Estimates for the two-dimensional Navier–Stokes equations in terms of the Reynolds number

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The tradition in Navier–Stokes analysis of finding estimates in terms of the Grashof number Gr, whose character depends on the ratio of the forcing to the viscosity ν, means that it is difficult to make comparisons with other results expressed in terms of Reynolds number Re, whose character depends on the fluid response to the forcing. The first task of this paper is to apply the approach of Doering and Foias [C. R. Doering and C. Foias, J. Fluid Mech. **467**, 289 (2002)] to the two-dimensional Navier–Stokes equations on a periodic domain $[0,L]^2$ by estimating quantities of physical relevance, particularly long-time averages $\langle\cdot\rangle$, in terms of the Reynolds number $\mathrm{Re}=U\ell/\nu$, where $U^2=L^{-2}\langle\|\boldsymbol{u}\|_2^2\rangle$ and ℓ is the forcing scale. In particular, the Constantin–Foias–Temam upper bound [P. Constantin, C. Foias, and R. Temam, Physica D **30**, 284 (1988)] on the attractor dimension converts to $a_\ell^2\mathrm{Re}(1+\ln\mathrm{Re})^{1/3}$, while the estimate for the inverse Kraichnan length is $(a_\ell^2\mathrm{Re})^{1/2}$, where a_ℓ is the aspect ratio of the forcing. Other inverse length scales, based on time averages, and associated with higher derivatives, are estimated in a similar manner. The second task is to address the issue of intermittency: it is shown how the time axis is broken up into very short intervals on which various quantities have lower bounds, larger than long time averages, which are themselves interspersed by longer, more quiescent, intervals of time. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2356912]

I. INTRODUCTION

A. General introduction

In the last two decades the notion of global attractors in parabolic partial differential equations has become a well-established concept.[1–4] The general nature of the dynamics on the attractor $\mathcal{A}$, in a time averaged sense, can roughly be captured by identifying sharp estimates of the Lyapunov (or fractal or Hausdorff) dimension of $\mathcal{A}$, or the number of determining modes,[5] with the number of degrees of freedom. Introduced by Landau,[6] this latter idea says that in a dynamical system of spatial dimension d of scale L, the number of degrees of freedom $\mathcal{N}$ is roughly defined to be that number of smallest eddies or features of scale λ and volume λ^d that fit into the system volume L^d,

$$\mathcal{N}\sim\left(\frac{L}{\lambda}\right)^d. \tag{1.1}$$

This is the origin of the much-quoted $\mathcal{N}\sim\mathrm{Re}^{9/4}$ result associated with the three-dimensional Navier–Stokes equations which rests on taking $\lambda\sim\lambda_k\sim L\,\mathrm{Re}^{-3/4}$, where λ_k is the Kolmogorov length scale. In the absence of a proof of existence and uniqueness of solutions of the three-dimensional Navier–Stokes equations, at best this is no more than a rule of thumb result. It rests on a more solid and rigorous foundation, however, for the closely related three-dimensional

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LANS-α equations for which Foias, Holm, and Titi[7] have proved existence and uniqueness of solutions. Following on from this, Gibbon and Holm[8] have demonstrated that the dimension of the global attractor for this system has an upper bound proportional to $\mathrm{Re}^{9/4}$. An important milestone has been passed recently in another closely related problem with the establishment by Cao and Titi[9] of an existence and uniqueness proof for Richardson's three-dimensional primitive equations for the atmosphere.

For the Navier–Stokes equations the idea sits more naturally in studies in the two-dimensional context. The existence and uniqueness of solutions has been a closed problem for many decades and the nature of the global attractor has been well-established.[1–5,10–14] While the two- and three-dimensional equations have the same velocity formulation, in reality, the former have a tenuous connection with the latter because of the absence of the drastic property of vortex stretching. As a result, the presence of vortex stretching in three dimensions, and perhaps other more subtle properties, have set up seemingly unsurmountable hurdles even on periodic boundary conditions. For problems on non-periodic boundaries, such as lid-driven flow, solving the two-dimensional Navier–Stokes equations is a technically more demanding problem—see some references in Refs. 10, 15, and 16.

The sharp estimate found by Constantin, Foias, and Temam[1] for the Lyapunov dimension of the global attractor $\mathcal{A}$ expressed in terms of the Grashof number Gr

$$d_L(\mathcal{A}) \leqslant c_1 \mathrm{Gr}^{2/3}(1 + \ln \mathrm{Gr})^{1/3}, \tag{1.2}$$

has been one of the most significant results in two-dimensional Navier–Stokes analysis on a periodic domain $\Omega = [0,L]^2_{\mathrm{per}}$. The traditional length scale in the two-dimensional Navier–Stokes equations is the Kraichnan length, η_k, which plays an equivalent role in two dimensions to that of the Kolmogorov length, λ_k, which is more important in three dimensions. In two dimensions, η_k and λ_k are defined respectively in terms of the enstrophy and energy dissipation rates ϵ_{ens} and ϵ,

$$\epsilon_{\mathrm{ens}} = \nu L^{-2}\left\langle \int_\Omega |\nabla \boldsymbol{\omega}|^2 dV \right\rangle, \quad \epsilon = \nu L^{-2}\left\langle \int_\Omega |\boldsymbol{\omega}|^2 dV \right\rangle, \tag{1.3}$$

where the pair of brackets $\langle\cdot\rangle$ denote a long-time average defined as[2,3,10–13]

$$\langle g(\cdot)\rangle = \lim_{t\to\infty} \limsup_{g(0)} \frac{1}{t}\int_0^t g(\tau)d\tau. \tag{1.4}$$

The inverse Kraichnan length η_k^{-1} and the inverse Kolmogorov length λ_k^{-1} are defined in terms of ϵ_{ens} and ϵ as

$$\eta_k^{-1} = \left(\frac{\epsilon_{\mathrm{ens}}}{\nu^3}\right)^{1/6}, \quad \lambda_k^{-1} = \left(\frac{\epsilon}{\nu^3}\right)^{1/4}. \tag{1.5}$$

It has been shown by Constantin, Foias, and Temam[1] that instead of using an estimate for ϵ_{ens} in terms of Gr, the upper bound for d_L can be re-expressed in terms of $L\eta_k^{-1}$ (see other literature on this topic[17–19])

$$d_L \leqslant c_2(L\eta_k^{-1})^2\{1 + \ln(L\eta_k^{-1})\}^{1/3}. \tag{1.6}$$

If d_L is identified with the number of degrees of freedom $\mathcal{N}$, this result is consistent with the idea expressed in Eq. (1.1) that in a two-dimensional domain, the average length scale of the smallest vortical feature λ can be identified with the Kraichnan length η_k, to within log corrections. The result in Eq. (1.2) has also been improved by Foias *et al.*[20,21] to an estimate proportional to $\mathrm{Gr}^{1/2}$ (to within logarithmic corrections) provided Kraichnan's theory of fully developed turbulence is implemented.[22]

