Quaternions and particle dynamics in the Euler fluid equations

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Abstract
Vorticity dynamics of the three-dimensional incompressible Euler equations are cast into a quaternionic representation governed by the Lagrangian evolution of the tetrad consisting of the growth rate and rotation rate of the vorticity. In turn, the Lagrangian evolution of this tetrad is governed by another that depends on the pressure Hessian. Together these form the basis for a direction of vorticity theorem on Lagrangian trajectories. Moreover, in this representation, fluid particles carry ortho-normal frames whose Lagrangian evolution in time are shown to be directly related to the Frenet–Serret equations for a vortex line. The frame dynamics suggest an elegant Lagrangian relation regarding the pressure Hessian tetrad. The equations for ideal MHD are similarly considered.

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1. Introductory and historical remarks

Hamilton's determined concentration on the idea of quaternions is often depicted by mathematical historians as an obsession. Lord Kelvin wrote that (O'Connor and Robertson 1998)

> *Quaternions came from Hamilton after his really good work had been done, and though beautifully ingenious, (they) have been an unmixed evil to those who have touched them in any way.*

Having fallen in and out of fashion over the last century and a half (Tait 1890), quaternions currently play an important part in the theory of 4-manifolds, through which it has been shown that the essential physics of particles and fields is governed by geometric principles. Fluid turbulence is one of the great unsolved problems of modern science. While viscosity plays a dominant role in the late development of an incompressible turbulent flow through the Navier–Stokes equations, the inviscid Euler equations determine the early and intermediate dynamics. The Euler fluid equations are known to be essentially geometrical, so it would not be surprising if quaternions were helpful in understanding their solutions.

A quaternion can be constructed from a scalar s and a 3-vector $\boldsymbol{r}$ by forming the tetrad[4] $\mathfrak{q} = [s, \boldsymbol{r}]$ that is defined by

$$\mathfrak{q} = [s, \boldsymbol{r}] = sI - \boldsymbol{r} \cdot \boldsymbol{\sigma}, \tag{1.1}$$

where $\boldsymbol{r} \cdot \boldsymbol{\sigma} = \sum_{i=1}^{3} r_i \sigma_i$ and I is the 2×2 unit matrix. $\{\sigma_1, \sigma_2, \sigma_3\}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{1.2}$$

that obey the relations $\sigma_i \sigma_j = -\delta_{ij} I - \epsilon_{ijk} \sigma_k$. A multiplication rule between two tetrads $\mathfrak{q}_1 = [s_1, \boldsymbol{r}_1]$ and $\mathfrak{q}_2 = [s_2, \boldsymbol{r}_2]$ can easily be determined from these properties

$$\mathfrak{q}_1 \circledast \mathfrak{q}_2 = [s_1 s_2 - \boldsymbol{r}_1 \cdot \boldsymbol{r}_2, s_1 \boldsymbol{r}_2 + s_2 \boldsymbol{r}_1 + \boldsymbol{r}_1 \times \boldsymbol{r}_2]. \tag{1.3}$$

This shows that quaternions are not commutative, although their associativity is easily demonstrated. They are found to be extremely useful in modern inertial navigation systems, robotics and graphics that are specifically designed to control or track rapidly moving objects undergoing three-axis rotations (Hanson 2006, Kuipers 1999). In fact, Hamilton discovered them in the context of an algorithm for rotating the telescope in his observatory. If Kelvin were alive today, he might be forced to revise his negative opinion of their importance.

Given the evidence, it is natural to reformulate Euler vorticity dynamics in terms of quaternions, particularly in tracking a fluid particle that carries its own ortho-normal co-ordinate system. Instead of setting Euler variables in standard function spaces, in which delicate geometric information might be lost, the principal aim of this paper is to investigate the Lagrangian evolution of these variables in appropriate quaternionic form in order to preserve their inherent geometric properties. The language of quaternions thus provides us with an alternative and unique look at the problem of Euler vortex dynamics. These manipulations are not specifically dependent upon the nature of the domain $\mathbb{D} \subset \mathbb{R}^3$ but for those parts of our work where the direction of vorticity is discussed, the local existence in time of classical solutions is necessary (Kato 1972). Thus, we restrict $\mathbb{D}$ to a three-dimensional periodic domain, although other more general forms of $\mathbb{D} \subset \mathbb{R}^3$ are also valid (see Majda and Bertozzi 2001). Otherwise our manipulations should be considered to be formal, particularly since Euler data gets rough very quickly.

Three-dimensional Euler vorticity growth is driven by the stretching vector $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$. This term plays a fundamental role in determining whether or not a singularity forms in finite time. Major computational studies can be found in Brachet *et al* (1983, 1992); Pumir and Siggia (1990); Kerr (1993, 2005); Grauer *et al* (1998), Pelz (2001) and Hou and Li (2006). The Beale–Kato–Majda theorem (Beale *et al* 1984) has been the main cornerstone of Euler analysis : one version of this theorem is the precise statement that $\int_0^t \|\boldsymbol{\omega}\|_{L^\infty(\mathbb{D})} d\tau$ must be finite to prevent singular behaviour on $\mathbb{D}$. A BMO-version of this theorem has been proved by Kozono and Taniuchi (2000). However, it has become clear that not only the magnitude but also the direction of vorticity is important. The papers by Constantin (1994), Constantin *et al* (1996), Cordoba and Fefferman (2001), Deng *et al* (2005, 2006) and Chae (2003a, 2005, 2006) are variations on this theme. References and a more global perspective on the Euler equations can be found in the book by Majda and Bertozzi (2001). Shnirelman (1997) has constructed very weak solutions which have some realistic features but whose kinetic energy monotonically decreases in time and which are everywhere discontinuous and unbounded. For work on Euler limits see Brenier (1999, 2000) and for its dynamics in the more exotic function spaces see the papers by Tadmor (2001) and Chae (2003b, 2004).

