D. Gibbon).

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The existence of quaternionic structures in the Euler equations makes it tempting to speculate that the advances that have been made in the geometry of 4-manifolds in recent years might be applicable to the long enduring and puzzling phenomena of fluid and MHD turbulence.

1.2. A relation between the strain matrix and the pressure Hessian

We begin this section by writing down the three-dimensional Euler equations that relate the velocity vector $\boldsymbol{u}(x, y, z, t)$ to the pressure $p(x, y, z, t)$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p \tag{1}$$

together with the incompressiblity condition $\operatorname{div} \boldsymbol{u} = 0$. The material derivative in (1) is defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla. \tag{2}$$

Eq. (1) can be re-formulated in terms of the vorticity vector $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = S\boldsymbol{\omega}. \tag{3}$$

The *ij*th element of the strain matrix S in (3) is given by

$$S_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{4}$$

which constitutes the symmetric part of the velocity gradient matrix $u_{i,j}$. Let us begin by defining the scalar α and the 3-vector $\boldsymbol{\chi}$ as (see [3,4])

$$\alpha(\boldsymbol{x}, t) = \frac{\boldsymbol{\omega} \cdot S\boldsymbol{\omega}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}, \qquad \boldsymbol{\chi}(\boldsymbol{x}, t) = \frac{\boldsymbol{\omega} \times S\boldsymbol{\omega}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}. \tag{5}$$

To find the evolution of $S\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ requires a result, generally credited to Ertel [5] in geophysical fluid dynamics, that says that if $\boldsymbol{\omega}$ evolves according to (3) then any arbitrary scalar μ satisfies

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega} \cdot \nabla \mu) = \boldsymbol{\omega} \cdot \nabla \left(\frac{\mathrm{D}\mu}{\mathrm{D}t}\right). \tag{6}$$

Consequently, if μ is a material constant then $\boldsymbol{\omega} \cdot \nabla \mu$ must also be a material constant. Depending on how μ is chosen, the scalar quantity $\boldsymbol{\omega} \cdot \nabla \mu$ is generally referred to as the potential vorticity.[1] For our purposes, however, let μ be chosen to be the ith component of the velocity field $\mu = u_i$. Because this evolves according to (1), the vortex stretching vector $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = S\boldsymbol{\omega}$ obeys

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}) = -P\boldsymbol{\omega}, \tag{7}$$

where $P = \{p_{,ij}\}$ is the Hessian matrix of the pressure. The form of the result stated in (7) can be found in [13] (see also [12]) Ohkitani also pointed out that $\boldsymbol{\omega}$ must satisfy

$$\frac{\mathrm{D}^2\boldsymbol{\omega}}{\mathrm{D}t^2} + P\boldsymbol{\omega} = 0. \tag{8}$$

[1] To understand the meaning of the concept of potential vorticity and its uses in geophysical fluid dynamics see the review by Hoskins et al. [6]. Two recent papers by Viudez [7,8] contain a discussion of the history of this result, its connection with the Cauchy formula, and the work of Ertel [5] and Rossby [9]. Viudez has suggested that Beltrami [10] introduced the idea of potential vorticity in a different vectorial form as far back as 1871, although the commutation formula (6) was first written down by Ertel [5] (see also [11,12]).

The apparent linearity of (8) is illusory because the dependence of P on the pressure means that it is connected to $\boldsymbol{\omega}$ through the Poisson equation

$$-\Delta p = u_{i,j}u_{j,i}, \tag{9}$$

which arises through application of the incompressibility condition $\operatorname{div}\boldsymbol{u} = 0$. Eq. (9) can be re-expressed as

$$\operatorname{Tr} P = -\operatorname{Tr} S^2 + \tfrac{1}{2}\omega^2, \tag{10}$$

illustrating the relation between P, $\boldsymbol{\omega}$ and S. While it is not obvious how to reconstruct $\boldsymbol{\omega}$ or $\boldsymbol{u}$ from a knowledge of α and $\boldsymbol{\chi}$ there is a more natural relationship between this latter pair and the spectrum of S. If S has exact eigenvalues $\lambda_3 \le \lambda_2 \le \lambda_1$ then the incompressibility condition insists that $\operatorname{Tr} S = \lambda_1 + \lambda_2 + \lambda_3 = 0$. Hence $\lambda_1 \ge 0$ and $\lambda_3 \le 0$ with λ_2 of variable sign. From its definition in (5), α is a Rayleigh's quotient estimate for an eigenvalue of S, which is bounded within the spectrum of S, such that $\lambda_3 \le \alpha \le \lambda_1$. Moreover, the combination

$$\alpha^2 + \chi^2 = \frac{|S\boldsymbol{\omega}|^2}{|\boldsymbol{\omega}|^2} = \frac{\boldsymbol{\omega}\cdot S^2\boldsymbol{\omega}}{\boldsymbol{\omega}\cdot\boldsymbol{\omega}} \tag{11}$$

is bounded by

$$|\lambda_2|^2 \le \alpha^2 + \chi^2 \le \max\{\lambda_1^2, |\lambda_3|^2\}. \tag{12}$$