While these results display a pleasing convergence between rigorous estimates and scaling methods in the two-dimensional case, the tradition in Navier–Stokes analysis of finding estimates in terms of the Grashof number Gr, whose character depends on the ratio of the forcing to the viscosity ν, means that it is difficult to compare with the results of scaling theories whose results are expressed in terms of Reynolds number. One of the tasks of this paper is to estimate quantities of physical relevance, particularly long-time averages, in terms of the Reynolds number, whose character depends on the fluid response to the forcing, and which is intrinsically a property of Navier–Stokes solutions. Doering and Foias[23] have addressed this problem and have shown that in the limit $\mathrm{Gr}\to\infty$, solutions of the d-dimensional Navier–Stokes equations must satisfy [This result is not advertised in Ref. 23 but follows immediately from their Eq. (48).]

$$\mathrm{Gr} \leqslant c(\mathrm{Re}^2 + \mathrm{Re}), \tag{1.7}$$

while the energy dissipation rate ϵ has a lower bound proportional to Gr. The problem, however, is not as simple as replacing standard estimates in terms of Gr by Re^2 from Eq. (1.7). Estimates such as that for d_L in Eq. (1.2) and the inverse Kraichnan and Kolmogorov lengths defined in Eq. (1.5), depend upon long time averages of the enstrophy and energy dissipation rates defined in Eq. (1.3). Other estimates of inverse length scales (to be discussed in Sec. I B) also depend upon long time averages. When estimated in terms of Re all these turn out to be better than straight substitution using Eq. (1.7). These results are summarized in Sec. I B and worked out in detail in Sec. II.

The second topic to be addressed in this paper is that of intermittency. Originally this important effect was considered to be a high Reynolds number phenomenon associated with three-dimensional Navier–Stokes flows. First discovered by Batchelor and Townsend,[24] it manifests itself in violent fluctuations of very short duration in the energy dissipation rate ϵ. These violent fluctuations away from the average are interspersed by quieter, longer periods in the dynamics. This is a well established, experimentally observable phenomenon;[25–27] its appearance in systems other than the Navier–Stokes equations has been discussed in an early and easily accessible paper by Frisch and Morf.[28] One symptom of its occurrence is the deviation of the "flatness" of a velocity signal (the ratio of the fourth order moment to the square of the second order moment) from the value of 3 that holds for Gaussian statistics.

Recent analysis discussing intermittency in three-dimensional Navier–Stokes flows shows that while it may be connected with loss of regularity, the two are subtly different issues.[29] This is reinforced by the fact that although solutions of the two-dimensional Navier–Stokes equations remain regular for arbitrarily long times, nevertheless many of its solutions at high Re are known to be intermittent.[30–34] While three-dimensional analysis of the problem is based on the assumption that a solution exists,[29,35] so that the higher norms can be differentiated, no such assumption is necessary in the two-dimensional case where existence and uniqueness are guaranteed. The result in both dimensions is such that the time-axis is broken up into good and bad intervals: on the latter there exist large lower bounds on certain quantities, necessarily resulting in their extreme narrowness and thus manifesting themselves as spikes in the data. This is summarized in Sec. I B and worked out in detail in Sec. IV.

B. Summary and interpretation of results

For simplicity the forcing $\boldsymbol{f}(\boldsymbol{x})$ in the two-dimensional Navier–Stokes equations $(\operatorname{div}\boldsymbol{u}=0)$

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f}(\boldsymbol{x}) \tag{1.8}$$

is taken to be divergence-free and smooth of narrow-band type, with a characteristic single length-scale ℓ such that[23,29,35]

$$\|\nabla^n \boldsymbol{f}\|_2 \approx \ell^{-n}\|\boldsymbol{f}\|_2. \tag{1.9}$$

Moreover, the aspect ratio of the forcing length scale to the box scale is defined as

$$a_\ell = L/\ell \, . \tag{1.10}$$

With $f_{\mathrm{rms}}=L^{-d/2}\|\boldsymbol{f}\|_2$, the usual definition of the Grashof number Gr appearing in Eq. (1.7) in d dimensions is

$$\mathrm{Gr} = \frac{\ell^3 f_{\mathrm{rms}}}{\nu^2}. \tag{1.11}$$

The Reynolds number Re in Eq. (1.7) is defined as

$$\mathrm{Re} = \frac{U\ell}{\nu}, \quad U^2 = L^{-d}\langle\|\boldsymbol{u}\|_2^2\rangle, \tag{1.12}$$

where $\langle\cdot\rangle$ is the long-time average defined in Eq. (1.4). One of the main results of this paper is the following theorem whose proof is given in Sec. II A. All generic constants are designated as c.

Theorem 1.1: *Let $\boldsymbol{u}(\boldsymbol{x},t)$ be a solution of the two-dimensional Navier–Stokes equations (1.8) on a periodic domain $[0,L]^2$, and subject to smooth, divergence-free, narrow-band forcing $\boldsymbol{f}(\boldsymbol{x})$. Then estimates in terms of the Reynolds number* Re *and the aspect ratio a_ℓ for the inverse Kraichnan length η_k^{-1}, the attractor dimension d_L, and the inverse Kolmogorov length λ_k^{-1} are given by*

$$L\eta_k^{-1} \leqslant c(a_\ell^2 \mathrm{Re})^{1/2}, \tag{1.13}$$

$$d_L \leqslant c a_\ell^2 \mathrm{Re}[1 + \ln \mathrm{Re}]^{1/3}, \tag{1.14}$$

$$L\lambda_k^{-1} \leqslant c a_\ell \mathrm{Re}^{5/8}. \tag{1.15}$$

In the short proof of this theorem in Sec. II A, the estimate for d_L in Eq. (1.14) is not reworked from first principles but is derived from a combination of Eqs. (1.13) and (1.14). The result in Eq. (1.15) comes from a $\mathrm{Re}^{5/2}$ bound on $\langle H_1\rangle$ and has also recently been found by Alexakis and Doering.[36] It implies that

$$\frac{L\epsilon}{U^3} \leqslant c a_\ell \mathrm{Re}^{-1/2}, \tag{1.16}$$

whereas in three dimensions the right hand side is $O(1)$. The estimate in Eq. (1.14) is also consistent with the result of Foias *et al.*[20] when their $\mathrm{Gr}^{1/2}$ estimate is converted to one proportioanl to Re. Their estimate, however, was based on the implementation of certain features of the Kraichnan model,[22] while Eq. (1.14) is true for all solutions and requires no assumption of fully developed turbulence.

