# Ten Lectures
# on
# Spatially Localized Structures

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# Binary fluid convection – again

We consider the equations

$$\begin{aligned}
\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla P + \sigma R[(1+S)\theta - S\eta]\hat{\mathbf{z}} + \sigma\nabla^2\mathbf{u},\\
\theta_t + (\mathbf{u}\cdot\nabla)\theta &= w + \nabla^2\theta,\\
\eta_t + (\mathbf{u}\cdot\nabla)\eta &= \tau\nabla^2\eta + \nabla^2\theta,
\end{aligned}$$

where $\nabla\cdot\mathbf{u} = 0$, so that $\mathbf{u} \equiv (u, w) = (-\psi_z, \psi_x)$ where $\psi$ is the streamfunction, with the boundary conditions $\beta = 1$:

$$\begin{aligned}
&\text{at} \quad z = 1: \quad u = w = \theta = \eta_z = 0.\\
&\text{at} \quad z = -1: \quad u = w = \theta = \eta_z = 0.
\end{aligned}$$

However, we now consider the case $S > 0$ for which steady convection sets in with a **long** wavelength. We therefore write $X = \epsilon x$, $T = \epsilon^4 t$, where $R = R_0(1+\mu\epsilon^2)$, $0 < \epsilon \ll 1$. In addition, we write

$$\psi = \epsilon\Psi(X, z, T), \quad \theta = \epsilon^2\Theta(X, z, T), \quad \eta = -\Phi(X, z, T).$$

# Binary fluid convection – again

Thus, with $D \equiv \partial/\partial z$,

$$\frac{1}{\sigma}[\epsilon^6\Psi_{XXT} + \epsilon^4 D^2\Psi_T + \epsilon^2\Psi_X(\epsilon^2 D\Psi_{XX} + D^3\Psi) - \epsilon^2 D\Psi(\epsilon^2\Psi_{XXX} + D^2\Psi_X)]$$

$$= R_0(1+\mu\epsilon^2)[(1+S)\epsilon^2\Theta_X + S\Phi_X] + \epsilon^4\Psi_{XXXX} + 2\epsilon^2 D^2\Psi_{XX} + D^4\Psi,$$

$$\epsilon^4\Theta_T + \epsilon^2(\Psi_X D\Theta - D\Psi\Theta_X) = \Psi_X + \epsilon^2\Theta_{XX} + D^2\Theta,$$

$$\epsilon^4\Phi_T + \epsilon^2(\Psi_X D\Phi - D\Psi\Phi_X) = \tau(\epsilon^2\Phi_{XX} + D^2\Phi) - \epsilon^2(\epsilon^2\Theta_{XX} + D^2\Theta).$$

These equations are solved by an asymptotic expansion of the form $\Psi = \Psi_0 + \epsilon^2\Psi_2 + \cdots$, $\Theta = \Theta_0 + \epsilon^2\Theta_2 + \cdots$, $\Phi = \Phi_0 + \epsilon^2\Phi_2 + \cdots$ as described in Knobloch, PRA **40**, 1549 (1989).

# Binary fluid convection – again

At $\mathcal{O}(1)$:

$$D^2\Phi_0 = 0, \qquad \text{with} \quad D\Phi_0 = 0 \quad \text{on} \quad z = \pm 1.$$

Hence $\Phi_0 = f(X, T)$, where $f$ is to be determined. Since

$$D^4\Psi_0 = -SR_0 f'$$

it follows that $\Psi_0 = SR_0 f' P(z)$, where $P$ is a fourth order polynomial in $z$ depending on the boundary conditions. In addition,

$$\Theta_0 = SR_0 f'' Q(z),$$

where $Q$ is a degree six polynomial in $z$.

