Early escape near tipping points

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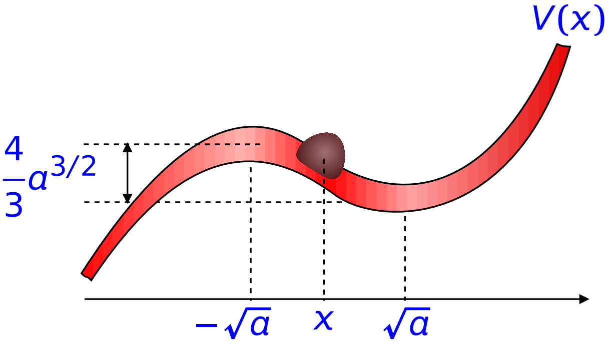
joint work with **J.M.T. Thompson**, FRS (Aberdeen)

Outline

- Noise induced escape near tipping
- Estimates of normal form parameters from time series

Noise induced escape

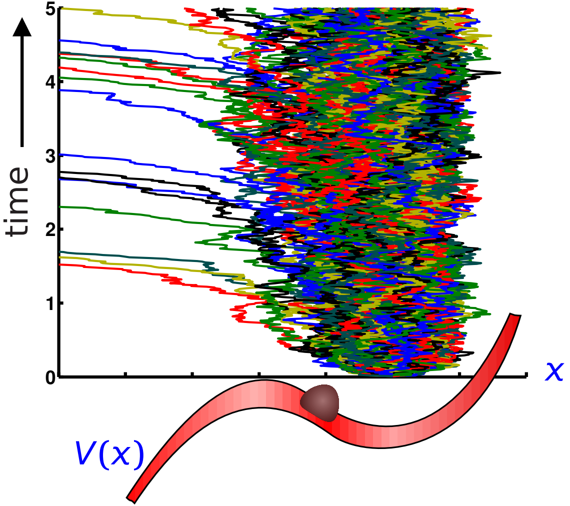
overdamped particle in a well



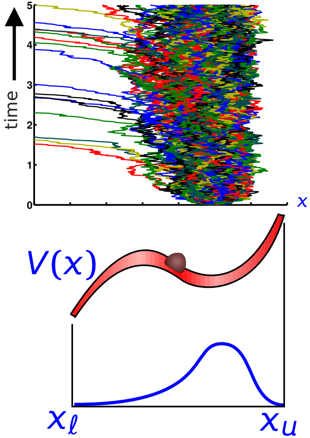
$$\frac{d}{dt}x = -V'(x) = a - x^2 + \textbf{noise}$$

Noise induced escape

Fixed well depth, Noise amplitude $\sigma > 0$



Noise induced escape

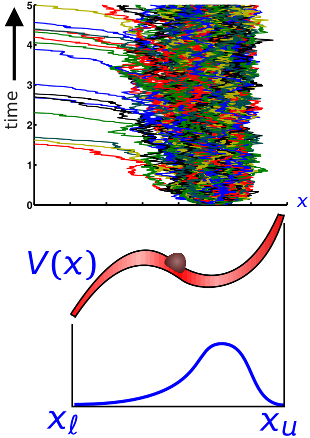


Conditional stationary density

SDE $\quad \mathrm{d}x = f(x)\mathrm{d}t + \sigma \mathrm{d}W_t$

with escape

Noise induced escape



Conditional stationary density

SDE $\mathrm{d}x = f(x)\mathrm{d}t + \sigma \mathrm{d}W_t$

with escape

$\Rightarrow$ density $p(x,t)$ satisfies

$$\partial_t p = \frac{\sigma^2}{2}\partial_{xx} p - \partial_x \left[f(x) p \right]$$

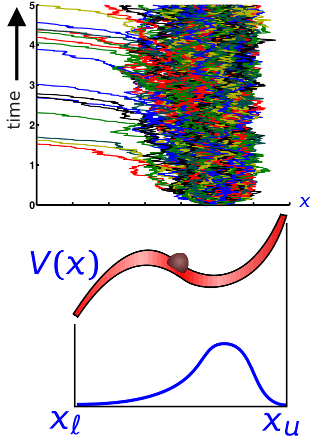
$$0 = p(x_\ell)$$

$$0 = p(x_u)$$

exit b.c.

