EE2 Mathematics : Complex Variables

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These notes are not identical word-for-word with my lectures which will be given on a BB/WB. Some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working. I will not be handing out copies of these notes – **you are therefore advised to attend lectures and take your own**.

1. The material in them is dependent upon the material on Complex Numbers you were taught at A-level and your 1st year.
2. Handouts are :

   (a) **Handout No 5** of the course on "Jordan's Lemma" in complex integration contains some of the material from §3.5.

Contents

[1]Do not confuse me with Dr J. Gibbons who is also in the Mathematics Dept.

1 Analyticity and the Cauchy-Riemann equations

1.1 Derivation of the Cauchy-Riemann equations

Functions of the complex variable $z = x + iy$

$$w = f(z) \tag{1.1}$$

are expressed in the usual manner except that the independent variable $z = x + iy$ is complex. Thus $f(z)$ has a real part $u(x, y)$ and an imaginary part $v(x, y)$

$$f(z) = u(x, y) + iv(x, y)\,. \tag{1.2}$$

Extra difficulties appear in differentiating and integrating such functions because z varies in a plane and not on a line. For functions of a single real variable the idea of an incremental change δx along the x-axis has to be replaced by an incremental change δz. Because δz is a vector the question of the direction of this limit becomes an issue.

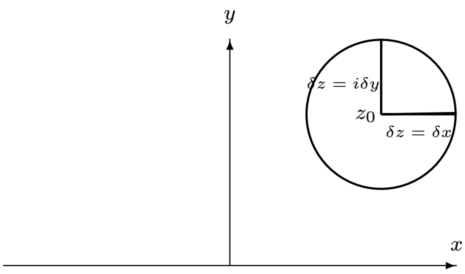
Firstly we look at the concept of differentiation. The definition of a derivative at a point z_0 remains the same as usual; namely

$$\left.\frac{df(z)}{dz}\right|_{z=z_0} = \lim_{\delta z\to 0}\left(\frac{f(z_0+\delta z)-f(z_0)}{\delta z}\right)\,. \tag{1.3}$$

The subtlety here lies in the limit $\delta z \to 0$ because δz is itself a vector and therefore the limit $\delta z \to 0$ may be taken in many directions. If the limit in (1.3) is to be unique (to make any sense) *it is required that it be independent of the direction in which the limit $\delta z \to 0$ is taken.* If this is the case then it is said that $f(z)$ is differentiable at the point z.

There is a general test on functions to determine whether (1.3) is independent of the direction of the limit. The simplest way is to firstly take the limit in the horizontal direction: that is $\delta z = \delta x$, in which case

$$\frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv u_x + iv_x\,. \tag{1.4}$$



The z-plane with a point at z_0 and a circle of radius $|\delta z|$ around it. The horizontal radius is drawn for the case when $\delta z = \delta x$ and the vertical for the case when $\delta z = i\delta y$.

Next we take the limit in the vertical direction : that is $\delta z = i\delta y$

$$\frac{df(z)}{dz} = \frac{\partial u}{\partial(iy)} + i\frac{\partial v}{\partial(iy)} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \equiv -iu_y + iv_y\,. \tag{1.5}$$

If the limits in both directions are to be equal df/dz in (1.4) and (1.5) must be equal, which makes

$$\boxed{u_x = v_y\,, \qquad\qquad u_y = -v_x\,.} \tag{1.6}$$

The boxed pair of equations above are known as **the Cauchy-Riemann equations**. If these hold at a point z then $f(z)$ is said to be differentiable at z. There is no such requirement in single variable calculus. Moreover the CR equations bring us to a further idea regarding differentiation in the complex plane :

Definition : *If $f(z)$ is differentiable at all points in a neighbourhood of a point z_0 then $f(z)$ is said to be analytic (regular) at z_0.*

Some functions are analytic everywhere in the complex plane except at certain points : these points are called *singularities*. Three examples illustrate this.

Example 1 : $f(z) = z^2$. Writing $z^2 = x^2 - y^2 + 2ixy$ we have

$$u(x,y) = x^2 - y^2\,, \qquad\qquad v(x,y) = 2xy\,. \tag{1.7}$$

Clearly four trivial partial derivatives show that $u_x = 2x$, $u_y = -2y$, $v_x = 2y$ and $v_y = 2x$ thus demonstrating that the CR equations hold for all values of x and y. It follows that $f(z) = z^2$ is differentiable at all points in the z-plane and every point in this plane has an (infinite) neighbourhood in which $f(z) = z^2$ is differentiable. Clearly $f(z) = z^2$ is analytic everywhere.

Example 2 : $f(z) = z^{-1}$. Writing $z^{-1} = (x - iy)/(x^2 + y^2)$ we have

$$u(x,y) = \frac{x}{x^2+y^2} \qquad\qquad v(x,y) = -\frac{y}{x^2+y^2}\,. \tag{1.8}$$

Without giving the working it is not difficult to show that the CR equations hold everywhere except at the origin $z = 0$ where the limit is indeterminate : $z = 0$ is the point where it fails to be differentiable. Hence $w = z^{-1}$ is analytic everywhere except at $z = 0$.

Example 3 : $f(z) = |z|^2$. We have

$$u(x,y) = x^2 + y^2\,, \qquad\qquad v(x,y) = 0\,, \tag{1.9}$$

and so

$$u_x = 2x\,, \qquad u_y = 2y\,, \qquad v_x = v_y = 0\,. \tag{1.10}$$

Clearly the CR equations do **not** hold anywhere except at $z = 0$. Therefore $f(z) = |z|^2$ is not differentiable anywhere except at $z = 0$ and there is no neighbourhood around $z = 0$ in which it is differentiable. Thus the function is analytic nowhere in the z-plane.

As a final remark on this example let us look again at the limit $\delta z \to 0$ in polar co-ordinates at a fixed point z_0 about which we describe a circle $z = z_0 + re^{i\theta}$.

$$\begin{aligned} \left.\frac{df}{dz}\right|_{z=z_0} &= \lim_{\delta z \to 0}\left(\frac{|z+\delta z|^2 - |z|^2}{\delta z}\right)_{z=z_0} \\ &= \lim_{\delta z \to 0}\left(\frac{(\delta z)z^* + (\delta z^*)z + (\delta z)(\delta z^*)}{\delta z}\right)_{z=z_0} \\ &= z_0^* + z_0 \lim_{\delta z \to 0}\left(\frac{\delta z^*}{\delta z}\right)_{z=z_0}. \end{aligned} \tag{1.11}$$

Then (1.11) can be written as

$$\left.\frac{df}{dz}\right|_{z=z_0} = z_0^* + z_0 e^{-2i\theta}. \tag{1.12}$$

This result illustrates the problem: as θ varies (and thus the direction of the limit) so does the limit. **This is clearly not unique except when $z_0 = 0$.**

1.2 Properties of analytic functions

Let us consider the CR equations $u_x = v_y$ and $u_y = -v_x$ as a condition for the analyticity of a function $w = u(x, y) + iv(x, y)$. Cross differentiation and elimination of first u and then v gives

$$u_{xx} + u_{yy} = 0 \qquad v_{xx} + v_{yy} = 0, \tag{1.13}$$

thus showing that u and v must always be a solution of Laplace's equation (without boundary conditions): these are called **harmonic functions**. It also said that $u(x, y)$ and $v(x, y)$ are **conjugate** to one another. In the following set of examples it will be shown how, given a harmonic function $u(x, y)$, its conjugate $v(x, y)$ can be constructed. The pair can then put together as $u + iv = f(z)$ to ultimately find $f(z)$.