1.3. Quaternionic formulation

The next task is to find evolution equations for α and $\boldsymbol{\chi}$. From (3) and (7) comes the simple pair of equations [3]

$$\frac{\mathrm{D}\alpha}{\mathrm{D}t} = \chi^2 - \alpha^2 - \alpha_p, \qquad \frac{\mathrm{D}\boldsymbol{\chi}}{\mathrm{D}t} = -2\boldsymbol{\chi}\alpha - \boldsymbol{\chi}_p, \tag{13}$$

where α_p and $\boldsymbol{\chi}_p$ are defined in terms of the Hessian matrix P

$$\alpha_p(\boldsymbol{x},t) = \frac{\boldsymbol{\omega}\cdot P\boldsymbol{\omega}}{\boldsymbol{\omega}\cdot\boldsymbol{\omega}}, \qquad \boldsymbol{\chi}_p(\boldsymbol{x},t) = \frac{\boldsymbol{\omega}\times P\boldsymbol{\omega}}{\boldsymbol{\omega}\cdot\boldsymbol{\omega}}. \tag{14}$$

The form of the right-hand sides of the two equations in (13) suggests an algebraic structure in $\mathbb{R}^4$ based on quaternions. Let unit 4-vectors be $\mathbf{1} = (1,0,0,0)^{\mathrm{T}}$, $\boldsymbol{i} = (0,1,0,0)^{\mathrm{T}}$, $\boldsymbol{j} = (0,0,1,0)^{\mathrm{T}}$ and $\boldsymbol{k} = (0,0,0,1)^{\mathrm{T}}$ with multiplication rules

$$\boldsymbol{m}\otimes\mathbf{1} = \mathbf{1}\otimes\boldsymbol{m} = \mathbf{1}, \quad \boldsymbol{m} = \mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \tag{15}$$

$$\boldsymbol{m}\otimes\boldsymbol{m} = -\mathbf{1}, \quad \boldsymbol{m} = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \tag{16}$$

and $\boldsymbol{i}\otimes\boldsymbol{j} = -\boldsymbol{j}\otimes\boldsymbol{i} = \boldsymbol{k}, \ldots$, cyclically. Based on these we can define 4-vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_p$ as

$$\boldsymbol{\zeta} = \begin{pmatrix} \alpha \\ \boldsymbol{\chi} \end{pmatrix}, \qquad \boldsymbol{\zeta}_p = \begin{pmatrix} \alpha_p \\ \boldsymbol{\chi}_p \end{pmatrix} \tag{17}$$

and the above rules mean that for any two 4-vectors $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$

$$\boldsymbol{\zeta}_1\otimes\boldsymbol{\zeta}_2 = \begin{pmatrix} \alpha_1\alpha_2 - \boldsymbol{\chi}_1\cdot\boldsymbol{\chi}_2 \\ \alpha_1\boldsymbol{\chi}_2 + \alpha_2\boldsymbol{\chi}_1 + \boldsymbol{\chi}_1\times\boldsymbol{\chi}_2 \end{pmatrix}. \tag{18}$$

It is obvious why this process is non-commutative. The conjugate $\boldsymbol{\zeta}^* = (\alpha, -\boldsymbol{\chi})^{\mathrm{T}}$ obeys the relation $\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^* = (\alpha^2+\chi^2)\mathbf{1}$. Hence the vector $\boldsymbol{\zeta}$ represents a point that must lie within the shell between the two concentric 4-spheres of radii $|\lambda_2|$ and $\max\{\lambda_1, |\lambda_3|\}$.

With the rules given in (18), and the labelling of the eigenvalues of S and P, respectively, as $\lambda_i(\boldsymbol{x}, t)$ and $\lambda_i^{(p)}(\boldsymbol{x}, t)$, the equations for α and $\boldsymbol{\chi}$ in (13) can formally be expressed in the following theorem.