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[4] We avoid the direct nomenclature '4-vector' because of the meaning assigned to this in gauge theories.
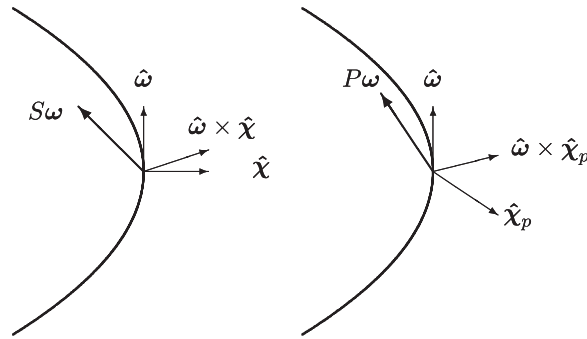
Figure 1. A vortex line with unit tangent vorticity vector $\hat{\omega}$. The normal vectors $\chi = \hat{\omega} \times S\hat{\omega}$ and $\chi_p = \hat{\omega} \times P\hat{\omega}$ are defined in (2.9) and (2.10). Thus, the three unit vectors $[\hat{\omega}, \hat{\chi}, \hat{\omega} \times \hat{\chi}]$ form an ortho-normal co-ordinate system. Moreover, $\hat{\omega}$, $S\hat{\omega}$ and $\hat{\omega} \times \hat{\chi}$ are co-planar, as are $\hat{\omega}$, $P\hat{\omega}$ and $\hat{\omega} \times \hat{\chi}_p$.

The new results in this paper displayed in sections 1–6 can be summarized as follows.

A well-known variable is the scalar growth rate $\alpha = \hat{\omega} \cdot S\hat{\omega}$ (Constantin 1994). Associated with this is the 3-vector rotation or swing rate $\chi = \hat{\omega} \times S\hat{\omega}$, where $\hat{\omega}$ is the unit vorticity and $S = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain matrix. Together these form a natural tetrad[5] $\mathfrak{q} = [\alpha, \chi]$. Theorem 1 of section 2 shows that the Lagrangian advection equation for the vorticity tetrad $\mathfrak{w} = [0, \omega]$ can then be written as

$$\frac{D\mathfrak{w}}{Dt} = \mathfrak{q} \circledast \mathfrak{w}. \tag{1.4}$$

All these quaternionic variables are Eulerian variables, i.e. point-wise functions of space and time, but undergoing Lagrangian evolution in time.

The tetrad $\mathfrak{q}$ satisfies its own Lagrangian advection equation driven by the effect of the pressure Hessian $P = \{p_{,ij}\}$ through the variables $\alpha_p = \hat{\omega} \cdot P\hat{\omega}$ and $\chi_p = \hat{\omega} \times P\hat{\omega}$. Together these also form a natural tetrad $\mathfrak{q}_p = [\alpha_p, \chi_p]$. Figure 1 shows how $S\omega$, $P\omega$ and the three orthonormal vectors $(\hat{\omega}, \hat{\chi}, \hat{\omega} \times \hat{\chi})$ are related. In addition to (1.4), theorem 1 also contains the results for the Lagrangian advection of $\mathfrak{q}$ and $\mathfrak{q}_p$. Simply stated this is

$$\frac{D\mathfrak{q}}{Dt} + \mathfrak{q} \circledast \mathfrak{q} + \mathfrak{q}_p = 0. \tag{1.5}$$

The result in (1.5) enables us to prove a theorem in section 2 on the direction of vorticity based on Lagrangian trajectories: '*Provided $\|\chi_p\|_{L^\infty(\mathbb{D})}$ is integrable in time up to $t^* > 0$ on a periodic domain $\mathbb{D}$, no Euler singularity is possible at t^*, with the exception of the case where $\hat{\omega}$ becomes collinear with an eigenvector of P at t^**'. Although different in detail, this result is in the same style as the direction of vorticity theorems cited above and is a variant of the BKM theorem. Ohkitani and Kishiba (1995) have observed in computations that at maximum points of enstrophy, ω becomes collinear with the most negative eigenvector of P. Collinearity may therefore be an important process in vorticity growth. The pressure Hessian P and its interplay with the strain matrix S has appeared in the Euler and Navier–Stokes literature in various places; see the references in Galanti *et al* (1997), Majda and Bertozzi (2001) and Chae (2006).

At each point in space-time a fluid particle carries its own ortho-normal co-ordinate system $(\hat{\omega}, \hat{\chi}, \hat{\omega} \times \hat{\chi})$: see figure 1. Explicit equations for Lagrangian time derivatives of this frame are

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[5] In Gibbon (2002) $\mathfrak{q} = [\alpha, \chi]$ was denoted as ζ. The change of notation to Gothic variables for tetrads has been introduced to avoid confusion between these and 3-vectors.

given in section 3. The corresponding Darboux vector is the particle rotation rate. The frame-equations are then shown to be directly related to the Frenet–Serret relations of differential geometry that govern the curvature and torsion of a vortex line through the arc-length derivative of its tangent, principal unit normal and bi-normal. Using Ertel's theorem, explicit differential equations for the curvature and torsion are then found.

It is shown in section 4 how to find Lagrangian differential equations for α_p and χ_p. The relation between $\mathfrak{q}_p$ and $\mathfrak{q}$ is given in theorem 3 where they are shown to satisfy

$$\frac{D\mathfrak{q}_p}{Dt} = \mathfrak{q} \circledast \mathfrak{q}_p + \mathfrak{P}. \tag{1.6}$$

$\mathfrak{P}$ is a tetrad linear in $\mathfrak{q}$ and $\mathfrak{q}_p$ whose arbitrary scalar coefficients, in principle, are determined by the Poisson pressure relation.

The vorticity vector-field $\boldsymbol{\omega} \cdot \nabla$ is frozen into the Euler flow. Any system with a frozen-in vector-field will also have an associated form of Ertel's theorem and a corresponding tetrad $\mathfrak{q} = [\alpha, \chi]$. Thus, the Lagrangian-quaternionic format displayed in this paper is more generally applicable, as illustrated by the equations for ideal MHD in section 5. Two time-clocks and two tetrads $\mathfrak{q}^{\pm} = [\alpha^{\pm}, \chi^{\pm}]$ appear as a result because of the two Lagrangian derivatives that naturally arise through the use of Elsasser variables.

Previous attempts at formulating Euler vorticity dynamics using quaternions have met with only partial success. Past results have appeared in reverse order: the relations between α and χ to be displayed in theorem 1 were derived first by Galanti *et al* (1997) (see also Gibbon *et al* 2000), which were then shown to be expressible in a quaternionic form (Gibbon 2002). That story was incomplete, however, because the Lagrangian advection equation for $\mathfrak{w}$ was missing, as were the ideas on particle frame dynamics, the pressure relation (1.6) and results on the direction of vorticity. Roubtsov and Roulstone (1997, 2001) have also formulated semi-geostrophic theory in terms of quaternions.