Example 1: Given that $u = x^2 - y^2$ show (i) that it is harmonic; (ii) find $v(x, y)$ and then (iii) construct the corresponding complex function $f(z)$.

With $u = x^2 - y^2$ we have $u_x = 2x$, $u_{xx} = 2$, $u_y = -2y$ and $u_{yy} = -2$. Therefore $u_{xx} + u_{yy} = 0$ so it satisfies Laplace's equation. This is a sufficient condition for v to exist and for us to write $v_y = u_x = 2x$ and $v_x = -u_y = 2y$. While there are two PDEs here there can only be one solution compatible with both. Integrating them both in turn gives

$$v = 2xy + A_(x), \qquad v = 2xy + B(y). \tag{1.14}$$

It is clear that they are compatible if $A(x) = B(y) = \text{const} = c$ making the result

$$v = 2xy + c, \tag{1.15}$$

with

$$f(z) = x^2 - y^2 = 2ixy + ic = z^2 + ic. \tag{1.16}$$

The ic simply moves $f(z)$ an arbitrary distance along the imaginary axis.

Example 2: Given that $u = x^3 - 3xy^2$ find its conjugate function $v(x, y)$ and the corresponding complex function $f(z)$.

We first check that $u = x^3 - 3xy^2$ satisfies Laplace's equation: $u_x = 3x^2 - 3y^2$; $u_{xx} = 6x$; $u_y = -6xy$ and $u_{yy} = -6x$. Thus $u_{xx} + u_{yy} = 0$ and so v exists and is found from the CR equations:

$$v_y = 3x^2 - 3y^2 \qquad\qquad v_x = 6xy\,. \tag{1.17}$$

Partially integrating these gives

$$v = 3x^2y - y^3 + A(x) \qquad\qquad v = 3x^2y + B(y)\,. \tag{1.18}$$

The way to make these compatible is to choose $B(y) = -y^3 + c$ and $A(x) = c$ finally giving

$$v = 3x^2y - y^3 + c \tag{1.19}$$

with

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + i(3x^2y - y^3 + c) \\ &= z^3 + ic\,. \end{aligned} \tag{1.20}$$

Example 3: Given that $u = e^x\big(x\cos y - y\sin y\big)$ show that it satisfies Laplace's equation. Also find its conjugate v and then $f(z)$.

We find that

$$u_{xx} = e^x\big[(x+2)\cos y - y\sin y\big]\,; \qquad u_{yy} = -e^x\big[(x+2)\cos y - y\sin y\big]\,, \tag{1.21}$$

and so Laplace's equation is satisfied. Then

$$v_y = u_x = e^x\big[(x+1)\cos y - y\sin y\big]\,; \qquad v_x = -u_y = e^x\big[(x+1)\sin y + y\sin y\big]\,. \tag{1.22}$$

Using the indefinite integrals $\int y\sin y\,dy = \sin y - y\cos y$ and $\int x\,e^x dx = e^x(x-1)$ we find

$$v = e^x\big(x\sin y + y\cos y\big) + A(x)\,; \qquad v = e^x\big(y\cos y + x\sin y\big) + B(y)\,. \tag{1.23}$$

For compatibility we take $A(x) = B(y) = \text{const} = c$. Then

$$\begin{aligned} w &= e^x\big[(x+iy)\cos y - (y-ix)\sin y\big] + ic \\ &= e^x\big[z\cos y + iz\sin y\big] + ic \\ &= ze^{x+iy} + ic \\ &= ze^z + ic\,. \end{aligned} \tag{1.24}$$

1.3 Orthogonality

Let us finally consider the family of curves on which $u =$ const. From the chain rule

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{1.25}$$

and therefore on curves of constant u we have $du = 0$, giving the gradient on this family as

$$\left.\frac{dy}{dx}\right|_{u=const} = -\frac{u_x}{u_y}\,. \tag{1.26}$$

Likewise, on the family of curves of constant v

$$\left.\frac{dy}{dx}\right|_{v=const} = -\frac{v_x}{v_y} \tag{1.27}$$

giving

$$\left.\frac{dy}{dx}\right|_{u=const} \times \left.\frac{dy}{dx}\right|_{v=const} = \frac{v_x u_x}{v_y u_y}\,. \tag{1.28}$$

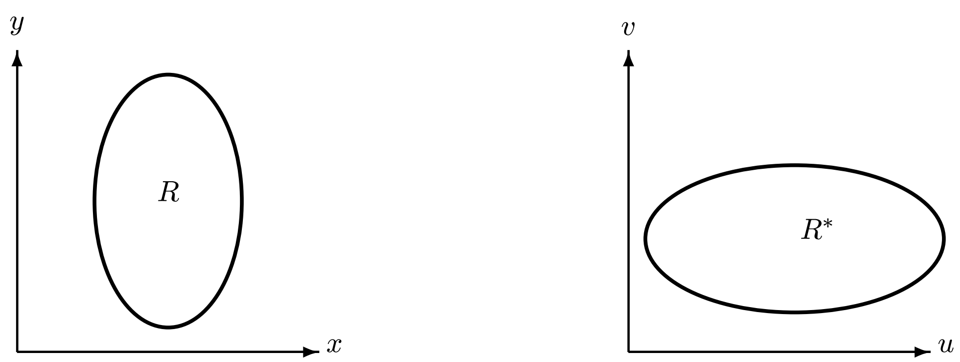
Now if $f(z)$ is analytic in a region R then the CR equations hold there, $u_x = v_y$ and $u_y = -v_x$, and (1.28) becomes

$$\left.\frac{dy}{dx}\right|_{u=const} \times \left.\frac{dy}{dx}\right|_{v=const} = -1\,. \tag{1.29}$$

The final result is that in regions of analyticity curves of constant u and curves of constant v are always orthogonal.

2 Mappings

2.1 Conformal mappings



A complex mapping $w = f(z)$ maps a region R in the z-plane to a different region R^* in the w-plane.

A complex function $w = f(z)$ can be thought of as a mapping from the z-plane to the w-plane. Depending on $f(z$ the mapping may not be unique. For instance, for $w = z^2$ for the values $\pm z_0$ there is one value w_0. Complex mappings do not necessarily behave in an expected way. The concept of analyticity intrudes into these ideas in the following way. A mapping is said to be *conformal* if it preserves angles in magnitude and sense. Moreover, a mapping has a *fixed point* when $w = f(z) = z$. The following theorem is now stated without proof:

Theorem 1 *The mapping defined by an analytic function $w = f(z)$ is conformal except at points where $f'(z) = 0$.*

Example 1: $w = z^2$ is conformal everywhere except at $z = 0$ because $f'(0) = 0$. Plotting contours of $u = x^2 - y^2$ and $v = 2xy$ shows that conformality fails at the origin.