look for $p(x,t) = \mathrm{e}^{\lambda t} p_s(x)$

Noise induced escape



SDE $\quad dx = f(x)dt + \sigma dW_t$

with escape

$$\lambda p_s = \frac{\sigma^2}{2}\partial_{xx} p_s - \partial_x \left[f(x) p_s \right]$$

$$0 = p_s(x_\ell)$$

$$0 = p_s(x_u)$$

Eigenvalue problem:

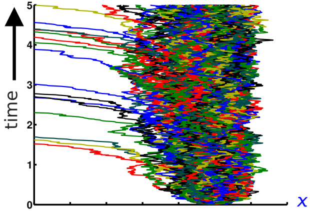
Eigenvalue $-\lambda$ $\Rightarrow$ escape rate

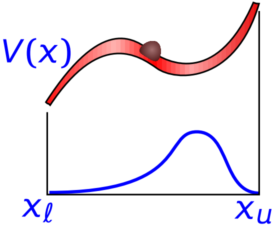
Eigenvector $p_s(x)$ $\Rightarrow$ cond. density

Conditional stationary density

(rescale) $\quad 1 = \int_{x_\ell}^{x_u} p_s(x)$

Noise induced escape





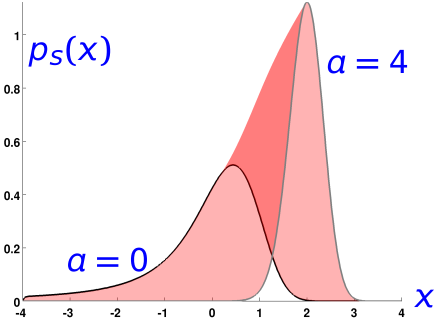
Conditional stationary density

SDE $\quad \mathrm{d}x = f(x)\mathrm{d}t + \sigma \mathrm{d}W_t$

with escape

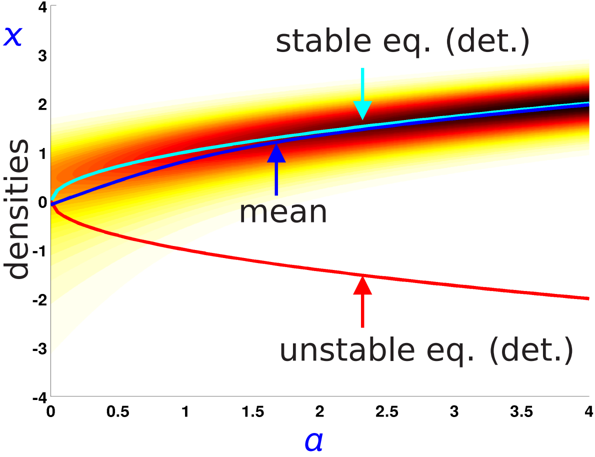
Example

$$\mathrm{d}x = \left[a - x^2\right]\mathrm{d}t + \mathrm{d}W_t$$



Noise induced escape

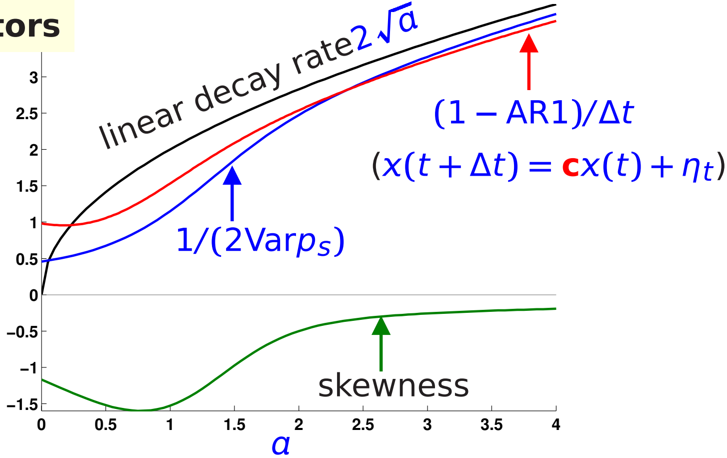
Example $\mathrm{d}x = \left[a - x^2\right] \mathrm{d}t + \mathrm{d}W_t$ ($\sigma = 1$ w.l.o.g.)