Theorem 1. *The 4-vector $\boldsymbol{\zeta}$ for the three-dimensional incompressible Euler equations evolves according to the quaternionic Riccati equation*

$$\frac{\mathrm{D}\boldsymbol{\zeta}}{\mathrm{D}t} + \boldsymbol{\zeta} \otimes \boldsymbol{\zeta} + \boldsymbol{\zeta}_p = 0 \tag{19}$$

subject to the relation between the exact eigenvalues of S and P

$$2\sum_{i=1}^{3}(\lambda_i^2 + \lambda_i^{(p)}) = \omega^2. \tag{20}$$

Remark. Note that the quaternionic structure in (19) lies in the *dependent* variable $\boldsymbol{\zeta}$ not in the independent variables. The Riccati structure in (19) can be linearized by introducing the 4-vector $\boldsymbol{\Psi}$ such that

$$\boldsymbol{\zeta} = \frac{\mathrm{D}\boldsymbol{\Psi}}{\mathrm{D}t} \otimes \boldsymbol{\Psi}^{-1}. \tag{21}$$

Unlike 3-vectors, these 4-vectors have inverses, so $\boldsymbol{\Psi}^{-1}$ exists. Moreover, while they are not multiplicatively commutative, they are associative. These properties can be used to show that

$$\frac{\mathrm{D}^2\boldsymbol{\Psi}}{\mathrm{D}t^2} + \boldsymbol{\zeta}_p \otimes \boldsymbol{\Psi} = 0. \tag{22}$$

This is a zero-eigenvalue quaternionic Schrödinger equation for the 4-vector $\boldsymbol{\Psi}$ with $\boldsymbol{\zeta}_p(\boldsymbol{x}, t)$ as a potential.

Proof. Eq. (19) is a re-expression of (13) in quaternionic form using the rules in (18). The relation between P, $\boldsymbol{\omega}$ and S expressed in (10) means that $\boldsymbol{\zeta}_p$ is not in itself completely independent of $\boldsymbol{\zeta}$. Eq. (10) can be re-written in terms of the eigenvalues of S and P, respectively, as in (20). □

1.4. An alternative complex formulation

There is a close connection between complex numbers and quaternions, the latter being a generalization of the former. It is therefore natural to ask whether the equations in (13) can be re-formulated in terms of complex variables. Define

$$\zeta_c = \alpha + \mathrm{i}\chi. \tag{23}$$

The evolution of scalar $\chi = |\boldsymbol{\chi}|$ can be found by dotting the $\boldsymbol{\chi}$ equation in (13) with $\boldsymbol{\chi}$ to obtain

$$\frac{\mathrm{D}\chi}{\mathrm{D}t} = -2\chi\alpha - \tilde{\chi}_p, \tag{24}$$

where $\tilde{\chi}_p = \hat{\boldsymbol{\chi}} \cdot \boldsymbol{\chi}_p$. Using (23), it is easily shown that ζ_c obeys the complex Riccati equation that is the obvious parallel to Eq. (19)

$$\frac{\mathrm{D}\zeta_c}{\mathrm{D}t} + \zeta_c^2 + \zeta_p^{(c)} = 0 \tag{25}$$

with

$$\zeta_p^{(c)} = \alpha_p + \mathrm{i}\tilde{\chi}_p. \tag{26}$$

The exact linearization of this Riccati equation is found from the substitution

$$\zeta_c = \frac{1}{\psi}\frac{D\psi}{Dt}, \tag{27}$$

giving the complex zero eigenvalue scalar Schrödinger problem

$$\frac{D^2\psi}{Dt^2} + \zeta_p^{(c)}\psi = 0 \tag{28}$$

with a potential $-\zeta_p^{(c)}$. Some information on the relative alignments of $\boldsymbol{\omega}$, $S\boldsymbol{\omega}$ and $P\boldsymbol{\omega}$ has been lost in solely using the lengths χ and $\tilde{\chi}_p$ and not the full vectors but the merit of this approach is that it reduces the problem from four to two components but still gives a zero-eigenvalue Schrödinger equation as before. More will be said about this class of problems in Section 3 where an exact class of solutions due to Adler and Moser [14] will be discussed.

1.5. A physical interpretation

The components of $\boldsymbol{\zeta}$, consisting of the scalar α and the 3-vector $\boldsymbol{\chi}$, have a physical interpretation. In addition to α being an estimate for an eigenvalue of S, it is also related to the evolution of the scalar vorticity ω by

$$\frac{D\omega}{Dt} = \alpha\omega. \tag{29}$$

Here α plays the role of the vorticity stretching rate: there will be some parts of the flow where $\alpha < 0$, indicating vortex compression, and other parts where $\alpha > 0$, indicating vortex stretching. In this context, Constantin [15] has written down a Biot–Savart-type integral formula that relates α to a prism of vectors that characterize the relative alignment of neighbouring vortex lines.