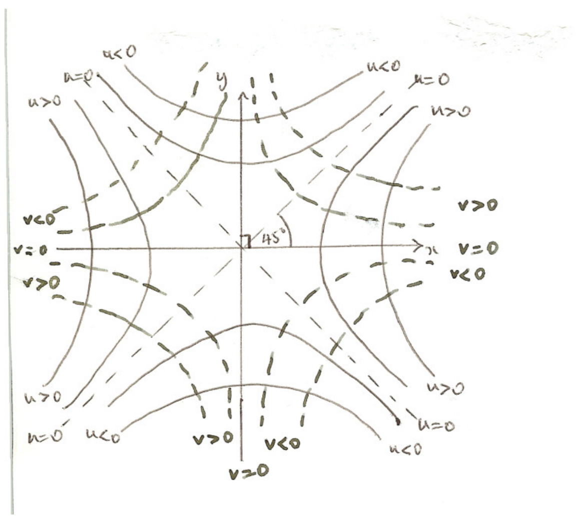


Figure 1: Contours of $u = const$ and $v = const$ in the z-plane: note their orthogonality except at $z = 0$ where conformality fails.

Example 2: Consider $w = \frac{1}{z-1}$ which is analytic everywhere except at $z = 1$. We have

$$w = \frac{1}{z-1} = \frac{1}{(x-1)+iy} = \frac{(x-1)-iy}{(x-1)^2+y^2} \tag{2.1}$$

in which case

$$u(x,y) = \frac{x-1}{(x-1)^2+y^2}\,, \qquad v(x,y) = -\frac{y}{(x-1)^2+y^2}\,. \tag{2.2}$$

It is clear from (2.2) that it is always true that

$$u^2 + v^2 = \frac{1}{(x-1)^2+y^2}\,. \tag{2.3}$$

So far we have specified no shape in the z-plane on which this map operates. Some examples of what this map will do are these:

1. Consider the family of circles in the z-plane: $(x-1)^2+y^2=a^2$. These circles are centred at $(1,\,0)$ of radius a. Clearly they map to
$$u^2+v^2=\frac{1}{a^2}\,, \tag{2.4}$$
which is a family of circles in the w-plane centred at $(0,\,0)$ of radius a^{-1}. As the value of a is increased the circles in the z-plane widen and those in the w-plane decrease. It is not difficult to show that the interior (exterior) of the circles in the z-plane map to the exterior (interior) of those in the w-plane. Thus we have
$$\begin{array}{cc} \underline{z\text{-plane}} & \underline{w\text{-plane}} \\ \text{interior} \to & \text{exterior} \\ \text{exterior} \to & \text{interior} \end{array} \tag{2.5}$$
The circle centre $(1,\,0)$ in the z-plane maps to *the point at infinity* in the w-plane.

2. The line $x=0$ in the z-plane maps to what? From (2.2) and (2.3) we know that
$$u(x,y)=-\frac{1}{1+y^2}\,,\qquad v(x,y)=-\frac{y}{1+y^2}\,,\qquad u^2+v^2=\frac{1}{1+y^2}\,. \tag{2.6}$$
$$u^2+v^2=\frac{1}{1+y^2}=-u\,,\qquad\Rightarrow\qquad \left(u+\tfrac{1}{2}\right)^2+v^2=\tfrac{1}{4}\,. \tag{2.7}$$
In the w-plane this is a circle of radius $\frac{1}{2}$ centred at $(-\frac{1}{2},\,0)$.

Thus we conclude that some circles can map to other circles but also straight lines can also map to circles. This is investigated in the next subsection.

2.2 $w=\frac{1}{z}$ maps lines/circles to lines/circles

The general equation for straight lines and circles in the z-plane can be written as
$$\alpha(x^2+y^2)+\beta x+\gamma y+\Delta=0. \tag{2.8}$$
where $\alpha,\ \beta,\ \gamma$ and Δ are constants. If $\alpha=0$ this represents a straight line but when $\alpha\neq 0$ (2.8) represents a circle. Writing (2.8) in terms of z
$$\alpha|z|^2+\frac{\beta}{2}(z+z^*)+\frac{\gamma}{2i}(z-z^*)+\Delta=0\,. \tag{2.9}$$
and then transforming to an equation w and w through $w=\frac{1}{z}$ and $w^*=\frac{1}{z^*}$, (2.9) becomes
$$\frac{\alpha}{ww^*}+\frac{\beta}{2}\left(\frac{1}{w}+\frac{1}{w^*}\right)+\frac{\gamma}{2i}\left(\frac{1}{w}-\frac{1}{w^*}\right)+\Delta=0\,, \tag{2.10}$$
or
$$\alpha+\frac{\beta}{2}(w+w^*)-\frac{i\gamma}{2}(w^*-w)+\Delta ww^*=0\,. \tag{2.11}$$

Since $w = u + iv$ we have

$$\alpha + \beta u - \gamma v + \Delta(u^2 + v^2) = 0\,. \tag{2.12}$$

This represents a family of circles in the $u - v$ plane when $\Delta \neq 0$ and a family of lines when $\Delta = 0$. Notice, that when $\alpha \neq 0$ and $\Delta \neq 0$ then the mapping maps circles to circles but a family of lines in the z-plane ($\alpha = 0$) also maps to a family of circles in the w-plane. However, there is also the case of a family of circles in the z-plane for which $\Delta = 0$ which map to a family of lines in the w-plane. **Thus we conclude that $w = \frac{1}{z}$ maps lines/circles to lines/circles but not necessarily lines to lines and circles to circles.**

In addition to this we now study the *fractional linear* or *Möbius* transformation

$$w = \frac{az + b}{cz + d}\,, \qquad ad \neq bc\,. \tag{2.13}$$

This includes cases such as:

(i) $w = \frac{1}{z}$ when $a = d = 0,\ b/c = 1\,.$

(ii) $w = \frac{1}{z-1}$ as in our example above where $a = 0,\ b = 1,\ c = 1,\ d = -1\,.$

(2.13) can be re-written as

$$w = c^{-1}\left\{a + \frac{bc - ad}{cz + d}\right\}\,. \tag{2.14}$$

For various special cases:

1. $w = z + b\,;\ \ (a = d = 1,\ c = 0)$ – translation.
2. $w = az\,;\ \ (b = c = 0,\ d = 1)$ – contraction/expansion + rotation
3. $w = \frac{1}{z}\,;\ \ (a = d = 0,\ b = c)$ – maps lines/circles to lines/circles.

Thus a Möbius transformation maps lines/circles to lines/circles with contraction/expansion, rotation and translation on top.