Noise induced escape

Example $\mathrm{d}x = \left[a - x^2\right] \mathrm{d}t + \mathrm{d}W_t$ ($\sigma = 1$ w.l.o.g.)

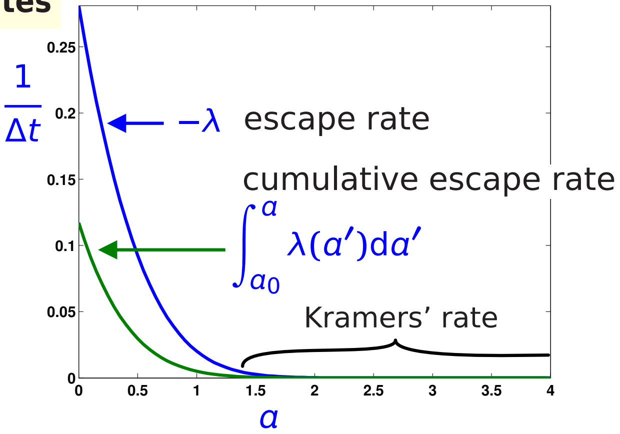
indicators



Noise induced escape

Example $\mathrm{d}x = \left[a - x^2\right] \mathrm{d}t + \mathrm{d}W_t$ ($\sigma = 1$ w.l.o.g.)

escape rates



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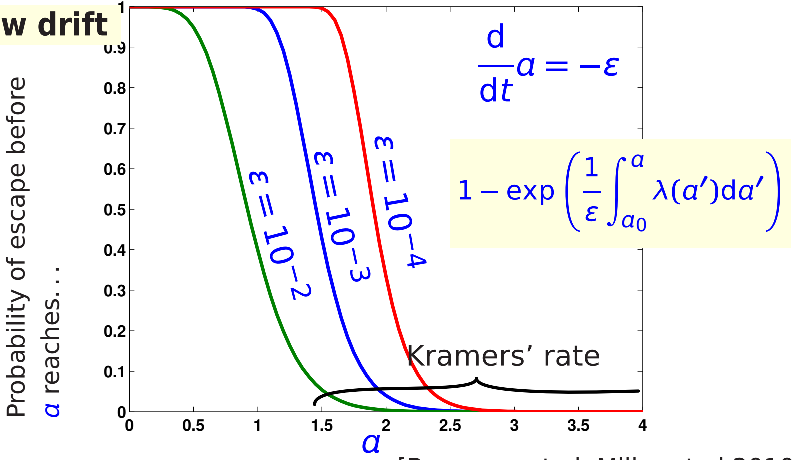
Noise induced escape

Example $dx = \left[a - x^2\right] dt + dW_t$ ($\sigma = 1$ w.l.o.g.)

slow drift



[Burgener et al, Miller et al 2010]

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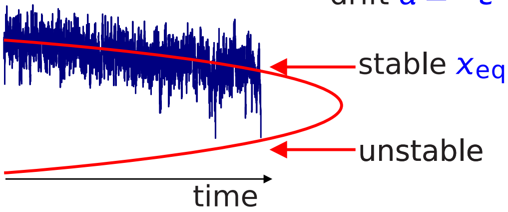
Estimate from time series