$\boldsymbol{\chi}$ has an association with the angle θ between $\boldsymbol{\omega}$ and $S\boldsymbol{\omega}$ such that

$$\tan\theta = \frac{\chi}{\alpha}. \tag{30}$$

Whereas, in general, α lies only within the spectrum of S, it is an exact eigenvalue when $\boldsymbol{\omega}$ aligns (anti-aligns) with one of the eigenvectors of S, in which case $\boldsymbol{\chi} = 0\,(\pi)$. Turbulent vorticity fields tend to be dominated by vortex tube-like and sheet-like features ([16,17]; see also references in [18]). Idealized, straight vortex tubes or shear layers would therefore be examples of the case $\boldsymbol{\chi} = 0$. In such a case $\boldsymbol{\zeta} = \alpha\mathbf{1}$ and the system reduces to a problem in the scalar variable α alone (see [19]). The full 4-vector $\boldsymbol{\zeta}$, and therefore its natural quaternionic structure, only becomes relevant when $\boldsymbol{\chi} \neq 0$. Because $\boldsymbol{\chi}$ gives the degree of misalignment between $\boldsymbol{\omega}$ and the eigenvectors of S, it is therefore some measure of the degree of local misalignment that occurs when tubes bend, knot and tangle, when the topology undergoes significant changes or when potentially singular behaviour starts to develop [17,20]. The tendency for $\boldsymbol{\omega}$ to align with certain eigenvectors of S, known as preferential alignment, has been one of the main themes of computational work in both inviscid and viscous turbulence within the last 15 years [16,21–24].

2. A quaternionic structure for ideal incompressible MHD

The quaternionic relationship expressed in Theorem 1 for the 3D Euler equations is degenerate in that a similar complex structure exists to the one in $\mathbb{R}^4$. The purpose of this section is to show that the equations of ideal incompressible MHD have a similar quaternionic structure in Elsasser variables but in a non-degenerate form. The equations of ideal incompressible MHD couple the inviscid fluid to a magnetic field $\boldsymbol{B}$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla p, \tag{31}$$

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}, \tag{32}$$

together with $\operatorname{div}\boldsymbol{u} = 0$ and $\operatorname{div}\boldsymbol{B} = 0$. The pressure p in (31) is the combination $p = p_{\mathrm{f}} + (1/2)B^2$, where p_{f} is the fluid pressure. Elsasser variables are defined by combining the $\boldsymbol{u}$ and $\boldsymbol{B}$ fields in a $\pm$-combination

$$\boldsymbol{v}^{\pm} = \boldsymbol{u} \pm \boldsymbol{B}. \tag{33}$$

The existence of two velocities $\boldsymbol{v}^{\pm}$ means that there are two material derivatives

$$\frac{\mathrm{D}^{\pm}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}^{\pm}\cdot\nabla. \tag{34}$$

In terms of these, (31) and (32) can be rewritten as

$$\frac{\mathrm{D}^{\pm}\boldsymbol{v}^{\mp}}{\mathrm{D}t} = -\nabla p \tag{35}$$

with the magnetic field $\boldsymbol{B}$ satisfying

$$\frac{\mathrm{D}^{\pm}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B}\cdot\nabla\boldsymbol{v}^{\pm} \tag{36}$$

together with $\operatorname{div}\boldsymbol{v}^{\pm} = 0$. Defining the $\pm$-stretching vectors in (36) as

$$\boldsymbol{\sigma}^{\pm} = \boldsymbol{B}\cdot\nabla\boldsymbol{v}^{\pm} \tag{37}$$

allows us to define

$$\alpha^{\pm} = \frac{\boldsymbol{B}\cdot\boldsymbol{\sigma}^{\pm}}{\boldsymbol{B}\cdot\boldsymbol{B}}, \qquad \boldsymbol{\chi}^{\pm} = \frac{\boldsymbol{B}\times\boldsymbol{\sigma}^{\pm}}{\boldsymbol{B}\cdot\boldsymbol{B}} \tag{38}$$

having used Moffatt's analogy between the vectors $\boldsymbol{\omega}$ and $\boldsymbol{B}$ [25,26]. Note that the numerators of $\alpha^{\pm}$, $\boldsymbol{\chi}^{\pm}$ automatically include the magnetic gradient matrix $B_{i,j}$ along with the velocity gradient matrix $u_{i,j}$ within $v_{i,j}$. The $\alpha^{\pm}$ clearly play the role of scalar magnetic field stretching rates

$$\frac{\mathrm{D}^{\pm}B}{\mathrm{D}t} = \alpha^{\pm}B. \tag{39}$$

It is also necessary, as in Section 1, to define equivalent variables based upon the Hessian matrix P

$$\alpha_p^{(m)} = \frac{\boldsymbol{B}\cdot P\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}, \qquad \boldsymbol{\chi}_p^{(m)} = \frac{\boldsymbol{B}\times P\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}. \tag{40}$$

As in (17), we define the 4-vectors $\boldsymbol{\zeta}^{\pm}$ and $\boldsymbol{\zeta}^{(m)}$ as follows:

$$\boldsymbol{\zeta}^{\pm} = \begin{pmatrix}\alpha^{\pm}\\ \boldsymbol{\chi}^{\pm}\end{pmatrix}, \qquad \boldsymbol{\zeta}^{(m)} = \begin{pmatrix}\alpha^{(m)}\\ \boldsymbol{\chi}^{(m)}\end{pmatrix}. \tag{41}$$

The main result of this section is the following.