2.3 Extra: Mappings of the type $w = \frac{e^z - 1}{e^z + 1}$

Consider a map $w = \frac{e^z - 1}{e^z + 1}$ which can be re-written as

$$e^z = \frac{1 + w}{1 - w} = \frac{(1 + u + iv)(1 - u + iv)}{(1 - u)^2 + v^2}\,. \tag{2.15}$$

Real and imaginary parts give

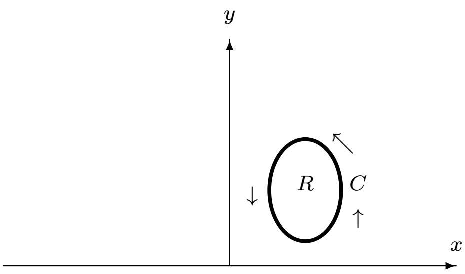
$$e^x \cos y = \frac{1 - u^2 - v^2}{(1 - u)^2 + v^2} \qquad e^x \sin y = \frac{2v}{(1 - u)^2 + v^2}\,. \tag{2.16}$$

From these we conclude that

1. The family of lines $y = n\pi$ in the z-plane map to the line $v = 0$ for n integer. Thus an infinite number of horizontal lines in the z-plane all map to the u-axis in the w-plane.
2. The family of lines $y = \frac{1}{2}(2n + 1)\pi$ in the z-plane map to the unit circle $u^2 + v^2 = 1$ in the w-plane.

3 Contour Integration

3.1 Cauchy's Theorem



A closed contour C enclosing a region R in the z-plane around which the line integral is considered in the counter-clockwise direction

$$\oint_C F(z)\,dz\,. \tag{3.1}$$

With

$$F(z) = u + iv \qquad\qquad z = x + iy \tag{3.2}$$

we have

$$\begin{aligned}\oint_C F(z)\,dz &= \oint_C (u+iv)(dx+idy)\\ &= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)\,.\end{aligned} \tag{3.3}$$

Now Green's Theorem in a plane says that for differentiable functions $P(x,y)$ and $Q(x,y)$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R (Q_x - P_y)\,dxdy\,. \tag{3.4}$$

Therefore we have

$$\begin{aligned}\oint (u\,dx - v\,dy) &= \int\int_R\int (-v_x - u_y)\,dxdy\,,\\ \oint (v\,dx + u\,dy) &= \int\int_R (u_x - v_y)\,dxdy\,,\end{aligned} \tag{3.5}$$

which turns (3.3) into

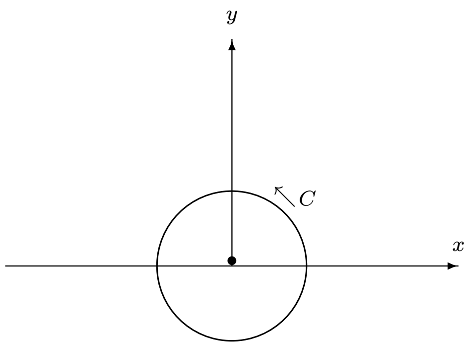
$$\oint_C F(z)\,dz = -\int\int_R (v_x + u_y)\,dxdy + i\int\int_R (u_x - v_y)\,dxdy\,. \tag{3.6}$$

If $F(z)$ is analytic everywhere within and on C then u and v must satisfy the CR equations: $u_x = v_y$ and $v_x = -u_y$, in which case both the real and imaginary parts on the RHS of (3.6) must be zero. We have established *Cauchy's Theorem:*

Theorem 2 *If $F(z)$ is analytic everywhere within and on a closed, piecewise smooth contour C then*

$$\oint_C F(z)\,dz = 0\,. \tag{3.7}$$

The key point is that provided $F(z)$ is analytic everywhere within and on C singularities in $F(z)$ *outside* of C are irrelevant.



For $F(z) = z^{-1}$, the circular contour C of radius a encloses a singularity • at the origin in the z-plane. The line integral is no longer zero because of this singularity.

Now write the circular contour C as $z = a\exp(i\theta)$ for $\theta\,:\,0 \to 2\pi$

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{ia\exp(i\theta)d\theta}{a\exp(i\theta)} = i\int_0^{2\pi} d\theta = 2\pi i\,. \tag{3.8}$$

The singularity at $z = 0$ contributes a non-zero value of $2\pi i$ to the integral. Note that it is independent of the value of a, which is consistent with this being the only non-zero contribution to the integral.

Given the powerful result of Cauchy's Theorem, our task is to see, in a more formal manner, how singularities contribute to complex integrals. Before this their nature and classification is necessary.

3.2 Poles and Residues

Singularities of complex functions can take many forms but the simplest class is what are called **simple poles**. A function $F(z)$ has a simple pole at $z = a$ (which could be real) if it can be written in the form

$$F(z) = \frac{f(z)}{z-a} \tag{3.9}$$

where $f(z)$ is an analytic function. $F(z)$ has a pole of multiplicity m at $z = a$ if it can be written in the form

$$F(z) = \frac{f(z)}{(z-a)^m} \tag{3.10}$$

where $m = 1, 2, 3, 4, \ldots$: when $m = 2$ we have a double pole. While poles are singularities not all singularities are poles. For instance, $\ln z$ has a singularity at $z = 0$ but this is not a pole nor is it a pole when m is not an integer.

Definition 1 : The **residue** of $F(z)$ at a simple pole at $z = a$ is

$$Residue\ of\ F(z) = \lim_{z\to a}\left\{(z-a)F(z)\right\}\,. \tag{3.11}$$

Definition 2 : The **residue** of $F(z)$ at a pole of multiplicity[2] m at $z = a$ is

$$Residue\ of\ F(z) = \lim_{z\to a}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m F(z)\right]\right\}. \tag{3.12}$$

Note that a function may have many poles and each pole has its own residue.

Example 1 : Consider $F(z) = \frac{2z}{(z-1)(z-2)}$ which has simple poles at $z = 1$ and $z = 2$

$$Residue\ at\ z = 1 = \lim_{z\to 1}\{(z-1)F(z)\} = -2\,. \tag{3.13}$$

$$Residue\ at\ z = 2 = \lim_{z\to 1}\{(z-2)F(z)\} = \ 4\,. \tag{3.14}$$

Example 2 : $F(z) = \frac{2z}{(z-1)^2(z+4)}$ has a double pole at $z = 1$ and a simple pole at $z = -4$.

$$Residue\ at\ z = -4 = \lim_{z\to -4}\{(z+4)F(z)\} = -8/25\,. \tag{3.15}$$

$$\begin{aligned} Residue\ at\ double\ pole\ at\ z = 1 &= \lim_{z\to 1}\frac{1}{1!}\left\{\frac{d}{dz}\left[(z-1^2)F(z)\right]\right\} \\ &= \lim_{z\to 1}\frac{d}{dz}\left[\frac{2z}{z+4}\right] \\ &= 2\lim_{z\to 1}\left[\frac{(z+4)-z}{(z+4)^2}\right] = 8/25\,. \end{aligned} \tag{3.16}$$

It is of no significance that the residues have opposite signs.