At $\mathcal{O}(\epsilon^2)$:

$$\tau D^2\Phi_2 = -(\tau + SR_0 P)f'' - SR_0 DP f'^2 \quad \text{with} \quad D\Phi_2 = 0 \quad \text{on} \quad z = \pm 1.$$

Thus

$$SR_0 \int_{-1}^{1} P\, dz + 2\tau = 0.$$

This equation determines $R_0$ leaving $\Phi_2 = f_2(X,T) + f''U(z) + f'^2V(z)$. Similarly, $\Psi_2 = SR_0 f_2' P(z) + \mu S f' P(z) + f'f''W(z) + f'''Z(z)$ and

$$D^2\Theta_2 = (SR_0)^2P)(f''^2PDQ - f'f'''DPQ) \quad - \quad SR_0 f_2''P - \mu S f'' P$$
$$-(f'f'')'W - f''''(Z + SR_0 Q).$$

Finally, at $\mathcal{O}(\epsilon^4)$ the solvability condition for $\Phi_4$ gives

$$f_T = -SR_0\mu A f'' - Bf'''' + C(f'^3)' + D(f'f'')'.$$

For the boundary conditions adopted $R_0 = 45\tau/S$, $A = 1/45$, $B = [34\tau - 131(1+S^{-1})\tau^2]/231$, $C = 10\tau/7$, $D = 0$.

The sign of the coefficient $B \equiv -\frac{1}{2}[\tau \int_{-1}^{1} U\,dz + \int_{-1}^{1} Z\,dz]$ plays an important role. If $B > 0$ $[S > S_0 \equiv 131\tau(34 - 131\tau)^{-1}]$ the minimum of the neutral stability curve is at $k = 0$ and we get Cahn-Hilliard dynamics; if $B < 0$ the minimum is at finite $k$ and the calculation needs to be taken to higher order: we take $S = S_0(1 - \nu\epsilon^2)$ so that $B = \mathcal{O}(\epsilon^2)$, and hence write $R = R_0(1 + \nu\epsilon^2 + \mu\epsilon^4)$, obtaining instead

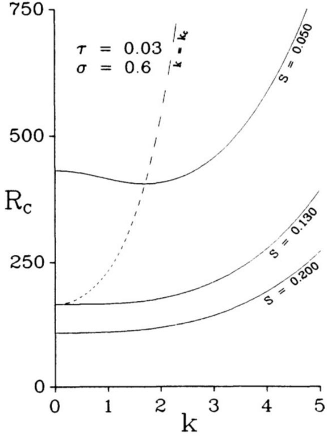
$$f_T = -SR_0(\mu - \nu^2)Af'' + \nu\tilde{B}f'''' + Ff'''''' + C(f'^3)'.$$

# Binary fluid convection – again

With $f' = \phi$ this equation takes the form

$$\phi_T = \partial_X^2[-SR_0(\mu - \nu^2)A\phi + \nu\tilde{B}\phi'' + F\phi'''' + C\phi^3].$$

This is the conserved Swift-Hohenberg equation.



# Rotating convection: Basic Equations

We consider a plane horizontal layer, heated from below and rotating uniformly with angular velocity $\Omega$ about the vertical axis. We consider two-dimensional convection with $\mathbf{u} \equiv (-\psi_z, v, \psi_x)$, where $\psi(x, z, t)$ is the streamfunction in the $(x, z)$ plane and $v(x, z, t)$ is the associated zonal velocity. The basic equations are (Veronis 1959)

$$
\begin{aligned}
Ra\theta_x - Tv_z + \nabla^4\psi &= \sigma^{-1}\left[\nabla^2\psi_t + J(\psi, \nabla^2\psi)\right], \\
\psi_x + \nabla^2\theta &= \theta_t + J\left(\psi, \theta\right), \\
T\psi_z + \nabla^2 v &= \sigma^{-1}\left[v_t + J(\psi, v)\right].
\end{aligned}
$$

The dimensionless parameters are

$$
\sigma = \frac{\nu}{\kappa}, \qquad Ra = \frac{g\alpha\triangle\Theta h^3}{\kappa\nu}, \qquad T = \frac{2\Omega h^2}{\nu}.
$$