Theorem 2. *The 4-vectors* $\boldsymbol{\zeta}^{\pm}$ *satisfy*

$$\frac{\mathrm{D}^{\mp}\boldsymbol{\zeta}^{\pm}}{\mathrm{D}t} + \boldsymbol{\zeta}^{\pm}\otimes\boldsymbol{\zeta}^{\mp} + \boldsymbol{\zeta}_p^{(m)} = 0 \tag{42}$$

subject to the Poisson relation

$$\mathrm{Tr}\, P = -\sum_{i,j} v^{\pm}_{i,j} v^{\mp}_{j,i}. \tag{43}$$

Remark. Despite the fact that there are now two material derivatives, a linearization can be achieved by introducing the 4-vector $\boldsymbol{\Psi}$ such that

$$\boldsymbol{\zeta}^{\pm} = \frac{\mathrm{D}^{\pm}\boldsymbol{\Psi}}{\mathrm{D}t}\otimes\boldsymbol{\Psi}^{-1} \Rightarrow \frac{\mathrm{D}^{\mp}}{\mathrm{D}t}\frac{\mathrm{D}^{\pm}\boldsymbol{\Psi}}{\mathrm{D}t} + \boldsymbol{\zeta}_p\otimes\boldsymbol{\Psi} = 0. \tag{44}$$

Proof. The first step in the proof is to calculate the ideal MHD equivalent of the Ertel–Ohkitani relations (7) and (8). This is found in the following subsidiary lemma.

Lemma 1. $\boldsymbol{\sigma}^{\pm}$ *are related to* P *and* $\boldsymbol{B}$ *by*

$$\frac{\mathrm{D}^{\pm}\boldsymbol{\sigma}^{\mp}}{\mathrm{D}t} = -P\boldsymbol{B}, \tag{45}$$

and $\boldsymbol{B}$ *satisfies*

$$\left(\frac{\mathrm{D}^{\pm}}{\mathrm{D}t}\frac{\mathrm{D}^{\mp}}{\mathrm{D}t} + P\right)\boldsymbol{B} = 0. \tag{46}$$

Proof. The $\pm$-material derivatives of $\boldsymbol{\sigma}^{\mp}$ in terms of components $\sigma_i^{\mp} = B_j v^{\mp}_{i,j}$ are

$$\frac{\mathrm{D}^{\pm}\sigma_i^{\mp}}{\mathrm{D}t} = \frac{\mathrm{D}^{\pm}B_j}{\mathrm{D}t}v^{\mp}_{i,j} + B_j\frac{\partial}{\partial x_j}\left(\frac{\mathrm{D}^{\pm}v_i^{\mp}}{\mathrm{D}t}\right) - B_j v^{\pm}_{k,j}v^{\mp}_{i,k} = (\sigma_j^{\pm}v^{\mp}_{i,j} - \sigma_k^{\pm}v^{\mp}_{i,k}) + B_j\frac{\partial}{\partial x_j}\left(\frac{\mathrm{D}^{\pm}v_i^{\mp}}{\mathrm{D}t}\right) = -B_j p_{,ij}. \tag{47}$$

The proof of (46) follows immediately.

The second step of the proof is to find the evolution of $\alpha^{\pm}$ and $\boldsymbol{\chi}^{\pm}$. The material $\mp$-derivatives of $\alpha^{\pm}$ are given by

$$\frac{\mathrm{D}^{\mp}\alpha^{\pm}}{\mathrm{D}t} = \frac{\boldsymbol{\sigma}^{\mp}\cdot\boldsymbol{\sigma}^{\pm}}{\boldsymbol{B}\cdot\boldsymbol{B}} - 2\alpha^{\mp}\alpha^{\pm} - \alpha_p^{(m)}. \tag{48}$$