3.3 The residue of $F(z) = \frac{h(z)}{g(z)}$ when $g(z)$ has a simple zero at $z = a$

We expand $g(z)$ about its zero at $z = a$ in a Taylor series

$$g(z) = g(a) + (z-a)g'(a) + \tfrac{1}{2}(z-a)^2 g''(a) + \ldots \tag{3.17}$$

Thus, noting that $g(a) = 0$ we have

$$\begin{aligned} Residue\ at\ z = a &= \lim_{z\to a}\left\{\frac{(z-a)h(z)}{g(z)}\right\} \\ &= \lim_{z\to a}\left\{\frac{(z-a)h(z)}{(z-a)g'(a) + \frac{1}{2}(z-a)^2 g''(a) + \ldots}\right\} \\ &= \lim_{z\to a}\left\{\frac{h(z)}{g'(a) + \frac{1}{2}(z-a)g''(a) + \ldots}\right\} = \frac{h(a)}{g'(a)}\,. \end{aligned} \tag{3.18}$$

[2] This formula will be quoted at the bottom of an exam question : it is found from a coefficient in what is known as a Laurent expansion – see Kreysig's book.

3.4 The Residue Theorem

Now consider a simple pole at $z = a$ as in the Figure below. As explained in the caption, the pole is isolated by a device which consists of taking a "cut" into the contour and inscribing a small circle of radius r around it. The full *closed* contour C consists of the two edges of the cuts $C^{\pm}$ running in opposite directions, the small circle C_a and then C_1 which is the rest of C with the small piece ϵ removed. Thus we have

$$C\ :\ \ C_1 + C_a + C^+ + C^-\,. \tag{3.19}$$

This device ensures that the pole lies outside of C as it has been constructed, in which case Cauchy's Theorem says that

$$\oint_C F(z)\,dz = 0 \tag{3.20}$$

in which case

$$0 = \oint_C F(z)\,dz = \left(\int_{C_1} + \int_{C_a} + \int_{C^+} + \int_{C^-}\right) F(z)\,dz\,. \tag{3.21}$$

Two points to note are:

1. The four integrals are not closed so they don't have the $\oint$ notation. In the limit $\epsilon \to 0$ the integrals $\int_{C^+}$ and $\int_{C^-}$ cancel as they go in opposite directions;
2. The integral over C_a is clockwise, not counter-clockwise.

We are left with

$$\lim_{\epsilon\to 0}\int_{C_1} F(z)\,dz = -\lim_{\epsilon\to 0}\int_{C_a\hookleftarrow} F(z)\,dz = \int_{C_a\hookrightarrow} F(z)\,dz \tag{3.22}$$

We now write

$$\int_{C_a\hookrightarrow} F(z)\,dz = \int_{C_a\hookrightarrow} \frac{f(z)}{z-a}\,dz\,. \tag{3.23}$$

As in the figure we write the equation of the circle of radius a in the complex plane as $z = a + re^{i\theta}$ so

$$\int_{C_a\hookrightarrow} F(z)\,dz = \int_{C_a\hookrightarrow} \frac{f(z)}{z-a}\,dz = \int_{C_a\hookrightarrow} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta\,. \tag{3.24}$$

Our next step at this stage is to take the limit $r \to 0$

$$\lim_{r\to 0}\int_{C_a\hookrightarrow} F(z)\,dz = \lim_{r\to 0}\int_{C_a\hookrightarrow} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta = 2\pi i f(a) \tag{3.25}$$

which gives **Cauchy's integral formula**

$$\oint_C F(z)\,dz = \lim_{r\to 0}\int_{C_a\hookrightarrow} F(z)\,dz = 2\pi i f(a)\,. \tag{3.26}$$

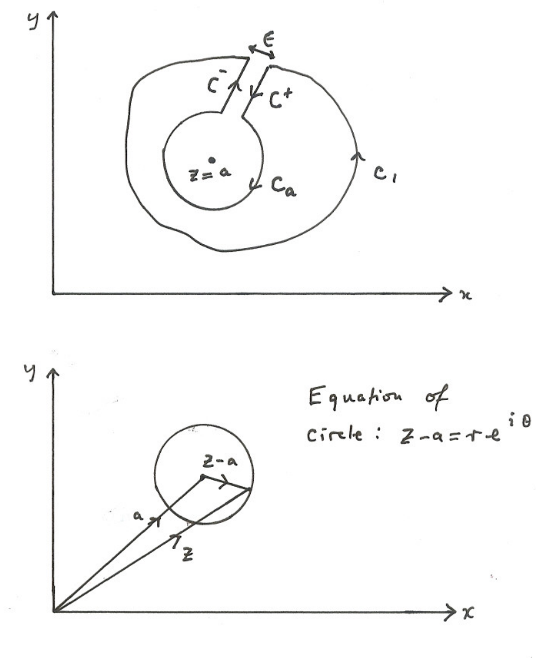


Figure 2: The top figure is the full contour C comprising the two edges of the cuts $C^{\pm}$ running in opposite directions, the small circle C_a and then C_1 which is the rest of C with the small piece ϵ removed. The bottom figure shows why the equation of the small circle of radius r is $z = a + r\exp i\theta$.

However, because $z = a$ is a simple pole

$$\textit{At the pole at } z = a \textit{ the residue of } F(z) = \lim_{z \to a} \{(z-a)F(z)\} = f(a) \tag{3.27}$$

We have proved

$$\oint_C F(z)\,dz = 2\pi i \times \{\text{Residue of } F(z) \text{ at the simple pole at } z = a\}\ . \tag{3.28}$$

It is clear that this procedure of making a cut and ring-fencing a pole can be performed for many simple poles and the individual residues added. The result can also be proved (not here) when poles have higher multiplicity. Altogether we have proved :

Theorem 3 (**Residue Theorem :**) *If the only singularities of $F(z)$ within C are poles then*

$$\oint_C F(z)\,dz = 2\pi i \times \{\textit{Sum of the residues of } F(z) \textit{ at its poles within } C\}\ . \tag{3.29}$$

Some examples of this immensely powerful theorem follow.

Example 1 : Find

$$\oint_{C_i} \frac{2z\,dz}{(z-1)(z-2)} \tag{3.30}$$

where (i) C_1 is the circle centred at $(0,0)$ of radius 3 and (ii) C_2 is the circle centred at $(0,0)$ of radius $3/2$.

$F(z)$ has two simple poles : the first at $z = 1$ and the second at $z = 2$. Their residues have been found in (3.13) and (3.14). For C_1 both poles lie inside C_1

$$\oint_{C_1} F(z)\,dz = 2\pi i \times (-2+4) = 4\pi i\,, \tag{3.31}$$

whereas for C_2 only the pole at $z = 1$ lie inside, thus

$$\oint_{C_2} F(z)\,dz = 2\pi i \times (-2) = -4\pi i\,. \tag{3.32}$$

Example 2 : Find

$$\oint_{C} \frac{2z\,dz}{(z-1)^2(z+4)} \tag{3.33}$$

where C is the circle of radius 5 centred at $z = 0$. For this C both poles lie inside so both must be included. From the residues computed in (3.15) and (3.16) we find that

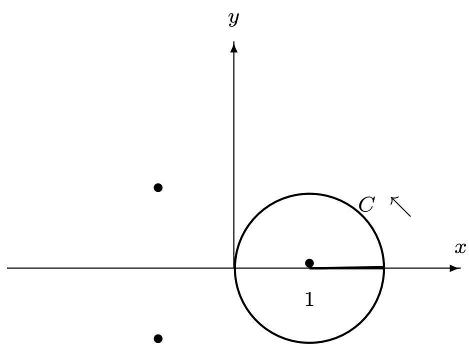
$$\oint_{C} \frac{2z\,dz}{(z-1)^2(z+4)} = 2\pi i \times \{-8/25 + 8/25\} = 0\,. \tag{3.34}$$

As remarked before, the zero sum of the residues has no significance.