2. Vorticity dynamics in quaternion form

In their vorticity form, the three-dimensional incompressible Euler equations are

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = S\boldsymbol{\omega}, \tag{2.1}$$

where the strain matrix is written as $S = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $\boldsymbol{\omega} = \text{curl}\,\boldsymbol{u}$ is the vorticity (Majda and Bertozzi 2001). Equation (2.1) arises from taking the curl of the Euler equations in their velocity formulation

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p, \qquad \text{div}\,\boldsymbol{u} = 0, \tag{2.2}$$

in which the Lagrangian (material) derivative is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla. \tag{2.3}$$

The vorticity can be expressed as a tetrad by taking the quaternionic curl of $\mathfrak{U} = [0, \boldsymbol{u}]$

$$\nabla \circledast \mathfrak{U} = [-\text{div}\,\boldsymbol{u},\ \text{curl}\,\boldsymbol{u}]. \tag{2.4}$$

Thus, there exists a natural vorticity tetrad $\mathfrak{w}$ which has the divergence-free constraint built into it

$$\mathfrak{w} = [0,\ \boldsymbol{\omega}]. \tag{2.5}$$

The results in this paper employ Ertel's theorem (Ertel 1942), which is widely used in geophysical fluid dynamics in the study of potential vorticity: see Hide (1983, 2004) and Hoskins, *et al* (1985). More generally it applies to any fluid system whose flow preserves a

vector field, as the Euler equations preserve $\omega \cdot \nabla$. For the extensive history behind this result, which seems to have originated with Cauchy, see Truesdell and Toupin (1960), Kuznetsov and Zakharov (1997) and Viudez (1999). The most general form of Ertel's theorem says that if ω satisfies (2.1) then for an arbitrary differentiable vector θ

$$\frac{D}{Dt}(\omega \cdot \nabla \theta) = \omega \cdot \nabla \left(\frac{D\theta}{Dt} \right). \tag{2.6}$$

The choice of θ as the Euler velocity field u (Ohkitani 1993) implies that the vortex stretching vector $\omega \cdot \nabla u = S\omega$ is governed by

$$\frac{D(S\omega)}{Dt} = -P\omega, \tag{2.7}$$

where $P = \{p_{,ij}\} = \{\partial^2 p / \partial x_i \partial x_j\}$ is the Hessian matrix of the pressure. Thus, the combination of (2.1) and (2.7) gives Ohkitani's relation (Ohkitani 1993):

$$\frac{D^2\omega}{Dt^2} + P\omega = 0. \tag{2.8}$$

To understand how the *direction* in which the vorticity vector stretches (compresses) in relation to its growth rate requires an understanding of its relationship with the matrices S and P. The scalar and vector variables α and χ are defined by

$$\alpha = \hat{\omega} \cdot S\hat{\omega}, \qquad \chi = \hat{\omega} \times S\hat{\omega}, \tag{2.9}$$

$$\alpha_p = \hat{\omega} \cdot P\hat{\omega}, \qquad \chi_p = \hat{\omega} \times P\hat{\omega}. \tag{2.10}$$

The left part of figure 1, based upon $S\hat{\omega}$, shows the ortho-normal co-ordinate system $\hat{\omega}$, $\hat{\chi}$ and $\hat{\omega} \times \hat{\chi}$; the right-hand part of the figure shows the same figure with S replaced by P. Thus, $S\hat{\omega}$ can be decomposed into its components along the two orthogonal vectors $\hat{\omega}$ and $\chi \times \hat{\omega}$

$$S\hat{\omega} = \alpha\hat{\omega} + \chi \times \hat{\omega}. \tag{2.11}$$

From (2.1) and (2.11), the Lagrangian derivatives of $|\omega|$ and $\hat{\omega}$ are given by

$$\frac{D|\omega|}{Dt} = \alpha|\omega|, \qquad \frac{D\hat{\omega}}{Dt} = \chi \times \hat{\omega}. \tag{2.12}$$

The quantities (α, χ) are, respectively, the rates of change in vorticity magnitude and direction; that is, one may, respectively, call α and χ the stretching rate[6] and the rotation or swing rate. These variables form natural tetrads associated with $\mathfrak{w} = [0, \omega]$

$$\mathfrak{q} = [\alpha, \chi], \qquad \mathfrak{q}_p = [\alpha_p, \chi_p]. \tag{2.13}$$

The following theorem shows how Euler vorticity dynamics can be formulated using quaternions.

Theorem 1 (Euler vorticity dynamics in terms of quaternions). *The vorticity tetrad $\mathfrak{w}(x, t)$ satisfies the relation*

$$\frac{D\mathfrak{w}}{Dt} = \mathfrak{q} \circledast \mathfrak{w}, \tag{2.14}$$

while Ohkitani's relation (2.8) becomes

$$\frac{D^2\mathfrak{w}}{Dt^2} + \mathfrak{q}_p \circledast \mathfrak{w} = 0. \tag{2.15}$$

The tetrads $\mathfrak{q}(x, t)$ and $\mathfrak{q}_p(x, t)$ defined in (2.13) satisfy the compatibility relation (Riccati equation)

$$\frac{D\mathfrak{q}}{Dt} + \mathfrak{q} \circledast \mathfrak{q} + \mathfrak{q}_p = 0. \tag{2.16}$$

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[6] α and α_p are Rayleigh quotient estimates for eigenvalues of S and P, respectively, although they are only exact eigenvalues when ω aligns with one of their eigenvectors. Constantin (1994) has a Biot–Savart formula for α.

Remark. In terms of α and χ, the components of (2.16) were originally calculated by an indirect route in Gibbon (2002), although at that time (2.14) was not yet available. Moreover the present formulation simplifies the proof.

Proof. Equation (2.14) follows from (2.1) and (2.11) by direct calculation

$$\frac{D\mathfrak{w}}{Dt} = [0, \alpha\boldsymbol{\omega} + \boldsymbol{\chi} \times \boldsymbol{\omega}] = [\alpha, \boldsymbol{\chi}] \circledast [0, \boldsymbol{\omega}] = \mathfrak{q} \circledast \mathfrak{w}. \tag{2.17}$$

Following (2.11) and figure 1, we have

$$P\boldsymbol{\omega} = \alpha_p\boldsymbol{\omega} + \boldsymbol{\chi}_p \times \boldsymbol{\omega} \Rightarrow [0, \ P\boldsymbol{\omega}] = \mathfrak{q}_p \circledast \mathfrak{w}. \tag{2.18}$$

Consequently, Ohkitani's relation (2.8) implies

$$\frac{D^2\mathfrak{w}}{Dt^2} = \frac{D}{Dt}[0, \ S\boldsymbol{\omega}] = -[0, \ P\boldsymbol{\omega}] = -\mathfrak{q}_p \circledast \mathfrak{w}, \tag{2.19}$$

which is (2.15). Differentiating (2.14) again and using (2.15) gives the compatibility relation

$$\frac{D\mathfrak{q}}{Dt} \circledast \mathfrak{w} + \mathfrak{q} \circledast (\mathfrak{q} \circledast \mathfrak{w}) + \mathfrak{q}_p \circledast \mathfrak{w} = 0. \tag{2.20}$$

The result (2.16) in theorem 1 follows because of the associativity property. ∎

The meaning of χ now becomes clear. For structures such as straight vortex tubes or flat sheets, $\boldsymbol{\omega}$ aligns with an eigenvector of S and thus $\chi = 0$, in which case α is an exact eigenvalue of S. The Ricatti equation for $\mathfrak{q}$ in (2.16) reduces to a simple scalar form. However, as soon as a tube or sheet bends, twists or tangles, $\chi \neq 0$ and the full tetrad is restored. Because all our variables are functions of $(\boldsymbol{x}, t)$, equations (2.14) and (2.16) govern the vorticity dynamics at all points and all times in the flow provided solutions remain finite.