The equations are to be solved subject to stress-free boundary conditions

$$
\psi = \psi_{zz} = \theta = v_z = 0 \text{ at } z \in \{0, 1\}
$$

# Basic Equations

With these boundary conditions

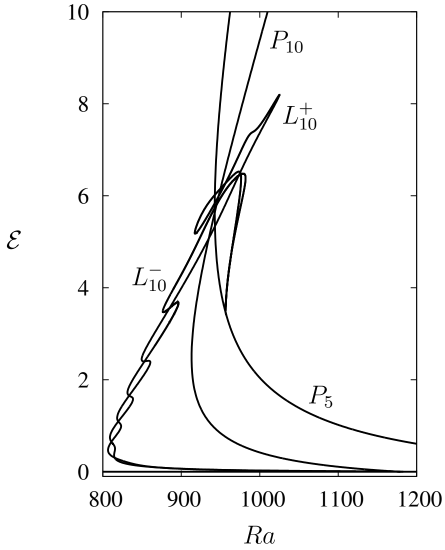
$$\frac{d}{dt}\bar{V} = 0, \qquad \bar{V} \equiv \int_D v(x,z,t)\,dx\,dz.$$

Thus $\bar{V}$ is a conserved quantity, and this fact exerts a profound influence on the behavior of this system. In the following we set wlog $\bar{V} = 0$. We also define $V(x) \equiv \int_0^1 v(x,z)\,dz$. Then

$$\sigma \frac{dV}{dx} = -\int_0^1 \psi_z v\,dz.$$

This is the Reynolds stress relation. The quantity $\Delta V \equiv V(x = L/2) - V(x = -L/2)$ measures the shear across a convecton of length $L$. This is always anticyclonic ($\Delta V < 0$).

# Subcritical case: $\Gamma = 10\lambda_c$, $T = 20$, $\sigma = 0.1$




Beaume et al., J. Fluid Mech. **717**, 417–448 (2013)

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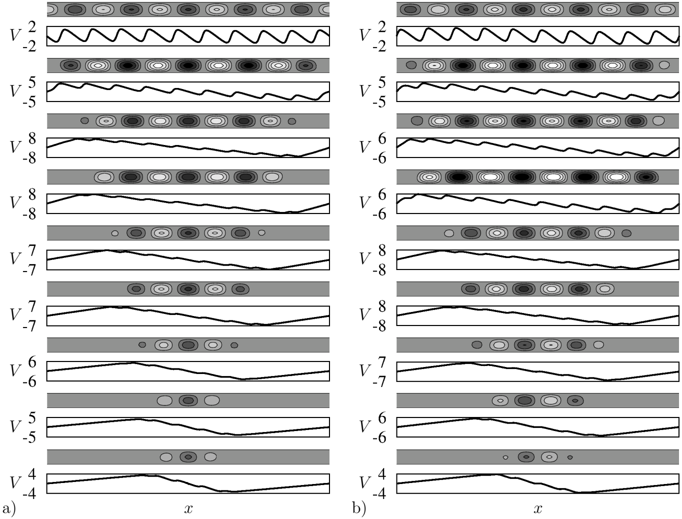


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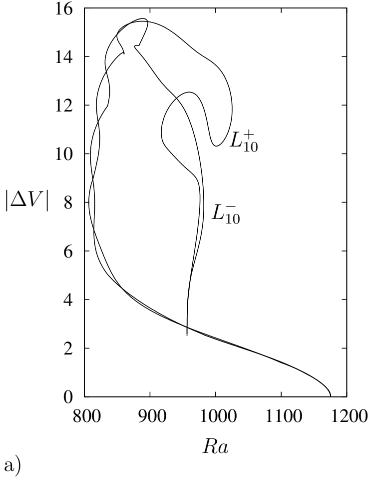