Example 3 : Find

$$\oint_{C} \frac{dz}{(z^3-1)^2} \tag{3.35}$$

where C is the circle $|z-1| = 1$.



The contour is the circle $|z-1| = 1$ in the z-plane. The double pole lies at $z = 1$ whereas the two other double poles lies outside C at $z = \exp 2\pi i/3$ and $z = \exp -2\pi i/3$.

$z^3-1=0$ factors into $(z-1)(z^2+z+1)=0$ so it has zeroes at 1, $z=\exp(\pm 2\pi i/3)$. These are double poles for $F(z)$ but only the double pole at $z=1$ lies inside C. Its residue there is

$$\begin{aligned}\lim_{z\to 1}\frac{d}{dz}\left\{\frac{(z-1)^2}{(z^3-1)^2}\right\} &= \lim_{z\to 1}\frac{d}{dz}\left\{\frac{1}{(z^2+z+1)^2}\right\}\\ &= -2\lim_{z\to 1}\left\{\frac{2z+1}{(z^2+z+1)^3}\right\} = -2/9\,. \end{aligned}\tag{3.36}$$

Therefore we deduce from the Residue Theorem that

$$\oint_C \frac{dz}{(z^3-1)^2} = -4\pi i/9\,. \tag{3.37}$$

Example 4 : (2006) Find

$$\oint_C \frac{z\,dz}{(z-1)^2(z-i)} \tag{3.38}$$

where C is the circle $|z|=2$.

For the double pole at $z=1$, the residue there is

$$\begin{aligned}\lim_{z\to 1}\frac{d}{dz}\left\{\frac{z(z-1)^2}{(z-1)^2(z-i)}\right\} &= \lim_{z\to 1}\frac{d}{dz}\left\{\frac{z}{(z-i)}\right\}\\ &= -\frac{i}{(1-i)^2} = \tfrac{1}{2}\,. \end{aligned}\tag{3.39}$$

For the simple pole at $z=i$ the residue there is

$$\lim_{z\to i}\left\{\frac{z(z-i)}{(z-1)^2(z-i)}\right\} = \frac{i}{(1-i)^2} = -\tfrac{1}{2}\,. \tag{3.40}$$

Both poles must be included within C so we conclude from the Residue Theorem that

$$\oint_C \frac{z\,dz}{(z-1)^2(z-i)} = (\tfrac{1}{2}-\tfrac{1}{2}) = 0\,. \tag{3.41}$$

Example 5 : (2006) Find

$$\oint_C \frac{z^2dz}{(z-i)^3} \tag{3.42}$$

where C is the circle $|z|=2$ as above.

For the triple pole at $z=i$ the residue there is

$$\lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{z^2(z-i)^3}{(z-1)^3}\right\} = 1\,. \tag{3.43}$$

Hence

$$\oint_C \frac{z^2dz}{(z-i)^3} = 2\pi i\,. \tag{3.44}$$

3.5 Improper integrals and Jordan's Lemma

The Residue Theorem can be used to evaluate *real integrals* of the type

$$\int_{-\infty}^{\infty} e^{imx} F(x)\, dx\,, \qquad\qquad m \geq 0\,, \tag{3.45}$$

provided $F(x)$ has certain convergence properties : these are called *improper integrals* because of the infinite nature of their limits. Formally we write them as

$$\int_{-\infty}^{\infty} e^{imx} F(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} e^{imx} F(x)\, dx\,. \tag{3.46}$$

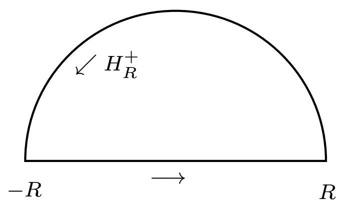
The main idea is to consider a class of *complex integrals*

$$\oint_C e^{imz} F(z)\, dz \tag{3.47}$$

where C consists of the semi-circular arc as in the figure below. The two essential parts are the arc of the semicircle of radius R denoted by H_R and the real axis $[-R,\, R]$. Hence we can re-write (3.47) as

$$\oint_C e^{imz} F(z)\, dz = \underbrace{\int_{-R}^{R} e^{imx} F(x)\, dx}_{real\ \ integral} + \underbrace{\int_{H_R} e^{imz} F(z)\, dz}_{complex\ \ integral} \tag{3.48}$$

In principle the closed complex integral over C on the LHS can be evaluated by the Residue Theorem. Our next aim is to evaluate the real integral on the RHS in the limit $R \to \infty$. This requires a result which is called Jordan's Lemma.



Jordan's Lemma deals with the problem of how a contour integral behaves on the semi-circular arc H_R^+ of a closed contour C.

Lemma 1 *(Jordan) If the only singularities of $F(z)$ are poles, then*

$$\lim_{R\to\infty} \int_{H_R} e^{imz} F(z)\, dz = 0 \tag{3.49}$$

provided that $m > 0$ and $|F(z)| \to 0$ as $R \to \infty$. If $m = 0$ then a faster convergence to zero is required for $F(z)$.

Proof: Since H_R is the semi-circle $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ and $dz = iRe^{i\theta}d\theta$

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{H_R} e^{imz}F(z)\,dz\right| &= \lim_{R\to\infty}\left|\int_{H_R} e^{imR\cos\theta - mR\sin\theta}F(z)R\,e^{i\theta}d\theta\right| \\ &\le \lim_{R\to\infty}\int_{H_R} e^{-mR\sin\theta}|F(z)|R\,d\theta \qquad (3.50)\end{aligned}$$

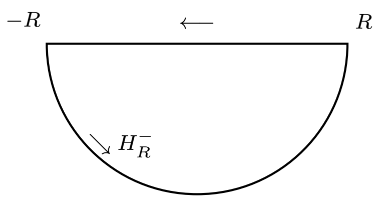
having recalled that $|e^{i\alpha}| = 1$ for any real α and $\left|\int f(z)\,dz\right| \le \int |f(z)|\,dz$. Note that in the exp-term on the RHS of (3.50), $\sin\theta > 0$ in the upper half plane. Hence, provided $m > 0$, the exponential ensures that the RHS is zero in the limit $R \to \infty$ (see remarks below). □

Remarks:

a) When $m > 0$ forms of $F(z)$ such as $F(z) = \frac{1}{z}$, $F(z) = \frac{1}{z+a}$ or rational functions of z such as $F(z) = \frac{z^p\dots}{z^q+\dots}$ (for $0 \le p < q$ and p and q integers) will all converge fast enough as these all have simple poles and $|F(z)| \to 0$ as $R \to \infty$.

b) If, however, $m = 0$ then a modification is needed: e.g. if $F(z) = \frac{1}{z}$ then $|F(z)| \to 0$ but $\lim_{R\to\infty} z|F(z)| = 1$. We need to alter the restriction on the integers p and q to $0 \le p < q-1$ which excludes cases like $F(z) = \frac{1}{z}$, $F(z) = \frac{1}{z+a}$.

c) What about $m < 0$? To ensure that the exponential is decreasing for $R \to \infty$ we need $\sin\theta < 0$. This is true in the *lower* half plane. Hence in this case we take our contour in the *lower* half plane (call this H_R^- as opposed to H_R^+ in the upper) but still in an anti-clockwise direction.