The following vector identity is useful:

$$(\boldsymbol{B}\cdot\boldsymbol{B})(\boldsymbol{\sigma}^{\pm}\cdot\boldsymbol{\sigma}^{\mp}) = (\boldsymbol{B}\times\boldsymbol{\sigma}^{\pm})\cdot(\boldsymbol{B}\times\boldsymbol{\sigma}^{\mp}) + (\boldsymbol{B}\cdot\boldsymbol{\sigma}^{\pm})(\boldsymbol{B}\cdot\boldsymbol{\sigma}^{\mp}) \tag{49}$$

which can be rewritten as

$$\frac{\boldsymbol{\sigma}^{\pm}\cdot\boldsymbol{\sigma}^{\mp}}{\boldsymbol{B}\cdot\boldsymbol{B}} = \boldsymbol{\chi}^{\pm}\cdot\boldsymbol{\chi}^{\mp} + \alpha^{\pm}\alpha^{\mp}. \tag{50}$$

This transforms (48) into

$$\frac{\mathrm{D}^{\mp}\alpha^{\pm}}{\mathrm{D}t} = \boldsymbol{\chi}^{+}\cdot\boldsymbol{\chi}^{-} - \alpha^{+}\alpha^{-} - \alpha_p^{(m)}. \tag{51}$$

For the evolution of $\boldsymbol{\chi}^{\pm}$, we find that

$$\frac{\mathrm{D}^{\mp}\boldsymbol{\chi}^{\pm}}{\mathrm{D}t} = \frac{\boldsymbol{\sigma}^{\mp}\times\boldsymbol{\sigma}^{\pm}}{\boldsymbol{B}\cdot\boldsymbol{B}} - 2\alpha^{\mp}\boldsymbol{\chi}^{\pm} - \boldsymbol{\chi}_p^{(m)}. \tag{52}$$

To handle the first term on the RHS of (52) it is desirable to express this in terms of the three vectors $\boldsymbol{\chi}^{+}$, $\boldsymbol{\chi}^{-}$ and $\boldsymbol{\chi}^{+}\times\boldsymbol{\chi}^{-}$. It is easy to show from the definition of $\boldsymbol{\chi}^{\pm}$ in (38) that

$$\boldsymbol{\sigma}^{\pm} = \alpha^{\pm}\boldsymbol{B} - \boldsymbol{B}\times\boldsymbol{\chi}^{\pm} \tag{53}$$

and so

$$\frac{\boldsymbol{\sigma}^{+}\times\boldsymbol{\sigma}^{-}}{\boldsymbol{B}\cdot\boldsymbol{B}} = -\alpha^{-}\boldsymbol{\chi}^{+} + \alpha^{+}\boldsymbol{\chi}^{-} + (\boldsymbol{B}\times\boldsymbol{\chi}^{+})\times(\boldsymbol{B}\times\boldsymbol{\chi}^{-})B^{-2}. \tag{54}$$

Now, $\boldsymbol{B}$ and $\boldsymbol{\chi}^{+}\times\boldsymbol{\chi}^{-}$ are orthogonal to both $\boldsymbol{\chi}^{+}$ and $\boldsymbol{\chi}^{-}$ so $\boldsymbol{\chi}^{+}\times\boldsymbol{\chi}^{-}$ can be written as $\boldsymbol{\chi}^{+}\times\boldsymbol{\chi}^{-} = \lambda_0\boldsymbol{B}$, where λ_0 is a scalar. The cross product in the third term on the RHS of (54) can be rewritten as

$$(\boldsymbol{B}\times\boldsymbol{\chi}^{+})\times(\boldsymbol{B}\times\boldsymbol{\chi}^{-}) = \boldsymbol{B}[\boldsymbol{B}\cdot(\boldsymbol{\chi}^{+}\times\boldsymbol{\chi}^{-})] = B^2(\boldsymbol{\chi}^{+}\times\boldsymbol{\chi}^{-}), \tag{55}$$

and so λ_0 can be identified as

$$\lambda_0 = B^{-2}(\boldsymbol{B}\cdot(\boldsymbol{\chi}^{+}\times\boldsymbol{\chi}^{-})). \tag{56}$$

Using this in (54) and then in (52) we find

$$\frac{\mathrm{D}^{\mp}\boldsymbol{\chi}^{\pm}}{\mathrm{D}t} = -(\alpha^{-}\boldsymbol{\chi}^{+} + \alpha^{+}\boldsymbol{\chi}^{-}) - (\boldsymbol{\chi}^{\pm}\times\boldsymbol{\chi}^{\mp}) - \boldsymbol{\chi}_p^{(m)}. \tag{57}$$

The third and final step in the theorem proof is accomplished by observing from (18) that the right-hand sides of equations (51) and (57) can be written as (42). The trace constraint (43) comes from the two divergence-free conditions div $\boldsymbol{v}^{\pm} = 0$ applied across (35). □

3. A complex Schrödinger equation and the work of Adler and Moser

In Section 1, we introduced three different zero-eigenvalue Schrödinger equations: a 3×3 matrix system involving the 3-vector $\boldsymbol{\omega}$ in (8), a quaternionic system for the 4-vector $\boldsymbol{\Psi}$ in (21) and a complex scalar system for ψ in (28).