The estimates for η_k^{-1} and d_L are consistent with the long-standing belief that $\mathrm{Re}^{1/2}\times\mathrm{Re}^{1/2}$ grid points are needed to numerically resolve a flow; indeed, when the aspect ratio is taken into account, Theorem 1.1 is consistent with $a_\ell \mathrm{Re}^{1/2}\times a_\ell \mathrm{Re}^{1/2}$. However, both these estimates are dependent upon only the time average of low moments of the velocity field. For non-Gaussian flows, low-order moments are not sufficient to uniquely determine the statistics of a flow. Thus it is necessary to find ways of estimating small length scales associated with higher-order moments. In Sec. II B we follow the way of defining inverse length scales associated with derivatives higher than 2, introduced elsewhere,[18] by combining the forcing with higher derivatives of the velocity field such that

$$F_n = \int_\Omega (|\nabla^n \boldsymbol{u}|^2 + \tau^2|\nabla^n \boldsymbol{f}|^2)dV, \tag{1.17}$$

where $\tau=\ell^2\nu^{-1}[\mathrm{Gr}(1+\ln \mathrm{Gr})]^{-1/2}$ is a characteristic time: this choice of τ is discussed in Appendix A. The gradient symbol ∇^n within Eq. (1.17) refers to all derivatives of every component of $\boldsymbol{u}$ of order n in $L^2(\Omega)$. The F_n are used to define a set of time-dependent inverse length scales

$$\kappa_{n,r}(t) = \left(\frac{F_n}{F_r}\right)^{1/2(n-r)}. \tag{1.18}$$

Actually, $\kappa_{n,0}^{2n}$ behaves as the $2n$th moment of the energy spectrum as shown by

$$\kappa_{n,0}^{2n} = \frac{\int_{2\pi/L}^{\infty} k^{2n}(|\hat{\boldsymbol{u}}|^2 + \tau^2|\hat{\boldsymbol{f}}|^2)dV_k}{\int_{2\pi/L}^{\infty}(|\hat{\boldsymbol{u}}|^2 + \tau^2|\hat{\boldsymbol{f}}|^2)dV_k}. \tag{1.19}$$

More relevant to the two-dimensional case, $\kappa_{n,1}^{2(n-1)}$ behaves as the $2(n-1)$th moment of the enstrophy spectrum. Using Landau's argument the dimension of the global attractor $d_L(\mathcal{A})$ was identified with the number of degrees of freedom $\mathcal{N}$. In Ref. 19 a definition was introduced to represent the number of degrees of freedom associated with all higher derivatives of the velocity field represented by $\kappa_{n,r}$, which is itself an inverse length. This naturally leads to the definition of the infinite set

$$\mathcal{N}_{n,r} = L^2\langle\kappa_{n,r}^2\rangle. \tag{1.20}$$

Using the definition of the quantities $\Lambda_{n,0}$ and $\Lambda_{n,1}$ $(n \geqslant 2)$,

$$\Lambda_{n,0} = \frac{3n-2}{2n}, \quad \Lambda_{n,1} = \frac{3n-4}{2(n-1)}, \tag{1.21}$$

the second main result of the paper is a theorem whose proof is given in Sec. II B.

Theorem 1.2: *Let $\kappa_{n,r}$ be the moments of a two-dimensional Navier–Stokes velocity field defined in Eq. (1.18). Then in a two-dimensional periodic box of side L the numbers of degrees of freedom $\mathcal{N}_{n,1}$ and $\mathcal{N}_{n,0}$ defined in Eq. (1.20) are estimated as* $(n \geqslant 2)$,

$$\mathcal{N}_{n,1} \leqslant c_{n,1}(a_\ell^2 \mathrm{Re})^{\Lambda_{n,1}}(1 + \ln \mathrm{Re})^{1/2}, \tag{1.22}$$

$$\mathcal{N}_{n,0} \leqslant c_{n,0}(a_\ell^2 \mathrm{Re})^{\Lambda_{n,0}}(1 + \ln \mathrm{Re})^{1/2}, \tag{1.23}$$

where $\Lambda_{n,0}$ and $\Lambda_{n,1}$ are defined in Eq. (1.21).

Note that $\Lambda_{2,0} = \Lambda_{2,1} = 1$. Thus the estimate for the first in each sequence, $\mathcal{N}_{2,1}$ and $\mathcal{N}_{1,0}$, are of the same order as the estimate for d_L, namely $a_\ell^2 \mathrm{Re}(1+\ln \mathrm{Re})^{1/3}$ except in the exponent of the logarithm. The exponents in Eqs. (1.22) and (1.23) provide an estimate of the extra resolution that is needed to take account of energy at sub-Kraichnan scales. Notice that in the limit $n \to \infty$ both exponents converge to $3/2$.

The intermittency results of Sec. IV show that there can exist small intervals of time where there are large *lower* bounds on $\kappa_{n,1}^2$ that are much larger than the upper bound on the long-time average for $\langle\kappa_{n,1}^2\rangle$. Translated into pictorial terms, Fig. 1 in Sec. IV is consistent with the existence of spiky data whose duration must be very short. Estimates are found for the width of these spikes which turn out to be in terms of a negative exponent of Re.

II. TIME AVERAGE ESTIMATES IN TERMS OF Re

A. Proof of Theorem 1.1

The first step in the proof of Theorem 1.1, which has been expressed in Sec. I B, is to find an upper bound on $\langle H_2\rangle$ in terms of Re. Consider the equation for the two-dimensional Navier–Stokes vorticity $\boldsymbol{\omega} = \omega\hat{\boldsymbol{k}}$,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{\omega} = \nu \boldsymbol{\Delta} \boldsymbol{\omega} + \mathrm{curl}\, \boldsymbol{f}, \tag{2.1}$$

and let H_n be defined by $(n \geqslant 0)$,

$$H_n = \int_\Omega |\boldsymbol{\nabla}^n \boldsymbol{u}|^2 dV. \tag{2.2}$$

For a periodic, divergence-free velocity field $\boldsymbol{u}$,

$$H_1 = \int_\Omega |\boldsymbol{\nabla} \boldsymbol{u}|^2 dV = \int_\Omega |\boldsymbol{\omega}|^2 dV. \tag{2.3}$$