The BKM-theorem (Beale *et al* 1984) shows that the time integral $\int_0^{t^*} \|\boldsymbol{\omega}\|_{L^\infty(\mathbb{D})} \, d\tau$ must be finite at a time t^* to rule out singular behaviour. Variations on this theme are the direction of vorticity theorems expressed in the work of Constantin *et al* (1996), Cordoba and Fefferman (2001), Deng *et al* (2005, 2006) and Chae (2006). Chae's result (his theorem 5.1) is based on control of the time integral of $\|S\hat{\boldsymbol{\omega}} \cdot P\hat{\boldsymbol{\omega}}\|_\infty$, which is derivable from (2.7). Here, a direct consequence of theorem 1 concerns the pressure Hessian and its associated variable $\boldsymbol{\chi}_p$.

Theorem 2. *On the domain* $\mathbb{D} = [0, L]_{per}^3$ *there exists a global solution of the Euler equations,* $\boldsymbol{u} \in C([0, \infty); H^s) \cap C^1([0, \infty); H^{s-1})$ *for* $s \geqslant 3$ *if, for every* $t^* > 0$,

$$\int_0^{t^*} \|\boldsymbol{\chi}_p\|_{L^\infty(\mathbb{D})} \, d\tau < \infty, \tag{2.21}$$

excepting the case where $\hat{\boldsymbol{\omega}}$ *becomes collinear with an eigenvector of* P *at* t^*.

Remark. The theorem does not imply that blow-up occurs when collinearity does; it simply implies that under condition (2.21) it is the only situation when it can happen. Ohkitani (1993) and Ohkitani and Kishiba (1995) have noted the collinearity mentioned above; they observed in Euler computations that at maximum points of enstrophy, $\boldsymbol{\omega}$ tends to align with the eigenvector corresponding to the most negative eigenvalue of P.

Proof. Consider the relation for $\mathfrak{q}$ in theorem 1 in (2.16) expressed in $\alpha-\chi$ components

$$\frac{D\alpha}{Dt} = \chi^2 - \alpha^2 - \alpha_p \qquad \frac{D\chi}{Dt} = -2\alpha\chi - \hat{\chi} \cdot \boldsymbol{\chi}_p, \tag{2.22}$$
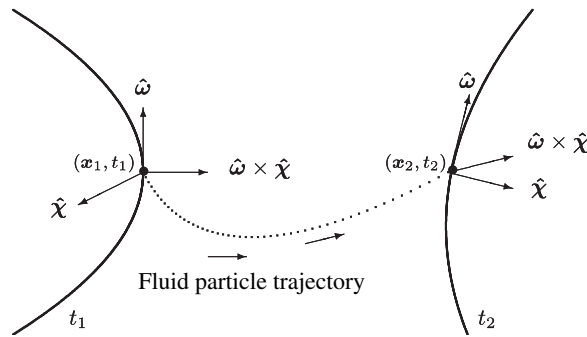
Figure 2. Vortex lines at two different times t_1 and t_2, with the dotted line representing the fluid particle ($\bullet$) trajectory moving from $(\boldsymbol{x}_1, t_1)$ to $(\boldsymbol{x}_2, t_2)$. The orientation of the ortho-normal unit vectors $(\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}})$ is shown at each of the two space-time points.

where $\alpha = \alpha(x, t)$ and $\chi = \chi(x, t)$. From (2.11) we know that

$$|S\hat{\boldsymbol{\omega}}|^2 = \alpha^2 + \chi^2 \tag{2.23}$$

and so

$$\frac{\mathrm{D}|S\hat{\boldsymbol{\omega}}|}{\mathrm{D}t} \leqslant -\alpha|S\hat{\boldsymbol{\omega}}| + \frac{|\alpha||\alpha_p| + |\chi||\chi_p|}{(\alpha^2 + \chi^2)^{1/2}}. \tag{2.24}$$

Because $\mathrm{D}|\boldsymbol{\omega}|/\mathrm{D}t = \alpha|\boldsymbol{\omega}|$ from (2.12), the magnitude of vorticity $|\boldsymbol{\omega}|$ cannot blow up for $\alpha < 0$. Thus our concern is with $\alpha \geqslant 0$. Inequality (2.24) becomes

$$\frac{\mathrm{D}|S\hat{\boldsymbol{\omega}}|}{\mathrm{D}t} \leqslant |\alpha_p| + |\chi_p|. \tag{2.25}$$

Lagrangian trajectories $X(t, \boldsymbol{x}_0)$ that satisfy $X_t = u(X(t, \boldsymbol{x}_0), t)$, are bounded in the specified space, so integrating along these we have

$$|S\hat{\boldsymbol{\omega}}(X(t, \boldsymbol{x}_0), t)| \leqslant |S\hat{\boldsymbol{\omega}}(X(0, \boldsymbol{x}_0), 0)| + \int_0^t \left(|\alpha_p(X(\tau, \boldsymbol{x}_0), \tau)| + |\chi_p(X(\tau, \boldsymbol{x}_0), \tau)| \right) \mathrm{d}\tau. \tag{2.26}$$

When there is no collinearity between $\hat{\boldsymbol{\omega}}$ and $P\hat{\boldsymbol{\omega}}$, the assumption of point-wise in space integrability in time of χ_p in (2.21) also extends to α_p. Thus $\|S\hat{\boldsymbol{\omega}}\|_\infty$ is bounded if (2.21) holds and, in consequence, so is $\|\boldsymbol{\omega}\|_\infty$. The Beale–Kato–Majda theorem then guarantees regularity of the Euler equations. However, there still exists the possibility that $|P\hat{\boldsymbol{\omega}}|$ could blow up simultaneously as the angle between $\hat{\boldsymbol{\omega}}$ and $P\hat{\boldsymbol{\omega}}$ approaches zero while keeping χ_p finite; under these circumstances $\int_0^t \|\chi_p\|_{L^\infty(\mathbb{D})} \mathrm{d}\tau < \infty$, whereas $\int_0^t \|\alpha_p\|_{L^\infty(\mathbb{D})} \mathrm{d}\tau \to \infty$; thus blow-up would still be theoretically possible. $\blacksquare$