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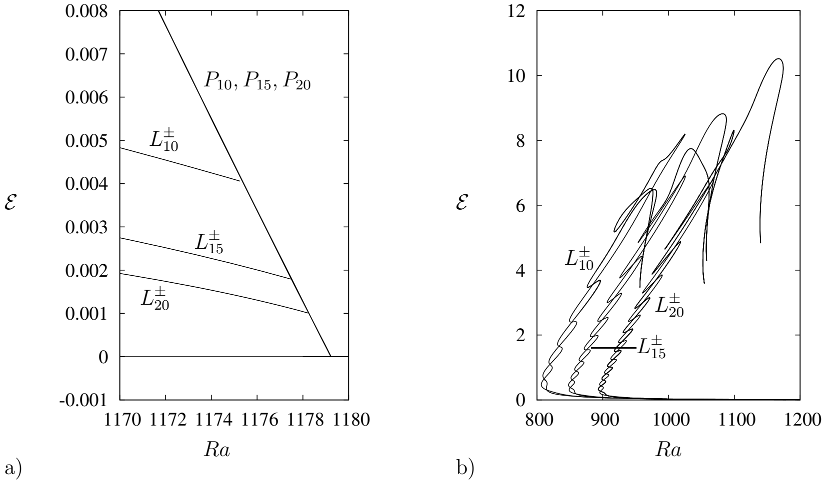
# Subcritical case: $\Gamma = 10\lambda_c, 15\lambda_c, 20\lambda_c$, $T = 20$, $\sigma = 0.1$

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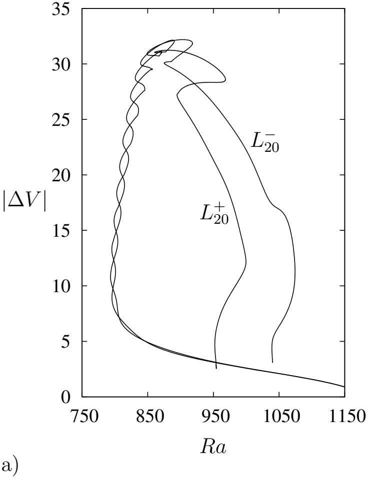
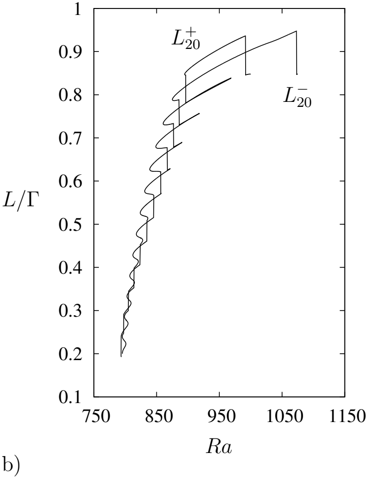
# Subcritical case: $\Gamma = 20\lambda_c$, $T = 20$, $\sigma = 0.1$

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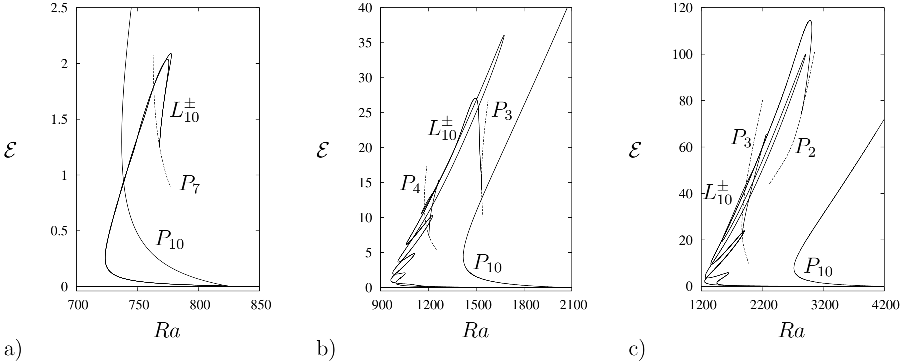
# Subcritical case: $\Gamma = 10\lambda_c$, $T = 10, 40, 80$, $\sigma = 0.1$

