TECHNISCHE UNIVERSITÄT DRESDEN

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Invariant manifolds in finite-time advection

Two Extensions of Haller's Hyperbolic LCS

Hyperbolic codimension-1 LCS

Recalling Haller [1], we consider the finite-time dynamical system φ generated by

$$\dot{x} = f(t,x), \quad t \in [t_-, t_+] =: \mathbb{I},\ x \in D \subseteq \mathbb{R}^n,\ f \in C^{0,3}(\mathbb{I} \times D, \mathbb{R}^n).$$

We investigate the dynamics of ensembles under $\varphi(t_+, t_-, \cdot)$. Denote the *Cauchy-Green strain tensor* by

$$C(x) := \partial_2\varphi(t_+, t_-, x)^* \partial_2\varphi(t_+, t_-, x), \qquad x \in D,$$

with eigenvalues $0 < \lambda_1(x) \leq \ldots \leq \lambda_n(x)$ and corresponding eigenvectors $v_1(x), \ldots, v_n(x)$. Let $\mathcal{M} \subseteq D$ denote a codimension-1 C^1-manifold of initial values ("material surface"). For $x \in \mathcal{M}$ define by

$$\rho(x) := \frac{1}{\langle n_0(x), C(x)^{-1} n_0(x)\rangle^{1/2}}, \qquad \nu(x) := \frac{\rho(x)}{\left\| \Phi(x)|_{T_x\mathcal{M}} \right\|},$$

the *repulsion rate* and the *repulsion ratio*, resp., where n_0 is the C^1 normal field of $\mathcal{M}$.

Definition 1 (Normally repelling material surface). $\mathcal{M}$ is called *normally repelling*, if there exists $c > 1$ such that $\rho(x) > c$ and $\nu(x) > c$ for all $x \in \mathcal{M}$.

Definition 2 (Repelling (W)LCS). A normally repelling material surface $\mathcal{M}$ is called *repelling weak LCS* if ρ admits stationary values on $\mathcal{M}$ for all smooth normal perturbations. A normally repelling material surface $\mathcal{M}$ is called *repelling LCS* if ρ admits non-degenerate maxima on $\mathcal{M}$ for all smooth normal perturbations.

Theorem 1 (cf. [1, Thm. 7]). *$\mathcal{M}$ is a repelling weak LCS if and only if for any $x \in \mathcal{M}$ the following conditions hold:*

$$\lambda_{n-1}(x) \neq \lambda_n(x) > 1; \quad v_n(x) \perp T_x\mathcal{M}; \quad \partial_{v_n(x)}\lambda_n(x) = 0.$$

$\mathcal{M}$ is a $(n-1)$-dimensional repelling LCS if and only if the following conditions hold:

1. *$\mathcal{M}$ is a $(n-1)$-dimensional repelling weak LCS;*
2. *for any $x \in \mathcal{M}$ either some matrix $L(x)$ is positive definite or, in case that $v_1, \ldots, v_n$ are continuously differentiable at x, the inequality $\partial^2_{v_n(x)}\lambda_n(x) < 0$ holds, see [2].*

Hyperbolic codimension-k LCS

The concepts of repulsion rate and repulsion ratio, i.e. the ration between minimal normal and maximal tangential repulsion, can be generalized to C^1-submanifolds of higher codimension as follows:

$$\rho(x) := \left\| \left(\Phi^{-1}\right)^*(x)|_{T_x^\perp\mathcal{M}} \right\|^{-1}, \qquad \nu(x) := \frac{\rho(x)}{\left\| \Phi(x)|_{T_x\mathcal{M}} \right\|}.$$

With Definitions 1 and 2 applied to the modified ρ and ν, Theorem 1 can be generalized to the following result.

Theorem 2 ([3]). *$\mathcal{M}$ with $\dim \mathcal{M} = k$ is a repelling weak LCS if and only if for any $x \in \mathcal{M}$ the following conditions hold:*

a) $\lambda_k(x) \neq \lambda_{k+1}(x) > 1$,

b) $\operatorname{span}\left\{v_{k+1}(x), \ldots, v_n(x)\right\} = T_x^\perp\mathcal{M}$;

c) $\partial_{v_i(y)}\lambda_{k+1}(x) = 0$ *for any* $i \in \{k+1, \ldots, n\}$.

The characterizing condition for $\mathcal{M}$ to be a repelling LCS is conjectured to be

$$d) \qquad \partial^2_{v_i(x)}\lambda_{k+1}(x) < 0 \qquad \text{for any } i \in \{k+1, \ldots, n\}.$$

Conditions c) and d) can be paraphrased: all $x \in \mathcal{M}$ are *generalized maximum points of λ_{k+1} w.r.t.* $\operatorname{span}\left\{v_{k+1}, \ldots, v_n\right\}$.

Filtrations of hyperbolic LCS

For an $(n-k-\ell)$-dimensional submanifold $\mathcal{N}$ of a hyperbolic $(n-k)$-dimensional LCS $\mathcal{M}$ define the normal space as the orthogonal complement of $T_x\mathcal{N}$ in $T_x\mathcal{M}$. Then the hypersurface approach and the codimension-k approach are both applicable, i.e. Theorems 1 and 2 characterize embedded hyperbolic LCS accordingly.

References

[1] G. Haller. A variational theory of hyperbolic Lagrangian Coherent Structures. *Physica D*, 240(7):574–598, 2011.

[2] D. Karrasch. Comment on [1]. 2012. submitted.

[3] D. Karrasch. Normally hyperbolic invariant manifolds in finite-time chaotic advection. in preparation.

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