3. Lagrangian frame dynamics for particles and the Frenet–Serret equations

3.1. Frame dynamics for particles

The Lagrangian dynamics of the ortho-normal frame $(\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}})$ can now be evaluated. Figure 2 illustrates the motion of a fluid particle from one co-ordinate point in space-time to another

To find a closed expression for the Lagrangian time derivatives of the ortho-normal set $(\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}})$ requires the derivative of $\hat{\boldsymbol{\chi}}$. To find this it is necessary to use the fact that the 3-vector $P\hat{\boldsymbol{\omega}}$ can be expressed in this ortho-normal frame as the linear combination

$$P\hat{\boldsymbol{\omega}} = \alpha_p\hat{\boldsymbol{\omega}} + c_1\hat{\boldsymbol{\chi}} + c_2(\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}}), \tag{3.1}$$
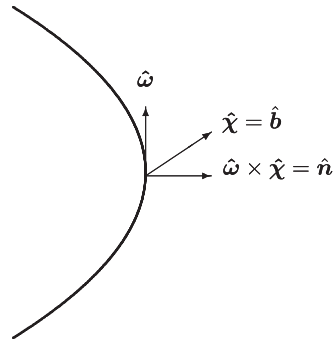
Figure 3. The ortho-normal frame $(\hat{\omega}, \hat{\omega} \times \hat{\chi}, \hat{\chi})$ as the Frenet–Serret frame.

where the coefficients c_1 and c_2 are defined by

$$c_1 = \hat{\omega} \cdot (\hat{\chi} \times \chi_p) \qquad c_2 = -(\hat{\chi} \cdot \chi_p). \tag{3.2}$$

The 3-vector product $\hat{\omega} \times P\hat{\omega}$ yields

$$\chi_p = c_1(\hat{\omega} \times \hat{\chi}) - c_2\hat{\chi}. \tag{3.3}$$

The Lagrangian time derivative of $\hat{\chi}$ comes from the 3-vector part of equation (2.16) for the tetrad $\mathfrak{q} = [\alpha, \chi]$ in theorem 1

$$\frac{D\chi}{Dt} = -2\alpha\chi - \chi_p \Rightarrow \frac{D\chi}{Dt} = -2\alpha\chi + c_2, \tag{3.4}$$

where $\chi = |\chi|$. Using (3.3) and (3.4) there follows

$$\frac{D\hat{\chi}}{Dt} = -c_1\chi^{-1}(\hat{\omega} \times \hat{\chi}), \qquad \frac{D(\hat{\omega} \times \hat{\chi})}{Dt} = \chi\,\hat{\omega} + c_1\chi^{-1}\hat{\chi}. \tag{3.5}$$

Thus, according to Euler's fluid equations, the Lagrangian time derivatives of the ortho-normal set can be expressed as

$$\frac{D\hat{\omega}}{Dt} = \mathcal{D} \times \hat{\omega}, \tag{3.6}$$

$$\frac{D(\hat{\omega} \times \hat{\chi})}{Dt} = \mathcal{D} \times (\hat{\omega} \times \hat{\chi}), \tag{3.7}$$

$$\frac{D\hat{\chi}}{Dt} = \mathcal{D} \times \hat{\chi}, \tag{3.8}$$

where the 'Darboux angular velocity vector' $\mathcal{D}$ for the ortho-normal frame is defined as

$$\mathcal{D} = \chi - \frac{c_1}{\chi}\hat{\omega} \qquad \text{with } |\mathcal{D}|^2 = \chi^2 + \frac{c_1^2}{\chi^2}. \tag{3.9}$$

3.2. Frame dynamics and the Frenet–Serret equations

Modulo a rotation around the unit tangent vector $\hat{\omega}$, with $\hat{\chi}$ as the unit bi-normal $\hat{b}$ and $\hat{\omega} \times \hat{\chi}$ as the unit principal normal $\hat{n}$ (figure 3), the matrix F can be formed

$$F = (\hat{\omega}^T, (\hat{\omega} \times \hat{\chi})^T, \hat{\chi}^T), \tag{3.10}$$

and (3.6)–(3.8) can be re-written as

$$\frac{DF}{Dt} = AF, \qquad A = \begin{pmatrix} 0 & -\chi & 0 \\ \chi & 0 & -c_1\chi^{-1} \\ 0 & c_1\chi^{-1} & 0 \end{pmatrix}. \tag{3.11}$$

For a space curve parametrized by arc-length s, then the Frenet–Serret equations relating dF/ds to the curvature κ and the torsion τ of a vortex line are

$$\frac{dF}{ds} = BF \qquad \text{where } B = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}. \tag{3.12}$$

It is now possible to relate the t and s derivatives of F given in (3.11) and (3.12). At any time t the integral curves of the vorticity vector field define a space-curve through each point $\boldsymbol{x}$: these space curves are called 'vortex lines'. The arc-length derivative d/ds is defined by

$$\frac{d}{ds} = \hat{\boldsymbol{\omega}} \cdot \nabla. \tag{3.13}$$

The evolution of the curvature κ and torsion τ of a vortex line may be obtained from Ertel's theorem in (2.6), expressed as the commutation of operators

$$\alpha \frac{d}{ds} + \left[\frac{D}{Dt}, \frac{d}{ds} \right] = 0. \tag{3.14}$$

Applying this to F and using the relations (3.11) and (3.12) gives the Lax pair

$$\alpha B + \frac{DB}{Dt} - \frac{dA}{ds} + [B, A] = 0. \tag{3.15}$$

Thus, Ertel's theorem gives explicit evolution equations for the curvature κ and torsion τ that lie within the matrix B and relates them to c_1 and χ.