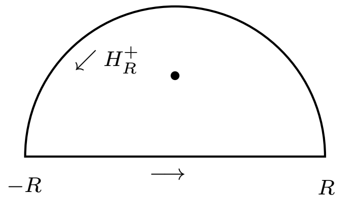


A contour in the lower $\frac{1}{2}$-plane with semi-circle H_R^- taken in the counter-clockwise direction which is used for cases when $m < 0$. See the notes on Fourier Transforms for cases when this is useful.

> The conclusion is that if $F(z)$ satisfies the conditions for Jordan's Lemma then
>
> $$\int_{-\infty}^{\infty} e^{imx}F(x)\,dx = 2\pi i\times\{\text{Sum of the residues of the poles of } e^{imz}F(z) \text{ in the upper } \tfrac{1}{2}\text{-plane}\}\,. \qquad (3.51)$$

Example 1 : Show that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi\,. \qquad (3.52)$$



C is comprised of a semi-circular arc H_R^+ and a section on the real axis from $-R$ to R. Only the simple pole at $z = i$ lies within C.

Thus we consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{dz}{1+z^2} \tag{3.53}$$

with $m = 0$. The simple pole at $z = i$ and the quadratic nature of the denominator is enough for convergence and so from Jordan's Lemma

$$\lim_{R\to\infty} \int_{H_R} \frac{dz}{1+z^2} = 0\,. \tag{3.54}$$

The residue of $F(z)$ at the pole in the upper-half-plane at $z = i$ is $1/2i$ and so from the Residue Theorem [3]

$$\oint_C \frac{dz}{1+z^2} = 2\pi i \times 1/2i = \pi\,. \tag{3.55}$$

Finally we have the result

$$\pi = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2}\,. \tag{3.56}$$

Example 2 : Show that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \pi/\sqrt{2}\,. \tag{3.57}$$

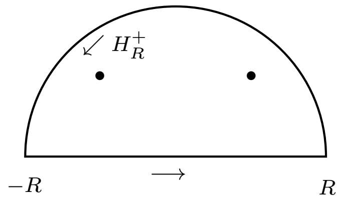
We consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{dz}{1+z^4} \tag{3.58}$$

with $m = 0$. The existence of poles as the only singularities and the quartic nature of the denominator allows us to appeal to Jordan's Lemma

$$\lim_{R\to\infty} \int_{H_R} \frac{dz}{1+z^4} = 0\,. \tag{3.59}$$

[3] Note that this result could have been found by direct integration but this can only be done for the case $n = 1$ when the denominator is $1 + x^{2n}$. See Examples 2 and 3.



The only simple poles in the upper half-plane at $e^{i\pi/4}$, $e^{3i\pi/4}$ lie within C.

$z^4 = -1$ has four zeroes lying at $e^{i\pi/4}$, $e^{3i\pi/4}$ in the upper half-plane and $e^{-i\pi/4}$, $e^{-3i\pi/4}$ in the lower half-plane. Only the first two are relevant. Now we use the trick in (3.18) to find the residues of the two poles in the upper half-plane. Using $h(z) = 1$ and $g(z) = 1 + z^4$ the residues at $e^{i\pi/4}$ and $e^{3i\pi/4}$ are

$$\left.\frac{1}{4z^3}\right|_{z=e^{i\pi/4}} = \tfrac{1}{4}e^{-3i\pi/4} \quad \text{and} \quad \left.\frac{1}{4z^3}\right|_{z=e^{3i\pi/4}} = \tfrac{1}{4}e^{-9i\pi/4} \tag{3.60}$$

Thus our final result is

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dx}{1+x^4} &= \tfrac{1}{2}\pi i\left(e^{-3i\pi/4} + e^{-9i\pi/4}\right)\\ &= \tfrac{1}{2}\pi i\left(-e^{i\pi/4} + e^{-i\pi/4}\right)\\ &= \pi\sin\left(\frac{\pi}{4}\right) = \pi/\sqrt{2}\,.\end{aligned} \tag{3.61}$$

Example 3: Show that

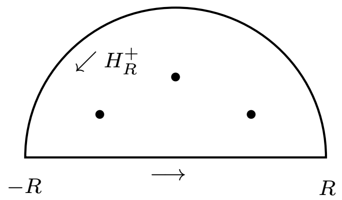
$$\int_{-\infty}^{\infty}\frac{dx}{1+x^6} = 2\pi/3\,. \tag{3.62}$$

Thus we consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{dz}{1+z^6} \tag{3.63}$$

with $m = 0$. The sextic nature of the denominator is enough for fast convergence and so from Jordan's Lemma

$$\lim_{R\to\infty}\int_{H_R}\frac{dz}{1+z^6} = 0\,. \tag{3.64}$$



The only simple poles in the upper half-plane at $e^{i\pi/6}$, $e^{i\pi/2}$ and $e^{5i\pi/6}$ lie within C.

$z^6 = -1$ has six zeroes lying at $e^{i\pi/6}$, $e^{i\pi/2}$ and $e^{5i\pi/6}$ in the upper half-plane and a further three in the lower half-plane. Now we use the trick in (3.18) to find the residues of the three poles in the upper half-plane. Using $h(z) = 1$ and $g(z) = 1 + z^6$ the residues at $e^{i\pi/6}$ and $e^{i\pi/2}$ and $e^{5i\pi/6}$ are

$$\left.\frac{1}{6z^5}\right|_{z=e^{i\pi/6}} = \tfrac{1}{6}e^{-5i\pi/6}\,; \quad \left.\frac{1}{6z^5}\right|_{z=e^{i\pi/2}} = \tfrac{1}{6}e^{-5i\pi/2}\,; \quad \left.\frac{1}{6z^5}\right|_{z=e^{5i\pi/6}} = \tfrac{1}{6}e^{-25i\pi/6} \tag{3.65}$$

Thus our final result is

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dx}{1+x^6} &= \frac{2\pi i}{6}\left(e^{-5i\pi/6} + e^{-5i\pi/2} + e^{-i\pi/6}\right) \\ &= -\frac{2\pi i^2}{6}\left(2\sin\tfrac{1}{6}\pi + \sin\pi/2\right) = 2\pi/3\,.\end{aligned} \tag{3.66}$$

Example 4 : For $m > 0$ show that

$$\int_{-\infty}^{\infty}\frac{\cos mx\,dx}{a^2+x^2} = \frac{\pi}{a}e^{-ma}\,. \tag{3.67}$$

We consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{e^{imz}dz}{a^2+z^2}\,. \tag{3.68}$$