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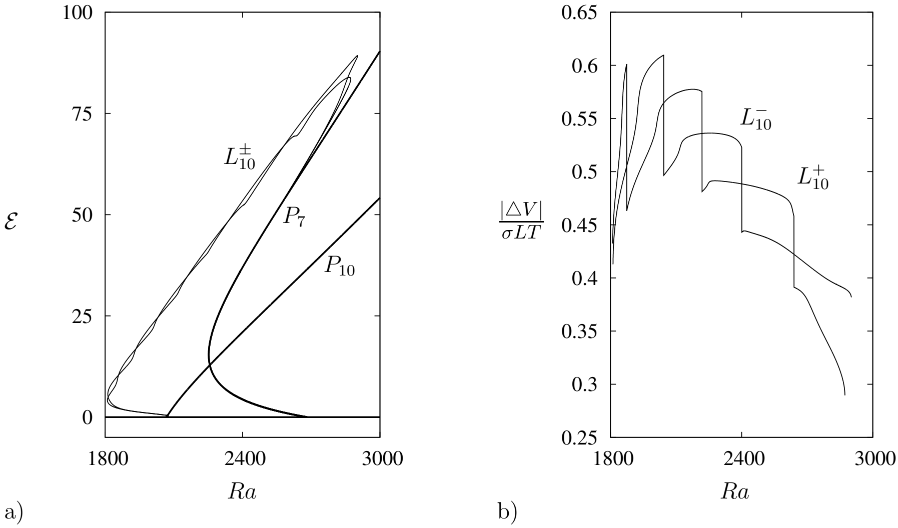
# Supercritical case: $\Gamma = 10\lambda_c$, $T = 40$, $\sigma = 0.6$

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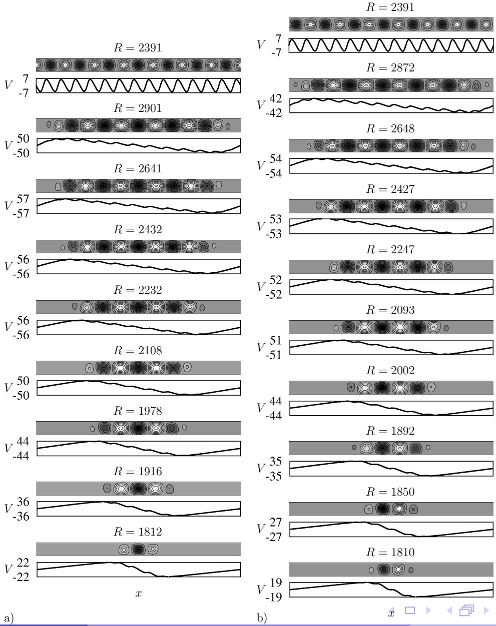

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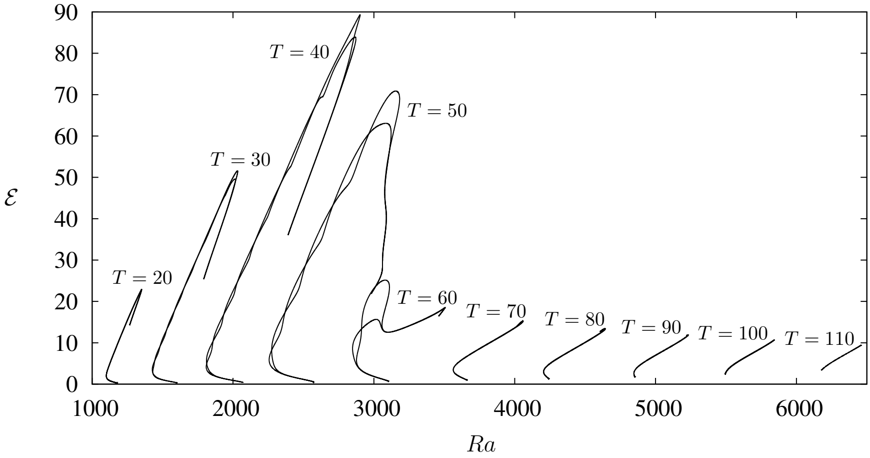