All of these contain slightly different information but it is convenient to choose the complex zero eigenvalue scalar Schrödinger problem (28)

$$\frac{\mathrm{D}^2\psi}{\mathrm{D}t^2} + \zeta_p^{(c)}\psi = 0 \tag{58}$$

with a potential $-\zeta_p^{(c)}$, and look at it in Lagrangian variables using fluid particles as the basis of the co-ordinate system. A convenient system for these particles (ξ_1, ξ_2, ξ_3) is given by their Eulerian position at some chosen instant (say t_0) such that $\boldsymbol{\xi} \equiv (\xi_1, \xi_2, \xi_3) = (x, y, z)$. Hence we have $u_1(\boldsymbol{\xi}, t) = \dot{x}(\boldsymbol{\xi}, t)$, $u_2(\boldsymbol{\xi}, t) = \dot{y}(\boldsymbol{\xi}, t)$ and $u_3(\boldsymbol{\xi}, t) = \dot{z}(\boldsymbol{\xi}, t)$. Let us therefore write $U = -\zeta_p^{(c)}(\boldsymbol{\xi}, t)$ and the zero-eigenvalue Schrodinger equation (58) as

$$-\ddot{\psi} + U\psi = 0, \tag{59}$$

where the double-dot refers to two Lagrangian time derivatives.

Given a complex potential $U(t)$ we seek to solve for ψ and hence for ζ_c. In any of these cases this is only half the solution; the most serious difficulty lies in determining the fluid particle trajectories that correspond to this formal set of solutions. As already discussed in Section 1, the Hessian matrix P is not independent of the other variables in the problem. Eq. (10) gives information only on the diagonal but not the off-diagonal elements of P. The particle paths corresponding to the set of solutions under discussion must be compatible with this and it is here where the problem still remains open. It is possible that there may be solutions of (59) for which this set is empty or is of measure zero.

Having established that solving equation (59) is only half the issue, nevertheless this Schrödinger equation has some interesting solutions which are worth discussing. It is well known that solutions of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{60}$$

for real $u(x,t)$ and with boundary conditions $u \to 0$ as $|x| \to \infty$, are associated with the isospectral solutions of a Schrödinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + u\psi = E\psi \tag{61}$$

in which the KdV dependent variable $u(x,t)$ plays the role of the potential and E the constant energy eigenvalue. Soliton solutions of (60) are associated with the discrete negative energy spectrum of the Schrödinger operator in (61) and the continuous spectrum is associated with the positive energy eigenvalues. Adler and Moser [14] showed that solutions rational in x and t correspond to the case when $E = 0$. For the KdV equation expressed in its traditional form in (60), the independent variable in the Schrödinger equation in (61) is x, with t held as a parameter, whereas in our Lagragian problem t is now the independent variable. Adler and Moser [14] proved the following result (with t and x exchanged) for the zero eigenvalue system (59), which can be generalized to the complex domain.

Theorem 3 (Adler and Moser [14]). *For potentials $U(t)$ in (59) that take the form*

$$U_k = -2\frac{\partial^2}{\partial t^2}\ln\theta_k, \tag{62}$$

the eigenfunctions ψ_k satisfy

$$\psi_k = \frac{\theta_{k+1}}{\theta_k}, \tag{63}$$

where the infinite set of polynomials θ_k of degree $n_k = (1/2)k(k+1)$ can be generated from the nonlinear Wronskian recurrence relation

$$\dot\theta_{k+1}\theta_{k-1} - \theta_{k+1}\dot\theta_{k-1} = (2k+1)\theta_k^2 \tag{64}$$

starting from $\theta_0(t) = 1$, $\theta_1(t) = t + \tau_1$.

Proof. The proof is virtually identical to that in Adler and Moser [14] where $U \in \mathbb{R}$, except that here we allow U to be complex. Summarising the main points, it is based around an idea that involves identifying eigenfunctions of the zero-eigenvalue Sturm–Liouville problem (59). If ψ is an eigenfunction of (59) with potential U then ψ^{-1} is also an eigenfunction with a potential $\tilde U$ provided that

$$\tilde U - U = -2\frac{\partial^2 \ln\psi}{\partial t^2}, \tag{65}$$

which is obviously true. One identifies the potential $U = U_k$ with the eigenfunction ψ_k and $\tilde U = U_{k+1}$ with the eigenfunction ψ_k^{-1}. If this identification is correct then there must be another eigenfunction ψ_{k-1}^{-1} associated with U_k in addition to ψ_k. To turn this into an induction proof, one assumes that the $\theta_0, \theta_1, \ldots, \theta_k$ have been found via