4. A Lagrangian advection equation for q_p

One of the hurdles in pursuing a Lagrangian approach to the Euler equations is the problem of the non-locality of the pressure field. Overtly, we have no Lagrangian differential equations for either α_p or χ_p: the usual numerical procedure is to up-date the pressure through its Poisson equation $-\Delta p = u_{i,j} u_{j,i}$. How to address this issue can be illustrated by an example. Differentiating the orthogonality relation $\chi \cdot \hat{\boldsymbol{\omega}} = 0$ and using the derivative of $\hat{\boldsymbol{\omega}}$ in (2.12) gives

$$\hat{\boldsymbol{\omega}} \cdot \frac{D\chi}{Dt} = 0 \Rightarrow \frac{D\chi}{Dt} = \boldsymbol{q}_0, \tag{4.1}$$

where $\boldsymbol{q}_0$ lies in the plane perpendicular to $\hat{\boldsymbol{\omega}}$ in which χ and χ_p also lie. Thus, $\boldsymbol{q}_0 = A(\boldsymbol{x}, t)\chi + B(\boldsymbol{x}, t)\chi_p$; independently it is known that $A = -2\alpha$ and $B = -1$ from (2.16). The same method may be used for χ_p by differentiating $\chi_p \cdot \hat{\boldsymbol{\omega}} = 0$, leading to

$$\frac{D\chi_p}{Dt} = \chi \times \chi_p + \boldsymbol{q} \qquad \text{where } \boldsymbol{q} = \mu\chi + \lambda\chi_p, \tag{4.2}$$

where $\mu = \mu(\boldsymbol{x}, t)$ and $\lambda = \lambda(\boldsymbol{x}, t)$ are unknown scalars. Explicitly differentiating $\chi_p = \hat{\boldsymbol{\omega}} \times P\hat{\boldsymbol{\omega}}$ in (4.2) gives

$$\hat{\boldsymbol{\omega}}(\chi \cdot P\hat{\boldsymbol{\omega}}) - \alpha_p\chi + \hat{\boldsymbol{\omega}} \times \frac{D(P\hat{\boldsymbol{\omega}})}{Dt} = \hat{\boldsymbol{\omega}}(\chi \cdot P\hat{\boldsymbol{\omega}}) + \boldsymbol{q}. \tag{4.3}$$

Using the cross product $\chi = \hat{\boldsymbol{\omega}} \times S\hat{\boldsymbol{\omega}}$, this can be manipulated into

$$\hat{\boldsymbol{\omega}} \times \left\{ \frac{D(P\hat{\boldsymbol{\omega}})}{Dt} - \alpha_p S\hat{\boldsymbol{\omega}} \right\} = \boldsymbol{q}, \tag{4.4}$$

which means that

$$\frac{D(P\hat{\boldsymbol{\omega}})}{Dt} = \alpha_p S\hat{\boldsymbol{\omega}} + \boldsymbol{q} \times \hat{\boldsymbol{\omega}} + \varepsilon\hat{\boldsymbol{\omega}}, \tag{4.5}$$

where $\varepsilon = \varepsilon(\boldsymbol{x}, t)$ is a third unknown scalar in addition to μ and λ in (4.2). Thus, the Lagrangian derivative of $\alpha_p = \hat{\boldsymbol{\omega}} \cdot P\hat{\boldsymbol{\omega}}$ is

$$\frac{\mathrm{D}\alpha_p}{\mathrm{D}t} = \alpha\alpha_p + \chi \cdot \chi_p + \varepsilon. \tag{4.6}$$

Lagrangian differential relations have now been found for χ_p and α_p, but at the price of introducing the triplet of unknown coefficients μ, λ, and ε which must adjust in a flow to take the Poisson pressure constraint into account—they cannot be regarded as arbitrary.

Dimensional analysis on the various Euler variables governed by equations[7] (4.6) for α_p and (4.2) for χ_p shows that $[\omega] = T^{-1}$, $[\alpha] = T^{-1}$, $[\chi] = T^{-1}$, whereas $[\alpha_p] = T^{-2}$, $[\chi_p] = T^{-2}$. This means that $[\lambda] = T^{-1}$, $[\mu] = T^{-2}$ and $[\varepsilon] = T^{-3}$. Since the Euler equations possess no other time scale μ, λ and ε must be expressible in terms of these units or their ratios

$$\mu = \mu(\omega, \alpha, \chi, \alpha_p, \chi_p) \qquad \text{such that } [\mu] = T^{-2}, \tag{4.7}$$

$$\lambda = \lambda(\omega, \alpha, \chi, \alpha_p, \chi_p) \qquad \text{such that } [\lambda] = T^{-1}, \tag{4.8}$$

$$\varepsilon = \varepsilon(\omega, \alpha, \chi, \alpha_p, \chi_p) \qquad \text{such that } [\varepsilon] = T^{-3}. \tag{4.9}$$

Now re-define the triplet such that

$$\lambda = \alpha + \lambda_1, \qquad \mu = \alpha_p + \mu_1, \qquad \varepsilon = -2\chi \cdot \chi_p + \mu_1\alpha + \lambda_1\alpha_p + \varepsilon_1 \tag{4.10}$$

where the new triplet is subsumed into the tetrad (the unit tetrad is $\mathbb{I} = [1, 0]$)

$$\mathfrak{P} = \mu_1\mathfrak{q} + \lambda_1\mathfrak{q}_p + \varepsilon_1\mathbb{I}. \tag{4.11}$$

Theorem 3 (dynamics of $\mathfrak{q}_p$). *The pressure tetrad $\mathfrak{q}_p = [\alpha_p, \chi_p]$ satisfies*

$$\frac{\mathrm{D}\mathfrak{q}_p}{\mathrm{D}t} = \mathfrak{q} \circledast \mathfrak{q}_p + \mathfrak{P}, \tag{4.12}$$

where the triplet of scalar variables μ_1, λ_1 and ε_1 within $\mathfrak{P}(\boldsymbol{x}, t)$ is determined by the Poisson equation for the pressure

$$-TrP = TrS^2 - \tfrac{1}{2}\omega^2. \tag{4.13}$$

Remark. It has yet to be understood what effect the Poisson pressure constraint has on the triplet of scalars μ_1, λ_1 and ε_1 within $\mathfrak{P}$. They are all not likely to be zero because, for example, in the simple case of Burgers vortex $\alpha = \alpha_0 = \text{const}$; $\alpha_p = -\alpha_0^2$; $\chi = \chi_p = 0 \implies \varepsilon_1 = \alpha_0^3$ and $\lambda_1 = \mu_1 = 0$.