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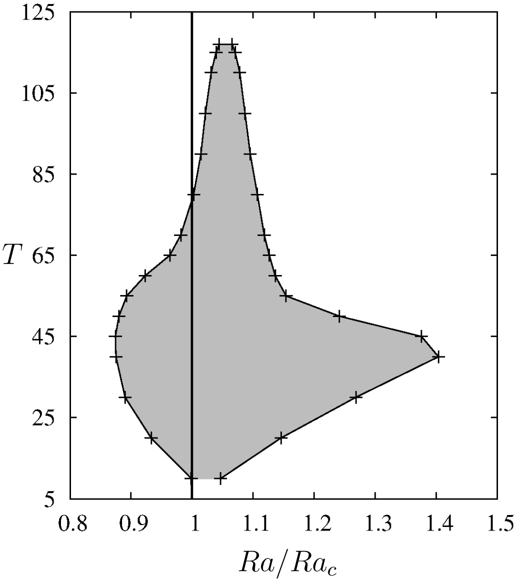


# Theory

We let $Ra = Ra_c + \epsilon^2 r$ and look for solutions in the form

$$\begin{aligned}
\psi &= \frac{\epsilon}{2}\left(a(X,T_2)e^{ikx} + c.c.\right)\sin(\pi z) + h.o.t., \\
\theta &= \frac{\epsilon k}{2p}\left(ia(X,T_2)e^{ikx} + c.c.\right)\sin(\pi z) + h.o.t., \\
v &= \epsilon V(X,T_2) + \frac{\epsilon T\pi}{2p}\left(a(X,T_2)e^{ikx} + c.c.\right)\cos(\pi z) + h.o.t.
\end{aligned}$$

where $X = \epsilon x,\ T_2 = \epsilon^2 t$. After rescaling

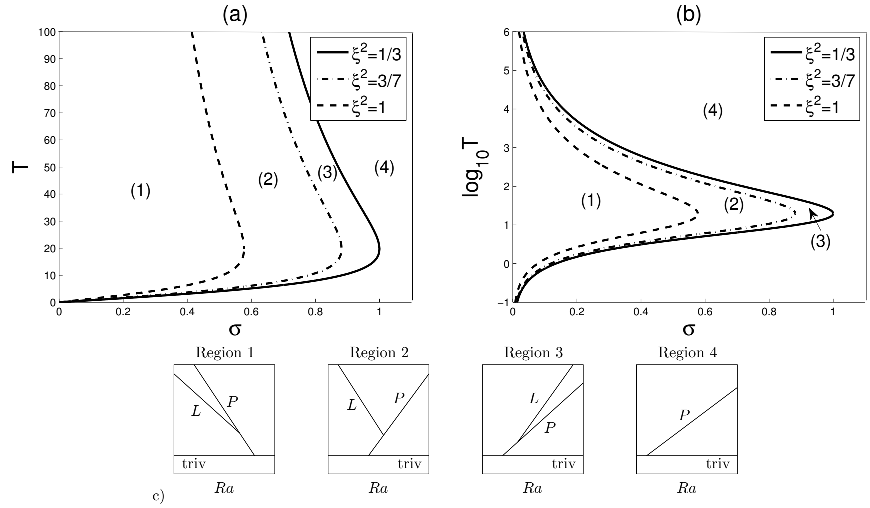
$$\begin{aligned}
\eta A_{T_2} &= rA + A_{XX} - \frac{1-\xi^2}{2}|A|^2 A - \xi A V_X, \\
V_{T_2} &= V_{XX} + \xi\left(|A|^2\right)_X,
\end{aligned}$$

where $\xi \equiv \frac{T\pi^2}{\sqrt{3}pk^2\sigma} > 0,\ p = k^2 + \pi^2$. Thus, in steady state,

$$V_X = \xi\left(\langle|A|^2\rangle - |A|^2\right),$$

where $\langle\cdot\rangle$ represents a spatial average over the domain. Thus $V_X < 0$ if $|A|^2 > \langle|A|^2\rangle$, i.e., inside the convecton, while $V_X < 0$ outside it.