Approach to **Saddle-node** $\dot{x} = f(x, \alpha) + \sigma \mathrm{d}W_t$

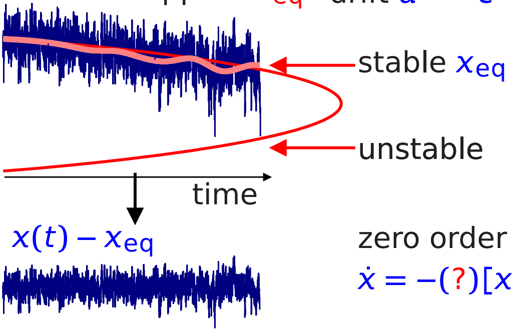
drift $\dot{\alpha} = -\varepsilon$



Estimate from time series

Approach to **Saddle-node** $\dot{x} = f(x, a) + \sigma \mathrm{d}W_t$

approx x_{eq} drift $\dot{a} = -\varepsilon$



zero order

$$\dot{x} = -(?)[x - x_{\mathrm{eq}}(\varepsilon t)] + ?$$

Estimate from time series

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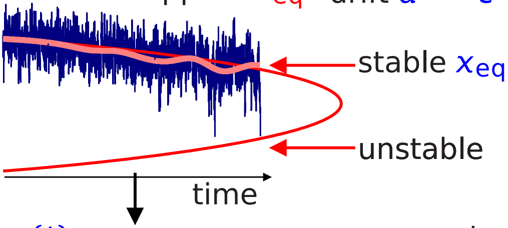
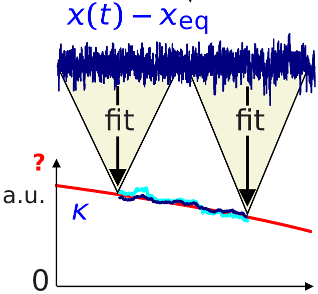




zero order

$$\dot{x} = -(?)[x - x_{\mathrm{eq}}(\varepsilon t)] + ?$$

first order

$$\dot{x} = -\kappa(\varepsilon t)[x - x_{\mathrm{eq}}(\varepsilon t)] + ?$$

κ_{AR} AR(1) (Held&Kleinen'04)

DFA (Livina&Lenton'07)

κ_{var} Var

$$\sigma^2 = \frac{\kappa_{\mathrm{AR}}}{\kappa_{\mathrm{var}}}$$

Estimate from time series

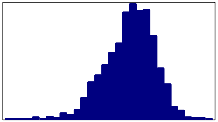
SDE $\dot{x} = -\kappa[x - x_{\text{eq}}] + \mathbf{c_2}[x - x_{\text{eq}}]^2 + \sigma \mathrm{d}W_t$

Conditional density

$$\lambda p_s = \frac{\sigma^2}{2}\partial_{xx} p_s - \partial_x \left[(\kappa[x - x_{\text{eq}}] + \mathbf{c_2}[x - x_{\text{eq}}]^2) p_s \right]$$

$p_s(x_\ell) = 0$

$p_s(x_u) = 0$



$\text{Skew}\, p_s = \text{Skew}_{\text{empirical}}$

Estimate from time series

SDE $\dot{x} = -\kappa[x - x_{\text{eq}}] + \mathbf{c_2}[x - x_{\text{eq}}]^2 + \sigma \mathrm{d}W_t$

Conditional density

$$\lambda p_s = \frac{\sigma^2}{2}\partial_{xx}p_s - \partial_x\left[(\kappa[x - x_{\text{eq}}] + \mathbf{c_2}[x - x_{\text{eq}}]^2)p_s\right]$$

$p_s(x_\ell) = 0$

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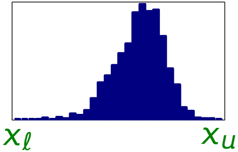


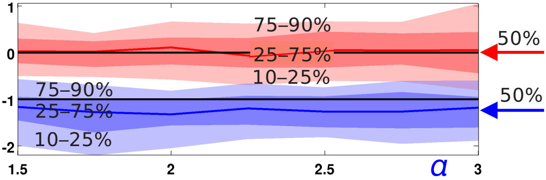
$\text{Skew}\,p_s = \text{Skew}_{\text{empirical}}$

$\Rightarrow$ est. $\mathbf{c_2}$

Test ($N = 2\,000$, a const)

$c_2 = 0$

$c_2 = -1$



est. c_2

est. c_2

Demo — early warning indicators for rate-induced tipping

[P. Ashwin, S. Wieczorek, C. Hobbs]

Summary

- noise causes early escape near tipping points
  [gen. theory: Gentz&Berglund, Kühn]
- effect of noise for a known model can be calculated via stationary conditional density p_s
- propensity to escape early can be estimated from time series
- alternatives to timeseries for sampling of density p_s

$\Rightarrow$ **[JMTT, JS]** on arxiv

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