Then the evolution equation for H_1 is

$$\frac{1}{2}\dot{H}_1 = -\nu H_2 + \int_\Omega \boldsymbol{\omega} \cdot \mathrm{curl}\, \boldsymbol{f} dV \tag{2.4}$$

$$\leqslant - \nu H_2 + \|\boldsymbol{u}\|_2 \|\boldsymbol{\nabla}^2 \boldsymbol{f}\|_2 \tag{2.5}$$

$$\leqslant - \nu H_2 + \ell^{-2} \|\boldsymbol{u}\|_2 \|\boldsymbol{f}\|_2, \tag{2.6}$$

where the forcing term has been integrated by parts in Eq. (2.4) and the narrow-band property has been used to move from Eq. (2.5) to Eq. (2.6). Using the definitions of Re, Gr, and a_ℓ in Eqs. (1.12), (1.11), and (1.10), the long-time average of H_2 is estimated as

$$\langle H_2 \rangle \leqslant L^2 \ell^{-6} \nu^2 \mathrm{Re}\ \mathrm{Gr} \tag{2.7}$$

$$\leqslant c a_\ell^2 \ell^{-4} \nu^2 \mathrm{Re}^3 + O(\mathrm{Re}^2). \tag{2.8}$$

This holds the key to the three results in Theorem 1.1.

The inverse Kraichnan length $\eta_k^{-6} = \epsilon_{\mathrm{ens}} / \nu^3$ with $\epsilon_{\mathrm{ens}} = \nu L^{-2} \langle H_2 \rangle$, can now be estimated by noting that

$$L^6 \epsilon_{\mathrm{ens}} \leqslant c a_\ell^6 \nu^3 \mathrm{Re}^3 \tag{2.9}$$

and so

$$L \eta_k^{-1} \leqslant c (a_\ell^2 \mathrm{Re})^{1/2}, \tag{2.10}$$

which is Eq. (1.13) of Theorem 1.1. The estimate for d_L in Eq. (1.14) then follows immediately from the relation between the estimate for d_L in Eqs. (1.6) and (2.10).

Finally, we turn to proving the estimate for $\langle H_1 \rangle$ in Eq. (1.15) which turns around the use of the simple inequality $H_1^2 \leqslant H_2 H_0$. The next step is to use the fact that

$$\langle H_1 \rangle \leqslant \langle H_2 \rangle^{1/2} \langle H_0 \rangle^{1/2} \tag{2.11}$$

$$= \nu a_\ell \mathrm{Re} \langle H_2 \rangle^{1/2}. \tag{2.12}$$

Using the upper bound in Eq. (2.7) gives

$$\langle H_1 \rangle \leqslant c \nu^2 a_\ell^2 \ell^2 \mathrm{Re}^{5/2}, \tag{2.13}$$

which then gives Eq. (1.15) in Theorem 1.1. In fact, Eq. (2.13) is an improvement in the bound for $\langle H_1 \rangle$ from Re^3 to $\mathrm{Re}^{5/2}$. This result has also been found recently by Alexakis and Doering.[36]

B. Proof of Theorem 1.2

Having introduced the notation for H_n in Eq. (2.2), similar quantities are used that contain the forcing,[35,29] namely

$$F_n = \int_\Omega (|\nabla^n \boldsymbol{u}|^2 + \tau^2 |\nabla^n \boldsymbol{f}|^2) dV, \tag{2.14}$$

defined first in Eq. (1.17), and the moments $\kappa_{n,r}$ defined in Eq. (1.18),

$$\kappa_{n,r}(t) := \left(\frac{F_n}{F_r}\right)^{1/2(n-r)}. \tag{2.15}$$

The parameter τ in Eq. (2.14) is a time scale and needs to be chosen appropriately. The idea is that it should be chosen in such a way that the forcing does not dominate the behavior of the moments of the velocity field. Defining $\omega_0 = \ell^{-2}\nu$, it is shown in Appendix A that this end is achieved if τ^{-1} is chosen as

$$\tau^{-1} = \omega_0 [\mathrm{Gr}(1 + \ln \mathrm{Gr})]^{1/2} \tag{2.16}$$

$$\leqslant c\omega_0 \mathrm{Re}(1 + \ln \mathrm{Re})^{1/2}. \tag{2.17}$$

As a preliminary to the proof of Theorem 1.2, we state the ladder theorem proved in Refs. 35 and 29.

Theorem 2.1: *The F_n satisfy the differential inequalities*

$$\tfrac{1}{2}\dot{F}_0 \leqslant -\nu F_1 + c\tau^{-1} F_0, \tag{2.18}$$

$$\tfrac{1}{2}\dot{F}_1 \leqslant -\nu F_2 + c\tau^{-1} F_1, \tag{2.19}$$

and, for $n \geqslant 2$, either

$$\tfrac{1}{2}\dot{F}_n \leqslant -\nu F_{n+1} + c_{n,1}(\|\nabla \boldsymbol{u}\|_\infty + \tau^{-1}) F_n, \tag{2.20}$$

or

$$\tfrac{1}{2}\dot{F}_n \leqslant -\tfrac{1}{2}\nu F_{n+1} + c_{n,2}(\nu^{-1}\|\boldsymbol{u}\|_\infty^2 + \tau^{-1}) F_n. \tag{2.21}$$

The L^∞ inequalities in Theorem 2.1, particularly $\|\nabla \boldsymbol{u}\|_\infty$ in Eq. (2.20), can be handled using a modified form of the L^∞ inequality of Brezis and Gallouet that has already been proved in Ref. 18.

Lemma 2.1: In terms of the F_n of Eq. (2.14) *and $\kappa_{3,2}$ of Eq.* (2.15), *a modified form of the two-dimensional L^∞ inequality of Brezis and Gallouet is*

$$\|\nabla \boldsymbol{u}\|_\infty \leqslant c F_2^{1/2} [1 + \ln(L\kappa_{3,2})]^{1/2}. \tag{2.22}$$

This lemma directly leads to an estimate for $\langle \kappa_{n,r}^2 \rangle$ for $r \geqslant 2$.