# Nonlocal amplitude equation

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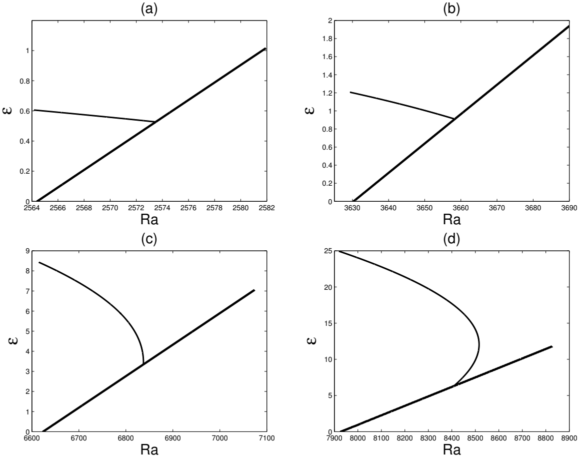
# Nonlocal amplitude equation




Bifurcation diagrams for $\sigma = 0.6$ and $\Gamma = 10\lambda_c$ and (a) $T = 50$ ($\xi^2 = 0.7032$, region (2)). (b) $T = 70$ ($\xi^2 = 0.5882$, region (2)). (c) $T = 120$ ($\xi^2 = 0.4269$, region (2)). (d) $T = 140$ ($\xi^2 = 0.3877$, region (3)).

# Homotopy to no-slip boundary conditions

With no-slip boundary conditions at $z = \pm 1/2$,

$$\psi = \psi_z = v = 0,$$

the zonal momentum equation contains a source term:

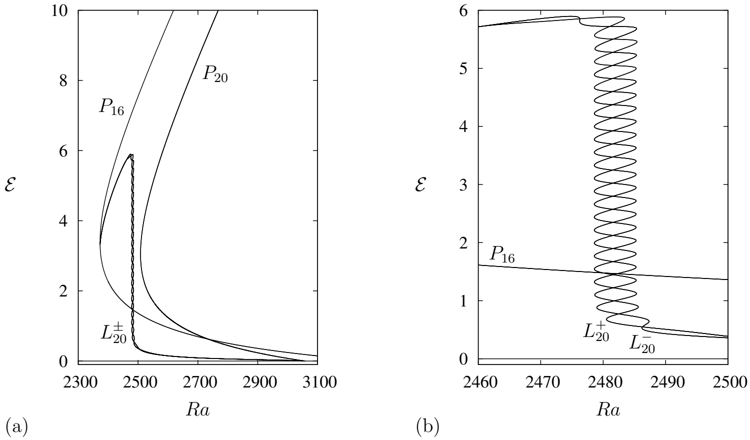
$$\frac{d}{dt}\overline{V} = \frac{\sigma}{\Gamma}\int_{-\Gamma/2}^{\Gamma/2} [\partial_z v]_{z=-1/2}^{1/2}\, dx,$$

To examine the effect of the breaking of this conserved quantity we perform homotopy between the stress-free case and the no-slip case:

$$\psi = (1-\beta)\psi_{zz} \pm \beta\psi_z = (1-\beta)v_z \pm \beta v = 0$$

at $z = \pm 1/2$, where $\beta \in [0,1]$ is a homotopy parameter. Thus $\beta = 0$ corresponds to stress-free boundaries and $\beta = 1$ to no-slip boundaries.

# Homotopy to no-slip boundary conditions: $\mathrm{Ta} = 60$, $\sigma = 0.1$, $\Gamma \approx 30.9711$

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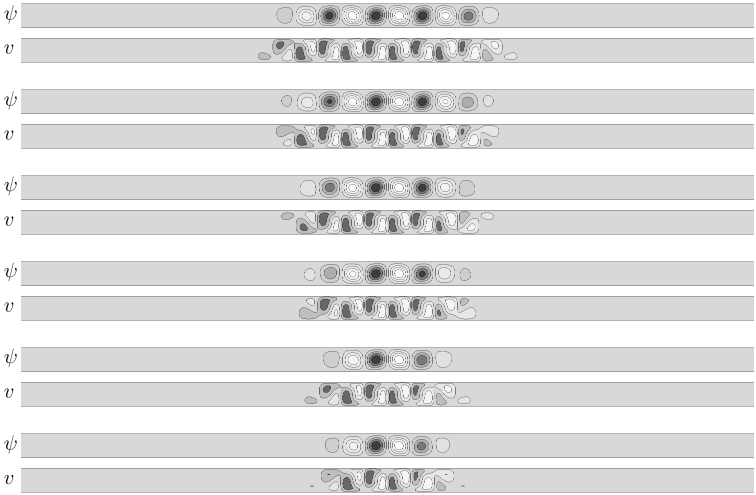
# Homotopy to no-slip boundary conditions: $L_{20}^{+}$ branch