(64) and therefore the ψ_k are available through (63). If ψ_k and ψ_{k-1}^{-1} are linearly independent eigenfunctions then their Wronskian must be a non-vanishing constant

$$W[\psi_k; \psi_{k-1}^{-1}] = c, \tag{66}$$

which becomes

$$\psi_k \dot{\psi}_{k-1} + \dot{\psi}_k \psi_{k-1} = c\psi_{k-1}^2. \tag{67}$$

In addition, this relation between ψ_k and ψ_{k-1}^{-1} must be consistent with (59) which insists that they must satisfy

$$\frac{\ddot{\psi}_k}{\psi_k} = \frac{(\ddot{\psi}_{k-1}^{-1})}{\psi_{k-1}^{-1}}, \tag{68}$$

where the double-dot on the right-hand side refers to two time derivatives on the inverse function inside the round brackets. Upon re-arrangement and integration this also gives exactly (67). Setting $\phi_k = \theta_{k+1}/\theta_k$ gives the recursion formula (64) with the normalization constant in (66) taken such that $c = 2k + 1$. To complete the induction one calculates

$$U_{k+1} = U_k - 2\frac{\partial^2 \ln \psi_k}{\partial t^2} = -2\frac{\partial^2 \ln \theta_k \psi_k}{\partial t^2} = -2\frac{\partial^2 \ln \theta_{k+1}}{\partial t^2}, \tag{69}$$

which completes the proof. The theorem is valid on the complex domain by simply allowing the τ_k to be complex constants. □

The recursion relation (64) will generate any θ_k of any desired order. The first four of these polynomials are

$$\theta_0(t) = 1, \qquad \theta_1(t) = t + \tau_1, \qquad \theta_2(t) = t^3 + \tau_2, \qquad \theta_3(t) = t^6 + 5\tau_2 t^3 + \tau_3 t - 5\tau_2^2, \tag{70}$$

where the τ_i are arbitrary complex constants. Adler and Moser have also shown that these polynomials are isobaric; i.e., they have the homogeneity property

$$\theta_k(\lambda\tau_1, \lambda^3\tau_2, \ldots, \lambda^{2k-1}\tau_k) = \lambda^{n_k}\theta_k(\tau_1, \tau_2, \ldots, \tau_k), \tag{71}$$

where $n_k = (1/2)k(k + 1)$ is the degree. Moreover, they have also shown that a generating function exists for these polynomials. With a k-label, the $\zeta^{(c)}$ in (27) can be expressed as

$$\zeta_k^{(c)} = \frac{\partial}{\partial t}(\ln \psi_k) = \frac{\partial}{\partial t} \ln \left(\frac{\theta_{k+1}}{\theta_k}\right), \tag{72}$$

and the real and imaginary parts of this give α_k and χ_k. The 'solutions' given above mean that the ψ_k expressed through the θ_k correspond to the class of potentials U_k given in (62).

As discussed in Section 1 and the beginning of this section, the particle paths corresponding to solutions (62) and (63) must be compatible with the trace equation (10) and it is here where the problem still remains open; the constants τ_k would need to be calculated in terms of particle path positions. As far as singularities in these solutions are concerned, they must lie off the real axis unless the τ_i are taken to be real constants. It is also possible that this infinite set of solutions can be generalized to the quaternionic Schrödinger case (19) because $\boldsymbol{\Psi}$ has an inverse. In this case the τ_i would be 4-vectors.

4. Conclusion

The two conspicuous open questions of mathematical fluid dynamics; namely, whether the Navier–Stokes equations are regular [27–31] and whether the three-dimensional Euler equations develop a finite time singularity [17,20], so far, have not completely yielded to methods of analysis, although great progress has been made. Attempts have been made to understand the dynamics of the vorticity field by studying the direction of vorticity [15,20]; this particularly paper is one variation on that theme. The natural quaternionic structure of Theorem 1 lies in the interplay between the 4-vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_p$, which are associated with the relative alignment of $\boldsymbol{\omega}$ with eigenvectors of the strain matrix S and the Hessian matrix of the pressure P, respectively. While this geometric formulation is not complete in that the independent variables need also to be brought into this formulation, nevertheless the problem has been cast in the form of a constrained (by incompressibility) Lagrangian flow on a quaternionic manifold. The Euler problem is degenerate in that the structure in $\mathbb{R}^4$ is almost identical to that in $\mathbb{R}^2$, which is not the case for the ideal MHD problem of Section 3 where no degeneracy occurs. There, the formulation in the $\alpha^{\pm}$, $\boldsymbol{\chi}^{\pm}$ variables is messy but is resolved into a the simple and elegant form in terms of the 4-vectors $\boldsymbol{\zeta}^{\pm}$ of Theorem 2.

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