Lemma 2.2: For $n > r \geqslant 2$, to leading order in Re,

$$L^2 \langle \kappa_{n,r}^2 \rangle \leqslant c(a_\ell^2 \mathrm{Re})^{3/2} (1 + \ln \mathrm{Re})^{1/2}. \tag{2.23}$$

Proof: By dividing Eq. (2.20) by F_n and time averaging, we have

$$\nu \langle \kappa_{n+1,n}^2 \rangle \leqslant c_{n,1} \langle \|\nabla \boldsymbol{u}\|_\infty \rangle + c\omega_0 \mathrm{Re}(1 + \ln \mathrm{Re})^{1/2}. \tag{2.24}$$

However, because $\kappa_{n,r} \leqslant \kappa_{n+1,n}$ for $r < n$, for every $2 \leqslant r < n$, in combination with Lemma 2.1, we have

$$\nu\langle\kappa_{n,r}^2\rangle \leqslant c\langle F_2^{1/2}[1+\ln(L\kappa_{3,2})]^{1/2}\rangle + c\omega_0\mathrm{Re}(1+\ln\mathrm{Re})^{1/2}. \tag{2.25}$$

The logarithm is a concave function and $\kappa_{3,2}\leqslant\kappa_{n,r}$ so Jensen's inequality gives

$$L^2\langle\kappa_{n,r}^2\rangle \leqslant L^2\nu^{-1}c\langle F_2\rangle^{1/2}\langle[1+\ln\{L^2\langle\kappa_{n,r}^2\rangle\}]\rangle^{1/2} + ca_\ell^2\mathrm{Re}(1+\ln\mathrm{Re})^{1/2}. \tag{2.26}$$

The estimate for $\langle F_2\rangle$ can be found from $\langle H_2\rangle$ in Eq. (2.7); the extra term $\tau^2\|\nabla^2\boldsymbol{f}\|_2^2$ is no more than $O(\mathrm{Re}^2)$. Standard properties of the logarithm turn inequality Eq. (2.26) into Eq. (2.23). ■

Lemma 2.2 gives estimates for $\langle\kappa_{n,r}^2\rangle$ for $r\geqslant 2$. These are used in the following theorem to give better estimates for the cases $r=0$ and $r=1$. Prior to this, it is necessary to state the results that immediately derive from Eqs. (2.18) and (2.19) by, respectively, dividing through by F_0 and F_1 before time averaging

$$\mathcal{N}_{1,0}\equiv L^2\langle\kappa_{1,0}^2\rangle \leqslant ca_\ell^2\mathrm{Re}(1+\ln\mathrm{Re})^{1/2}, \quad \mathcal{N}_{2,1}\equiv L^2\langle\kappa_{2,1}^2\rangle \leqslant ca_\ell^2\mathrm{Re}(1+\ln\mathrm{Re})^{1/2}. \tag{2.27}$$

With the estimates in Eq. (2.27) we are now ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2: Let us return to Eq. (2.23) in Lemma 2.2 and use the fact that

$$\langle\kappa_{n,1}^2\rangle = \left\langle\left(\frac{F_n}{F_2}\right)^{1/n-1}\left(\frac{F_2}{F_1}\right)^{1/n-1}\right\rangle = \left\langle\kappa_{n,2}^{2(n-2)/n-1}\kappa_{2,1}^{2/n-1}\right\rangle, \tag{2.28}$$

and thus

$$\langle\kappa_{n,1}^2\rangle \leqslant \langle\kappa_{n,2}^2\rangle^{n-2/n-1}\langle\kappa_{2,1}^2\rangle^{1/n-1}. \tag{2.29}$$

Using Eq. (2.23) in Lemma 2.2, together with Eq. (2.27), for $n\geqslant 2$,

$$\mathcal{N}_{n,1} = L^2\langle\kappa_{n,1}^2\rangle \leqslant c_{n,1}(a_\ell^2\mathrm{Re})^{(3n-4)/2(n-1)}[1+\ln\mathrm{Re}]^{1/2}, \tag{2.30}$$

which coincides with $a_\ell^2\mathrm{Re}(1+\ln\mathrm{Re})^{1/2}$ at $n=2$ but converges to $\mathrm{Re}^{3/2}(1+\ln\mathrm{Re})^{1/2}$ as $n\to\infty$. The exponent $\Lambda_{n,1}$ is defined in Eq. (1.21).

Likewise, in the same manner as Eq. (2.28) we have

$$\langle\kappa_{n,0}^2\rangle \leqslant \langle\kappa_{n,1}^2\rangle^{(n-1)/n}\langle\kappa_{1,0}^2\rangle^{1/n}. \tag{2.31}$$

Thus we find that for $n\geqslant 2$,

$$\mathcal{N}_{n,0} = L^2\langle\kappa_{n,0}^2\rangle \leqslant c_{n,0}(a_\ell^2\mathrm{Re})^{(3n-2)/2n}[1+\ln\mathrm{Re}]^{1/2}. \tag{2.32}$$

The exponent $\Lambda_{n,0}$ is defined in Eq. (1.21). ■

III. POINT-WISE ESTIMATES

Let us consider the differential inequalities for H_0 and H_1:

$$\tfrac{1}{2}\dot{H}_0 \leqslant -\nu H_1 + \|\boldsymbol{f}\|_2 H_0^{1/2}, \tag{3.1}$$

$$\tfrac{1}{2}\dot{H}_1 \leqslant -\nu H_2 + \ell^{-2}\|\boldsymbol{f}\|_2 H_0^{1/2}, \tag{3.2}$$

having used the narrow-band property on Eq. (3.2). Upon combining Poincaré's inequality with Lemmas B.1 and B.2 in Appendix B we obtain

$$\overline{\lim}_{t\to\infty} H_0 \leqslant ca_\ell^6\nu^2\mathrm{Gr}^2 \leqslant ca_\ell^6\nu^2\mathrm{Re}^4, \tag{3.3}$$

and

$$\overline{\lim}_{t\to\infty} H_1 \leqslant c\ell^{-2}a_\ell^6\nu^2\mathrm{Gr}^2 \leqslant c\ell^{-2}a_\ell^6\nu^2\mathrm{Re}^4. \tag{3.4}$$

The additive forcing terms in F_1 and F_0 are of a lower order in Re so we end up with

$$\overline{\lim}_{t\to\infty} F_0 \leqslant c a_\ell^6 \nu^2 \mathrm{Re}^4 + O(\mathrm{Re}^2), \tag{3.5}$$

$$\overline{\lim}_{t\to\infty} F_1 \leqslant c \ell^{-2} a_\ell^6 \nu^2 \mathrm{Re}^4 + O(\mathrm{Re}^2). \tag{3.6}$$

The estimate for F_1 enables us to obtain point-wise estimates on F_n, $n \geqslant 2$ (Ref. 18, Sec. 7.2). In fact we have the following lemma.