The integrand has only one simple pole at $z = ia$ in the upper half-plane whose residue is

$$\lim_{z\to ia}\left\{\left(\frac{z-ia}{a^2+z^2}\right)e^{imz}\right\} = \frac{e^{-ma}}{2ia}\,. \tag{3.69}$$

Therefore, from the Residue Theorem

$$\oint_C \frac{e^{imz}\,dz}{a^2+z^2} = 2\pi i \times \frac{e^{-ma}}{2ia} = \frac{\pi}{a}e^{-ma}\,. \tag{3.70}$$

Moreover, from Jordan's Lemma

$$\lim_{R\to\infty}\int_{H_R}\frac{e^{imz}dz}{a^2+z^2} = 0\,. \tag{3.71}$$

Therefore

$$\frac{\pi}{a}e^{-ma} = \int_{-\infty}^{\infty}\frac{e^{imx}\,dx}{a^2+x^2} + 0\,. \tag{3.72}$$

What happens to the imaginary part

$$\int_{-\infty}^{\infty}\frac{\sin mx\,dx}{a^2+x^2}\,? \tag{3.73}$$

Notice that the integrand is an *odd function* : therefore, over $(-\infty,\,\infty)$ the part over $(-\infty,\,0)$ will cancel with that over $(0,\,\infty)$, leaving zero as a result. Thus we have

$$\int_{-\infty}^{\infty}\frac{\cos mx\,dx}{a^2+x^2} = \frac{\pi}{a}e^{-ma}\,. \tag{3.74}$$

Example 5 : For $m > 0$ show that

$$\int_{-\infty}^{\infty} \frac{\cos mx\, dx}{(a^2+x^2)^2} = \frac{\pi e^{-ma}}{2a^3}(1+ma)\,. \tag{3.75}$$

We consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{e^{imz}dz}{(a^2+z^2)^2}\,. \tag{3.76}$$

The integrand has only one double pole at $z = ia$ in the upper half-plane whose residue is

$$\lim_{z\to ia} \frac{d}{dz}\left[\frac{(z-ia)^2 e^{imz}}{(a^2+z^2)^2}\right] = -\frac{ie^{-ma}}{4a^3}(1+ma)\,. \tag{3.77}$$

Therefore, from the Residue Theorem

$$\begin{aligned} \oint_C \frac{e^{imz}\,dz}{(a^2+z^2)^2} &= -2\pi i \times \frac{ie^{-ma}}{4a^3}(1+ma) \\ &= \frac{\pi e^{-ma}}{2a^3}(1+ma) \end{aligned} \tag{3.78}$$

Moreover, from Jordan's Lemma, with $m > 0$

$$\lim_{R\to\infty} \int_{H_R} \frac{e^{imz}dz}{(a^2+z^2)^2} = 0\,. \tag{3.79}$$

Therefore

$$\frac{\pi e^{-ma}}{2a^3}(1+ma) = \int_{-\infty}^{\infty} \frac{e^{imx}\,dx}{(a^2+x^2)^2} + 0\,. \tag{3.80}$$

Aa in Example 4, the imaginary part is zero because the integrand is an odd function leaving

$$\int_{-\infty}^{\infty} \frac{\cos mx\, dx}{(a^2+x^2)^2} = \frac{\pi e^{-ma}}{2a^3}(1+ma)\,. \tag{3.81}$$

3.6 Integrals around the unit circle

We consider here integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$. Let us do this by example :

$$I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta}\,, \qquad a > 1\,. \tag{3.82}$$

Take C as the unit circle $z = e^{i\theta}$. Therefore $dz = ie^{i\theta}\,d\theta$ and

$$\begin{aligned} I &= \oint_C \frac{dz}{iz(a+\frac{1}{2}(z+z^{-1})} \\ &= -2i \oint_C \frac{dz}{z^2+2az+1}\,. \end{aligned} \tag{3.83}$$

The next task is to determine the roots of $z^2 + 2az + 1 = 0$.

$$z^2 + 2az + 1 = (z - \alpha^+)(z - \alpha^-) \tag{3.84}$$

where $\alpha^\pm = -a \pm \sqrt{a^2 - 1}$. Note that when $a > 1$, while α^+ lies *within* C, α^- lies *without*. Therefore we exclude the pole at $z = \alpha^-$ and compute the residue of the integrand at $z = \alpha^-$, which is

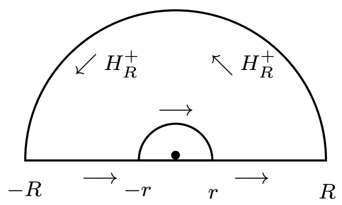
$$\frac{1}{\alpha^+ - \alpha^-} . \tag{3.85}$$

The Residue Theorem then gives

$$T = -2i \times 2\pi i \times \frac{1}{\alpha^+ - \alpha^-} = \frac{2\pi}{\sqrt{a^2 - 1}} . \tag{3.86}$$

3.7 Poles on the real axis

When an integrand has a pole on the real axis this means that it causes problems by sitting on the semicircular contour. Let us do this by example.



The contour is deformed by a small semi-circle of radius r centred at the origin that excludes the pole at $z = 0$. Following the direction of the arrows, the big semicircle of radius R is designated as H_R ($\theta : 0 \to \pi$) and the little semicircle of radius r is designated as H_r ($\theta : \pi \to 0$).

To calculate

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x} \tag{3.87}$$

we consider the complex integral

$$\oint_C \frac{e^{iz} \, dz}{z} . \tag{3.88}$$

The integrand has no poles in C because that $z = 0$ is excluded in the above construction. Cauchy's Theorem can be invoked to give

$$0 = \oint_C \frac{e^{iz} \, dz}{z} = \int_{-R}^{-r} \frac{e^{ix} \, dx}{x} + \int_{H_r \hookleftarrow} \frac{e^{iz} \, dz}{z} + \int_r^R \frac{e^{ix} \, dx}{x} + \int_{H_R \hookrightarrow} \frac{e^{iz} \, dz}{z} . \tag{3.89}$$

Now we take the limit $R \to \infty$ and, with $m = 1$ Jordan's Lemma tell us that $\int_{H_R} = 0$ because the only singularity is a pole and the integrand decays to zero as $R \to \infty$. Given that the small circle has the equation $z = r(\cos\theta + i\sin\theta)$ for $\theta : \pi \to 0$, and noting that $\sin\theta \geq 0$ in this range

$$\lim_{r \to 0} \int_{H_r} \frac{e^{iz}}{z} dz = i \lim_{r \to 0} \int_{\pi}^{0} e^{-r\sin\theta} e^{ir\cos\theta} d\theta = -\pi i . \tag{3.90}$$

Taking the two limits $R \to \infty$ and $r \to 0$ together, we have

$$0 = \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - \pi i + 0\,. \tag{3.91}$$

The real part of the integrand $\cos x/x$ is odd so the contributions on $(-\infty,\, 0)$ and $(0,\, \infty)$ cancel leaving us with

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi\,. \tag{3.92}$$

If the small circle is taken below the origin indented into the lower half-plane then its contribution is πi and, because the pole at $z = 0$ is now included with residue unity, the contribution from the Residue Theorem is $2\pi i$. Thus we end up with the same result, as we should.