5. Ideal MHD

As already indicated, these ideas can be pursued for other systems that possess vortex stretching. The equations of ideal incompressible MHD couple an ideal fluid to a magnetic field $\boldsymbol{B}$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{B} \cdot \nabla\boldsymbol{B} - \nabla p, \tag{5.1}$$

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \nabla\boldsymbol{u}, \tag{5.2}$$

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[7] S and P are subsumed into the scalars α, α_p, χ and χ_p so there is no need to consider them separately.

together with $\mathrm{div}\boldsymbol{u} = 0$ and $\mathrm{div}\boldsymbol{B} = 0$. The pressure p in (5.1) is the combination $p = p_f + \frac{1}{2}B^2$ where p_f is the fluid pressure. Elsasser variables are defined by combining the $\boldsymbol{u}$ and $\boldsymbol{B}$ fields such that

$$\boldsymbol{v}^\pm = \boldsymbol{u} \pm \boldsymbol{B}. \tag{5.3}$$

The existence of two velocities $\boldsymbol{v}^\pm$ means that there are two material derivatives (and two time clocks)

$$\frac{\mathrm{D}^\pm}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}^\pm \cdot \nabla. \tag{5.4}$$

In terms of these, (5.1) and (5.2) can be rewritten as

$$\frac{\mathrm{D}^\pm \boldsymbol{v}^\mp}{\mathrm{D}t} = -\nabla p, \tag{5.5}$$

with the magnetic field $\boldsymbol{B}$ satisfying

$$\frac{\mathrm{D}^\pm \boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \nabla \boldsymbol{v}^\pm \equiv \boldsymbol{\sigma}^\pm, \tag{5.6}$$

together with $\mathrm{div}\boldsymbol{v}^\pm = 0$. The $\boldsymbol{\sigma}^\pm$-stretching vectors defined in (5.6) obey an Ertel's relation already proved in Gibbon (2002)

$$\frac{\mathrm{D}^\pm \boldsymbol{\sigma}^\mp}{\mathrm{D}t} = -P\boldsymbol{B}. \tag{5.7}$$

The relations in (5.6) thus allow us to define

$$\alpha^\pm = \hat{\boldsymbol{B}} \cdot (\hat{\boldsymbol{B}} \cdot \nabla \boldsymbol{v}^\pm) \qquad \chi^\pm = \hat{\boldsymbol{B}} \times (\hat{\boldsymbol{B}} \cdot \nabla \boldsymbol{v}^\pm), \tag{5.8}$$

having used Moffatt's analogy between the vectors $\boldsymbol{\omega}$ and $\boldsymbol{B}$ (Moffatt 1978). Moreover, because $\boldsymbol{\sigma}^\pm$ defined in (5.6) lie in the plane of the unit vectors $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{B}} \times \hat{\boldsymbol{\chi}}^\pm$ we have the decomposition

$$\boldsymbol{\sigma}^\pm = \alpha^\pm \boldsymbol{B} + \chi^\pm \times \boldsymbol{B}. \tag{5.9}$$

Thus, it is easy to prove that

$$\frac{\mathrm{D}^\pm |\boldsymbol{B}|}{\mathrm{D}t} = \alpha^\pm |\boldsymbol{B}|, \qquad \frac{\mathrm{D}^\pm \hat{\boldsymbol{B}}}{\mathrm{D}t} = \chi^\pm \times \hat{\boldsymbol{B}}, \tag{5.10}$$

which correspond to (2.12) for the Euler equations. The $\alpha^\pm$ play the role(s) of scalar Elsasser stretching rates, with $\chi^\pm$ as the rotation rates. One may also define corresponding variables based upon the Hessian matrix P

$$\alpha_{pB} = \hat{\boldsymbol{B}} \cdot P\hat{\boldsymbol{B}}, \qquad \chi_{pB} = \hat{\boldsymbol{B}} \times P\hat{\boldsymbol{B}}. \tag{5.11}$$

We define the tetrads $\mathfrak{q}^\pm$ and $\mathfrak{q}_{pB}$ as follows

$$\mathfrak{w}_B = [0, \boldsymbol{B}], \qquad \mathfrak{q}^\pm = [\alpha^\pm, \chi^\pm], \qquad \mathfrak{q}_{pB} = [\alpha_{pB}, \chi_{pB}]. \tag{5.12}$$

Theorem 4. *The magnetic field tetrad $\mathfrak{w}_B$ satisfies the two relations*

$$\frac{\mathrm{D}^\pm \mathfrak{w}_B}{\mathrm{D}t} = \mathfrak{q}^\pm \circledast \mathfrak{w}_B, \tag{5.13}$$

$$\frac{\mathrm{D}^\mp}{\mathrm{D}t} \left(\frac{\mathrm{D}^\pm \mathfrak{w}_B}{\mathrm{D}t} \right) + \mathfrak{q}_{pB} \circledast \mathfrak{w}_B = 0. \tag{5.14}$$

The tetrads $\mathfrak{q}^\pm$ satisfies the compatibility relation

$$\frac{\mathrm{D}^\mp \mathfrak{q}^\pm}{\mathrm{D}t} + \mathfrak{q}^\pm \circledast \mathfrak{q}^\mp + \mathfrak{q}_{pB} = 0. \tag{5.15}$$

Proof. The proof of (5.13) follows immediately from (5.6)

$$\frac{D^{\pm}\mathfrak{w}_B}{Dt} = \left[0, \frac{D^{\pm}\boldsymbol{B}}{Dt}\right] = [0, \boldsymbol{\sigma}^{\pm}] = \mathfrak{q}^{\pm} \circledast \mathfrak{w}_B, \tag{5.16}$$

where we have used (5.9) at the last step. The proof of (5.14) follows by combining (5.6) and (5.7) together with the fact that $P\boldsymbol{B}$ lies in the plane of the unit vectors $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{B}} \times \hat{\boldsymbol{\chi}}_{pB}$. Thus, we have the decomposition

$$P\boldsymbol{B} = \alpha_{pB}\boldsymbol{B} + \boldsymbol{\chi}_{pB} \times \boldsymbol{B}, \Rightarrow [0, P\boldsymbol{B}] = \mathfrak{q}_{pB} \circledast \mathfrak{w}_B. \tag{5.17}$$

The proof of (5.15) follows as a compatibility relation between (5.13) and (5.14). ∎