Lemma 3.1: As $\mathrm{Gr} \to \infty$,

$$\overline{\lim}_{t\to\infty} F_n \leqslant c_n \nu^2 \ell^{-2n} a_\ell^{6n} \mathrm{Re}^{4n}. \tag{3.7}$$

Proof: Applying a Gagliardo–Nirenberg inequality in two-dimensions to $\nabla \boldsymbol{u}$ we obtain

$$\|\nabla \boldsymbol{u}\|_\infty \leqslant c \|\nabla^n \boldsymbol{u}\|_2^a \|\nabla \boldsymbol{u}\|_2^{1-a} \leqslant c F_n^{a/2} F_1^{(1-a)/2}, \tag{3.8}$$

with $a = 1/(n-1)$. Using this in Eq. (2.20) gives

$$\tfrac{1}{2}\dot{F}_n \leqslant -\nu F_{n+1} + c_n F_n^{1+a/2} F_1^{1-a/2} + c\omega_0 \mathrm{Re}(1+\ln \mathrm{Re})^{1/2} F_n. \tag{3.9}$$

Moreover, the following inequality can easily be proved using Fourier transforms:

$$F_N^{p+q} \leqslant F_{N-p}^q F_{N+q}^p, \tag{3.10}$$

from which, with $N=n$, $p=n-1$, $q=1$, it can be deduced that

$$-F_{n+1} \leqslant -\frac{F_n^{n/(n-1)}}{F_1^{1/(n-1)}}. \tag{3.11}$$

We now use Eq. (3.11) in Eq. (3.9) to obtain

$$\frac{1}{2}\dot{F}_n \leqslant -\nu \frac{F_n^{n/(n-1)}}{F_1^{1/(n-1)}} + c_n F_n^{1+a/2} F_1^{1-a/2} + c\omega_0 \mathrm{Re}(1+\ln \mathrm{Re})^{1/2} F_n, \tag{3.12}$$

with $a = 1/(n-1)$. We use now estimate Eq. (3.6) in Eq. (3.12) with the further use of Lemma B.2 to obtain

$$\overline{\lim}_{t\to\infty} F_n \leqslant c_n \nu^2 \ell^{-2n} a_\ell^{6n} \mathrm{Gr}^{2n}, \tag{3.13}$$

which leads to the result. ■

The above Lemma enables us to obtain an estimate on the wave numbers $\kappa_{n,r}$.

Lemma 3.2: For $n > r \geqslant 0$, *as* $\mathrm{Gr} \to \infty$,

$$\overline{\lim}_{t\to\infty} (L\kappa_{n,r}) \leqslant c_n a_\ell^{(4n-r-1)/(n-r)} \mathrm{Re}^{(2n-1)/(n-r)} (1+\ln \mathrm{Re})^{1/2(n-r)}. \tag{3.14}$$

Proof: Essentially one uses the upper bound on F_n and the lower bound on F_r which can be calculated from the forcing part in terms of Gr, leading to the result (see also Ref. 18, Chap. 7). ■

IV. INTERMITTENCY: GOOD AND BAD INTERVALS

The issue of intermittency in solutions of the two-dimensional Navier–Stokes equations is now addressed. While the F_n and $\kappa_{n,r}$ are bounded from above for all time, nevertheless it is possible that their behaviour could be spiky in an erratic manner. To show how this might come about, consider the definition of $\kappa_{n,r}$ in Eq. (1.18) from which we find

$$F_{n+1} = \kappa_{n,r}^2 \left(\frac{\kappa_{n+1,r}}{\kappa_{n,r}} \right)^{2(n+1-r)} F_n. \tag{4.1}$$

Now consider inequality (3.9) rewritten as

$$\frac{1}{2}\frac{\dot{F}_n}{F_n} \leqslant -\nu\kappa_{n,1}^2\left(\frac{\kappa_{n+1,1}}{\kappa_{n,1}}\right)^{2n} + c_n\left(\frac{\kappa_{n+1,1}}{\kappa_{n,1}}\right)^{n}\kappa_{n,1}F_1^{1/2} + c\omega_0\mathrm{Re}(1+\ln\mathrm{Re})^{1/2}, \tag{4.2}$$

where we have used Eq. (4.1) and the fact that $\kappa_{n,1} \leqslant \kappa_{n+1,1}$ in the middle term. Using Young's inequality on this same term we end up with

$$\frac{1}{2}\frac{\dot{F}_n}{F_n} \leqslant -\frac{1}{2}\nu\kappa_{n,1}^2\left(\frac{\kappa_{n+1,1}}{\kappa_{n,1}}\right)^{2n} + c_n\nu^{-1}F_1 + c\omega_0\mathrm{Re}(1+\ln\mathrm{Re})^{1/2}. \tag{4.3}$$

The main question is whether, for Navier–Stokes solutions, the lower bound on

$$\frac{\kappa_{n+1,1}}{\kappa_{n,1}} \geqslant 1 \tag{4.4}$$

can be raised from unity. A variation on the interval theorem proved in Ref. 29 is used.

Theorem 4.1: *For any value of the parameter $\mu \in (0,1)$, the ratio $\kappa_{n+1,1}/\kappa_{n,1}$ obeys the long-time averaged inequality* $(n \geqslant 2)$,

$$\left\langle \left[c_n\left(\frac{\kappa_{n+1,1}}{\kappa_{n,1}}\right)^2\right]^{1/\mu-1} - \left[\frac{(L^2\kappa_{n,1}^2)^{\mu}}{(a_\ell^2\mathrm{Re})^{\Lambda_{n,1}}(1+\ln\mathrm{Re})^{1/2}}\right]^{1/\mu-1}\right\rangle \geqslant 0, \tag{4.5}$$

where the c_n are the same as those in Theorem 1.2. Hence there exists at least one interval of time, designated as a "good interval", on which the inequality

$$c_n\left(\frac{\kappa_{n+1,1}}{\kappa_{n,1}}\right)^2 \geqslant \frac{(L^2\kappa_{n,1}^2)^{\mu}}{(a_\ell^2\mathrm{Re})^{\Lambda_{n,1}}(1+\ln\mathrm{Re})^{1/2}} \tag{4.6}$$

holds. Those other parts of the time-axis on which the reverse inequality

$$c_n\left(\frac{\kappa_{n+1,1}}{\kappa_{n}}\right)^2 < \frac{(L^2\kappa_{n,1}^2)^{\mu}}{(a_\ell^2\mathrm{Re})^{\Lambda_{n,1}}(1+\ln\mathrm{Re})^{1/2}} \tag{4.7}$$

holds are designated as "bad intervals".