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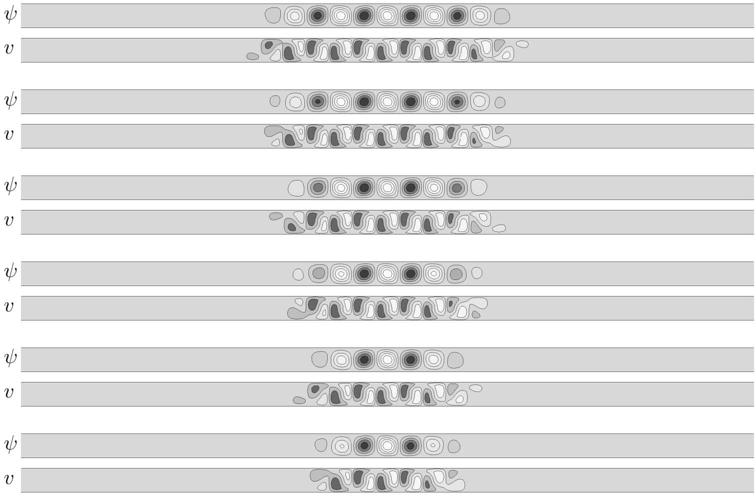
# Homotopy to no-slip boundary conditions: $L_{20}^{-}$ branch

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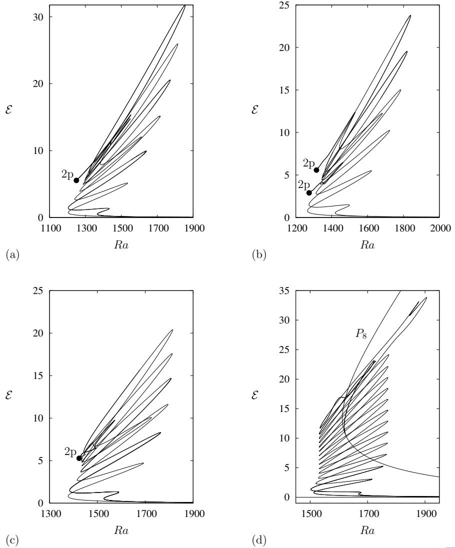
# Homotopy: (a) $\beta = 0.1$, (b) $\beta = 0.2$, (c) $\beta = 0.4$ and (d) $\beta = 0.6$



# Conclusions

We have seen that on a periodic domain with finite period $\Gamma$ the presence of zonal momentum conservation changes dramatically the standard snaking scenario:

- localized states exist outside the bistability region
- snaking becomes slanted
- snaking occurs even in the supercritical case

In the limit $\Gamma \to \infty$ the momentum conservation constraint is lost and snaking becomes vertical as in the standard scenario.

Similar behavior occurs in other systems with a conserved quantity, including the conserved Swift-Hohenberg equation and convection in an imposed vertical magnetic field.

# Take-home message from the lecture series

I have

- introduced the notion of spatial dynamics
- demonstrated its usefulness for understanding the origin of LS in 1D
- described the snakes-and-ladders structure of the bifurcation diagram
- discussed the corresponding 2D results
- explained how the presence of a conserved quantity modifies this structure
- shown how to determine the speed of a front invading an unstable state
- shown how to determine the speed of a front invading a stable state
- demonstrated how these results inform our understanding of more complicated systems described by Navier-Stokes dynamics

There is a large number of other systems exhibiting this type of behavior because it is **generic**.

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