Finally, MHD–Lagrangian frame dynamics, in the spirit of section 3, needs to be interpreted in terms of two sets of ortho-normal vectors $\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{\chi}}^{\pm}$, $(\hat{\boldsymbol{B}} \times \hat{\boldsymbol{\chi}}^{\pm})$ acted on by their opposite Lagrangian time derivatives. After some calculation we find the equivalent of (3.6)–(3.8) and (3.9) is

$$\frac{D^{\mp}\hat{\boldsymbol{B}}}{Dt} = \boldsymbol{\mathcal{D}}^{\mp} \times \hat{\boldsymbol{B}}, \tag{5.18}$$

$$\frac{D^{\mp}}{Dt}(\hat{\boldsymbol{B}} \times \hat{\boldsymbol{\chi}}^{\pm}) = \boldsymbol{\mathcal{D}}^{\mp} \times (\hat{\boldsymbol{B}} \times \hat{\boldsymbol{\chi}}^{\pm}), \tag{5.19}$$

$$\frac{D^{\mp}\hat{\boldsymbol{\chi}}^{\pm}}{Dt} = \boldsymbol{\mathcal{D}}^{\mp} \times \hat{\boldsymbol{\chi}}^{\pm}, \tag{5.20}$$

where the pair of Elsasser Darboux vectors $\boldsymbol{\mathcal{D}}^{\mp}$ are defined as

$$\boldsymbol{\mathcal{D}}^{\mp} = \boldsymbol{\chi}^{\mp} - \frac{c_1^{\mp}}{\chi^{\mp}}\hat{\boldsymbol{B}}, \qquad c_1^{\mp} = \hat{\boldsymbol{B}} \cdot [\hat{\boldsymbol{\chi}}^{\pm} \times (\boldsymbol{\chi}_{pB} + \alpha^{\pm}\boldsymbol{\chi}^{\mp})]. \tag{5.21}$$

6. Summary

The tetrad reformulation of Euler's equations in this paper appears to be completely natural, giving results that are remarkably simple in their expression. It also provides a new hybrid picture of ideal fluid dynamics in which the Lagrangian fluid parcels carry ortho-normal frames, whose rotation velocity depends on the local Eulerian values of the pressure and vorticity. These frames are defined by three ortho-normal vectors: (a) along the vorticity, (b) along its rate of change following the Lagrangian trajectory and (c) along the cross product of these two unit vectors. This frame is governed by the Darboux vector that has components that lie only in the $\hat{\boldsymbol{\chi}}$–$\hat{\boldsymbol{\omega}}$ plane.

Remarkably, a picture similar to that for Euler fluids also emerges for magnetic fluids described by the ideal MHD equations. The MHD equations have two characteristic velocities, corresponding to the two Elsässer variables. Thus, MHD–Elsässer variables summon two Lagrangian characteristics along which the evolutionary equations reduce to ortho-normal frame dynamics. Instead of being attached to the vorticity vector, both of these MHD frames are attached to the magnetic field vector (Moffatt 1978). The second vector in each moving frame is obtained by the rate of change in magnetic field along the Elsasser characteristic. The two frames are then completed by taking the cross product of the first two unit vectors in each frame. Again the rates of rotation of these Elsasser frames depends on local Eulerian properties and the respective Darboux angular velocity vectors have only two components in each frame.

An interesting direction of future work would be to use the tetrad relations between $\mathfrak{q}$ and $\mathfrak{q}_p$ as diagnostics in postprocessing numerical simulations of fluid dynamics. To remind the reader of the relation between them, we proved in theorem 1 that $\mathfrak{q}$ satisfies

$$\frac{\mathrm{D}\mathfrak{q}}{\mathrm{D}t} + \mathfrak{q} \circledast \mathfrak{q} + \mathfrak{q}_p = 0, \tag{6.1}$$

and in theorem 3 it was shown that if the triplet of scalars is chosen in a certain way then $\mathfrak{q}_p$ satisfies

$$\frac{\mathrm{D}\mathfrak{q}_p}{\mathrm{D}t} = \mathfrak{q} \circledast \mathfrak{q}_p + \mathfrak{P}, \tag{6.2}$$

where $\mathfrak{P}$ is the tetrad, linear in $\mathfrak{q}$ and $\mathfrak{q}_p$, defined in (4.11). Eliminating $\mathfrak{q}_p$ between (6.1) and (6.2) gives

$$\frac{\mathrm{D}^2\mathfrak{q}}{\mathrm{D}t^2} + \frac{\mathrm{D}\mathfrak{q}}{\mathrm{D}t} \circledast \mathfrak{q} + \mathfrak{P} = \mathfrak{q} \circledast \mathfrak{q} \circledast \mathfrak{q}, \tag{6.3}$$

which is not completely closed because of the triplet of coefficients μ_1, λ_1, ε_1 in $\mathfrak{P}$ and the need to respect the Poisson equation. Through this, the vorticity is related to P and S by

$$\frac{1}{2}\omega^2 = Tr(P + S^2) = \sum_{i=1}^{3} \left[\lambda_P^{(i)} + (\lambda_S^{(i)})^2 \right]. \tag{6.4}$$

The associated eigenvectors of P and S project onto the the ortho-normal frame $(\hat{\omega}, \hat{\chi}, \hat{\omega} \times \hat{\chi})$ to yield the coefficients $\alpha, \chi, \alpha_p, c_1$ and c_2. In this ortho-normal basis P has six components

$$P = \begin{pmatrix} \alpha_p & c_1 & c_2 \\ c_1 & \beta & c_3 \\ c_2 & c_3 & \gamma \end{pmatrix}, \tag{6.5}$$

but for the present formulation, only α_p, c_1 and c_2 are required from P with $\alpha = \hat{\omega} \cdot S\hat{\omega}$ and $\chi = (\hat{\omega} \times \hat{\chi}) \cdot S\hat{\omega}$.

An alternative way of looking at the coupling between $\mathfrak{q}$ and $\mathfrak{q}_p$ is to define the $\pm$-operators as $\mathcal{D}_\mathfrak{q}^\pm = D \pm \mathfrak{q} \circledast$, where $D = D/Dt$. Then (2.14), (6.1) and (6.2) can be written as

$$\mathcal{D}_\mathfrak{q}^- \mathfrak{w} = 0, \qquad \mathcal{D}_\mathfrak{q}^+ \mathfrak{q} = -\mathfrak{q}_p, \qquad \mathcal{D}_\mathfrak{q}^- \mathfrak{q}_p = -\mathfrak{P}, \tag{6.6}$$

from which we conclude that $\mathcal{D}_\mathfrak{q}^- \mathcal{D}_\mathfrak{q}^+ \mathfrak{q} = \mathfrak{P}$.

Finally, recent developments in experimental and numerical capabilities also address the hybrid Lagrangian and Eulerian descriptions of fluid dynamics while theoretical developments include the Lagrangian averaged Navier–Stokes-alpha equations. The latter have been reviewed in Holm *et al* (2005). The implications for Lagrangian averaging of the hybrid picture of rotating frames in ideal fluids presented here will be discussed elsewhere but we note that the rotating frame representation may suggest a natural decomposition into fast and slow variables involving rapid rotations with slow modulations.

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