Remark: In principle, the whole time-axis could be a good interval, whereas the positive time average in Eq. (4.5) ensures that the complete time axis cannot be "bad". This paper is based on the worst-case supposition that bad intervals exist, that they could be multiple in number, and that the good and the bad are interspersed. The precise distribution and occurrence of the good/bad intervals and how they depend on n remains an open question. The contrast between the two-dimensional and three-dimensional Navier–Stokes equations is prominent; while no singularities can occur in the $\kappa_{n,1}$ in the two-dimensional case, in three dimensions it is within these bad intervals that they can potentially occur.

Proof: Take two parameters $0<\mu<1$ and $0<\alpha<1$ such that $\mu+\alpha=1$. The inverses μ^{-1} and α^{-1} will be used as exponents in the Hölder inequality on the far right-hand side of

$$\langle\kappa_{n,1}^{2\alpha}\rangle \leqslant \langle\kappa_{n+1,1}^{2\alpha}\rangle = \left\langle\left(\frac{\kappa_{n+1,1}}{\kappa_{n,1}}\right)^{2\alpha}\kappa_{n,1}^{2\alpha}\right\rangle \leqslant \left\langle\left(\frac{\kappa_{n+1,1}}{\kappa_{n,1}}\right)^{2\alpha/\mu}\right\rangle^{\mu}\langle\kappa_{n,1}^2\rangle^{\alpha}, \tag{4.8}$$

thereby giving

$$\left\langle\left(\frac{\kappa_{n+1,1}}{\kappa_{n,1}}\right)^{2\alpha/\mu}\right\rangle \geqslant \left(\frac{\langle\kappa_{n,1}^{2\alpha}\rangle}{\langle\kappa_{n,1}^2\rangle^{\alpha}}\right)^{1/\mu} = \langle\kappa_{n,1}^{2\alpha}\rangle\left(\frac{\langle\kappa_{n,1}^{2\alpha}\rangle}{\langle\kappa_{n,1}^2\rangle}\right)^{\alpha/\mu}. \tag{4.9}$$

Two-dimensional Navier–Stokes information can be injected into these formal manipulations: the upper bound on $\langle\kappa_{n,1}^2\rangle$ from Theorem 1.2 and the lower bound $L\kappa_{n,1} \geqslant 1$ are used in the ratio on the far right-hand side of Eq. (4.9) to give Eq. (4.5), with the same c_n as in Theorem 1.2. ■

Now consider what must happen on bad intervals. It is always true that $\kappa_{n+1,1}/\kappa_{n,1}\geqslant 1$, so Eq. (4.7) implies that on these intervals there is a lower bound

$$L^2\kappa_{n,1}^2 > c_n(a_\ell^2\mathrm{Re})^{\Lambda_{n,1}/\mu}(1+\ln \mathrm{Re})^{1/2\mu}. \tag{4.10}$$

This lower bound cannot be greater than the upper point-wise bound in Eq. (3.14), which means that μ is restricted by

$$\frac{\Lambda_{n,1}}{\mu} < 2\left(\frac{2n-1}{n-1}\right). \tag{4.11}$$

Moreover, the factor of $1/\mu$ in the exponent makes the lower bound in Eq. (4.10) much larger than the *upper* bound on the average $\langle \kappa_{n,1}^2\rangle$ given in Theorem 1.2. *These intervals must therefore be very short*. To estimate how large they can be requires an integration of Eq. (4.3) over short times $\Delta t = t-t_0$ which, in turn, requires the time integral of H_1 for short times Δt. We use the notation $\int_{\Delta t}=\int_{t_0}^{t}$, with the definition $\omega_0=\nu\ell^{-2}$.

Lemma 4.1: To leading order in Re

$$\int_{\Delta t} F_1 dt \leqslant \nu a_\ell^4[c_1 a_\ell^2 + c_2\omega_0\Delta t]\mathrm{Re}^4. \tag{4.12}$$

Proof: Integrating Eq. (3.1) over a short time Δt gives

$$\nu\int_{\Delta t} H_1 dt \leqslant \frac{1}{2}H_0(t_0) + \Delta t[\ell^{-2}\nu^3 a_\ell^4 \mathrm{Gr}^2] \leqslant c_1 a_\ell^6\nu^2\mathrm{Re}^4 + \Delta t[c_2\ell^{-2}\nu^3 a_\ell^4\mathrm{Re}^4], \tag{4.13}$$

having used Eq. (3.3) for the $\frac{1}{2}H_0(t_0)$ term. The forcing term in F_1 is only $O(\mathrm{Re}^2)$. ■

Now we wish to estimate $\omega_0\Delta t$ in terms of Re. Integrating Eq. (4.3), using (4.13) and the lower bound Eq. (4.10) and multiplying by ℓ^2, we have

$$\tfrac{1}{2}\ell^2[\ln F_n(t) - \ln F_n(t_0)] + \tfrac{1}{2}c_n\nu a_\ell^{-2}(a_\ell^2\mathrm{Re})^{\Lambda_{n,1}/\mu}(1+\ln \mathrm{Re})^{1/2\mu}\Delta t \leqslant \ell^2 a_\ell^4[c_1a_\ell^2 + c_2\omega_0\Delta t]\mathrm{Re}^4 + c\ell^2\omega_0\Delta t\mathrm{Re}(1+\ln\mathrm{Re})^{1/2}. \tag{4.14}$$

As $\mathrm{Gr}\to\infty$, the dominant terms are

$$\omega_0\Delta t\{a_\ell^{-2}(a_\ell^2\mathrm{Re})^{\Lambda_{n,1}/\mu}(1+\ln\mathrm{Re})^{1/2\mu} - a_\ell^6\mathrm{Re}^4\} \leqslant c_1 a_\ell^6\mathrm{Re}^4. \tag{4.15}$$

Choosing μ in the range, to leading order we have

$$\mu < \tfrac{1}{4}\Lambda_{n,1}, \tag{4.16}$$

then Δt must satisfy

$$\omega_0\Delta t \leqslant c(a_\ell^2\mathrm{Re})^{4-\Lambda_{n,1}/\mu}. \tag{4.17}$$

Because the exponent in Eq. (4.17) is necessarily negative these intervals are very small and decreasing with increasing Re. Combining Eq. (4.11) with Eq. (4.16) we have

$$\frac{(n-1)}{2(2n-1)}\Lambda_{n,1} < \mu < \frac{1}{4}\Lambda_{n,1}, \tag{4.18}$$

which actually holds for every $n\geqslant 1$. Figure 1 is a cartoon-like figure displaying the lower bound on the bad intervals of width $(\Delta t)_b$ and also the maximum of $\kappa_{n,1}$ allowed by Eq. (3.14) in Lemma 3.2. The full dynamics of two-dimensional Navier–Stokes is actually determined by the intersection of all cartoons for every $n\geqslant 3$ on the grounds that the position and occurrence of the bad intervals varies with n. Thus we are interested in the limit $n\to\infty$ which determines that the range of μ is squeezed between

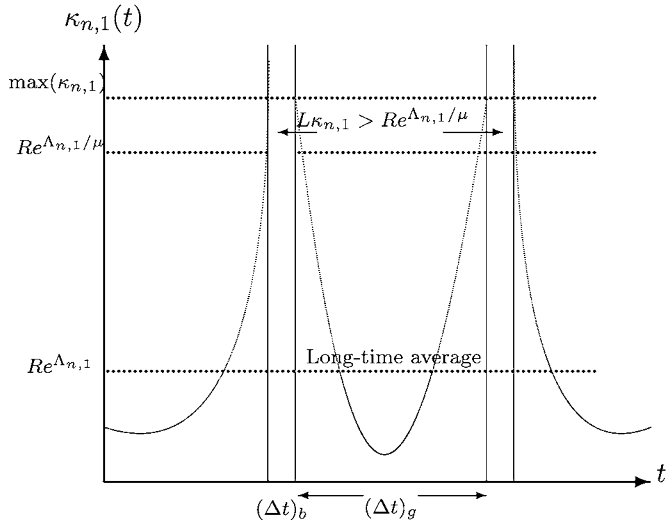

FIG. 1. A cartoon, not to scale, of good/bad intervals for some value of $n \geqslant 3$.

$$\frac{3}{8}\left(1-\frac{5}{6n}\right) < \mu < \frac{3}{8}\left(1-\frac{1}{3n}\right). \tag{4.19}$$

Thus, in the limit, μ takes a value just under $3/8$. We conclude that the interval theorem (Theorem 4.1) reproduces the effects of intermittency in a two-dimensional flow by manifesting very large lower bounds within bad intervals and suppressing spiky behavior within the good intervals which must be quiescent for long intervals, otherwise the long-time average would be violated.

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APPENDIX A: FORCING AND THE FLUID RESPONSE

For technical reasons, we must address the possibility that in their evolution the quantities H_n might take small values. Thus we need to circumvent problems that may arise when dividing by these (squared) seminorms. We follow Doering and Gibbon[35] who introduced the modified quantities

$$F_n = H_n + \tau^2 \|\nabla^n f\|_2^2, \tag{A1}$$

where the "time scale" τ is to be chosen for our convenience. So long as $\tau \neq 0$, the F_n are bounded away from zero by the explicit value $\tau^2 L^3 \ell^{-2n} f_{\rm rms}^2$. Moreover, we may choose τ to depend on the parameters of the problem such that $\langle F_n\rangle - \langle H_n\rangle$ as $\mathrm{Gr} \to \infty$. To see how to achieve this, let us define

$$\tau = \ell^2 \nu^{-1} [\mathrm{Gr}(1+\ln \mathrm{Gr})]^{-1/2}. \tag{A2}$$

Then the additional term in Eq. (A1) is

$$\tau^2 \|\nabla^n f\|_2^2 = L^3 \nu^{-2} \ell^{4-2n} f_{\rm rms}^2 [\mathrm{Gr}(1+\ln \mathrm{Gr})]^{-1} = \nu^2 \ell^{-(2n+2)} L^3 \mathrm{Gr}(1+\ln \mathrm{Gr})^{-1}. \tag{A3}$$

Now Doering and Foias[23] proved that in d dimensions, the energy dissipation rate ϵ has a lower bound of the form

$$\epsilon \geqslant c\nu^3 \ell^{-3} L^{-1} \mathrm{Gr}. \tag{A4}$$

Using this on the far right-hand side of Eq. (A3) we arrive at

$$\tau^2 \|\nabla^n \boldsymbol{f}\|_2^2 \leqslant c_6 \epsilon \ell^{-(2n-1)} L^4 \nu^{-1} (1+\ln \mathrm{Gr})^{-1} = c_6 \left(\frac{L}{\ell}\right)^{(2n-1)} L^{-2(n-1)} \langle H_1 \rangle > (1+\ln \mathrm{Gr})^{-1}. \tag{A5}$$

Using Poincaré's inequality in the form $H_1 \leqslant (2\pi L)^{2(n-1)} H_n$, as $\mathrm{Gr} \to \infty$ we have

$$\frac{\tau^2 \|\nabla^n \boldsymbol{f}\|_2^2}{\langle H_n \rangle} \leqslant c_6 a_\ell^{(2n-1)} (1+\ln \mathrm{Gr})^{-1}. \tag{A6}$$

Hence, the additional forcing term in Eq. (A1) becomes negligible with respect to $\langle H_n \rangle$ as $\mathrm{Gr} \to \infty$, so the forcing does not dominate the response.

APPENDIX B: COMPARISON THEOREMS FOR ODE

We present a comparison theorem for ODE which is useful for obtaining various estimates. We start with the following classical result.

Lemma B.1: Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function which is locally Lipschitz uniformly in t: for all intervals* $[a,b]\subset\mathbb{R}$ *there exists a constant such that* $|f(s,x)-f(s,y)| \leqslant C|x-y|$ *for all* $x,y\in[a,b]$ *and all* $s\in[0,T]$. *Furthermore, let* $x\in AC([0,T],\mathbb{R})$ *be such that*

$$\dot{x}(t) \leqslant f[t,x(t)]$$

for all $t\in[0,T]$ *and let* $y(t)$ *be the solution of* $\dot{y}(t)=f[t,y(t)]$ *on* $[0,T]$. *Assume further that* $x(0) \leqslant y(0)$. *Then,* $x(t)\leqslant y(t)$ *for all* $t\in[0,T]$.

We can use this Lemma to prove the following useful result.

Lemma B.2: Let $x:[0,T]\to[0,\infty)$ *be an absolutely continuous function with* $x(0)>0$ *which satisfies*

$$\dot{x} \leqslant \Delta_0 x + F x^{n_1} - E x^{n_2}, \tag{B1}$$

where $\Delta_0, F, E>0$ *and* $1<n_1<n_2$. *Then*

$$\limsup_{t\to\infty} x(t) \leqslant (4\Delta_0 E^{-1})^{1/n_2-1} + (2FE^{-1})^{1/n_2-n_1}. \tag{B2}$$

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