Novel Types of Tipping Points.

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Outline:

1. Tipping types

2. Rate-dependent Tipping:
   - Basic examples and existence of critical rates
   - Does it occur in real systems?
   - Towards a general theory of R-tipping

3. Conclusions

United Nations Framework Convention on Climate Change (UNFCCC)

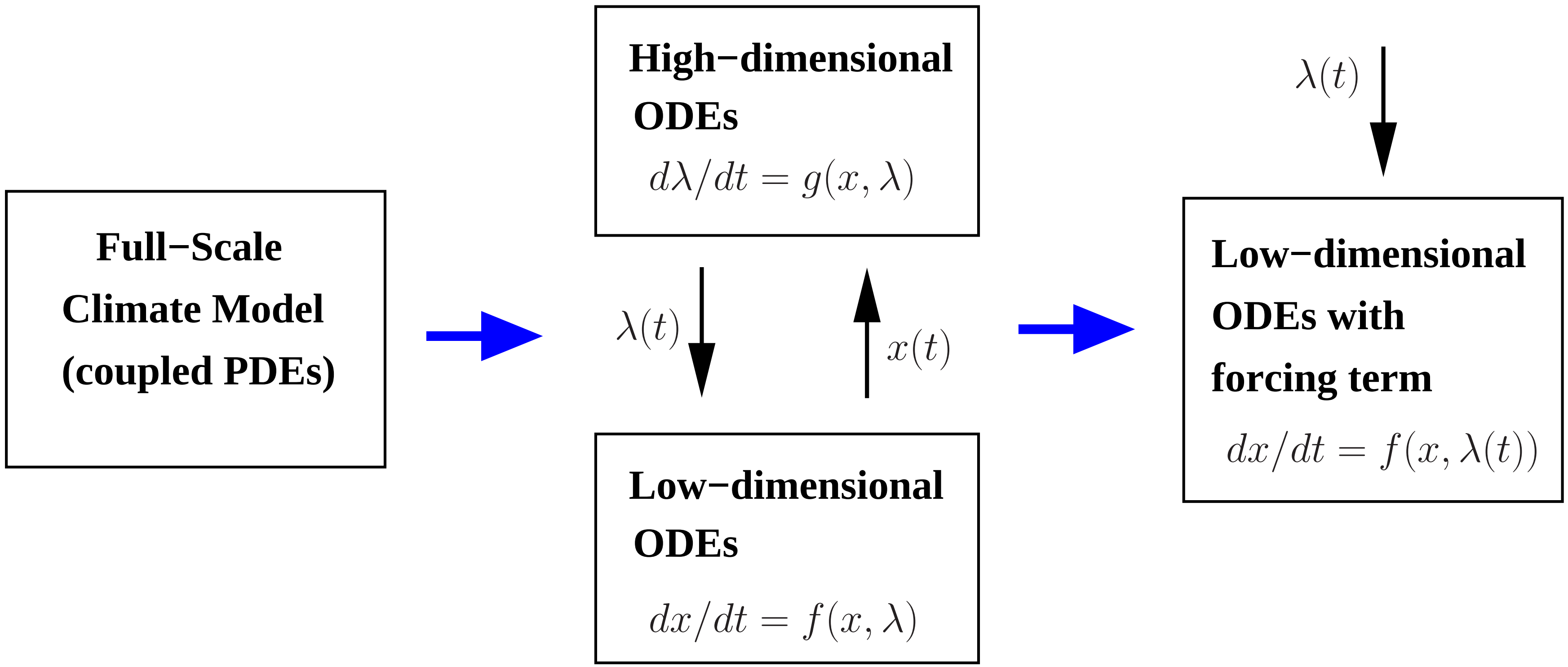
"The ultimate objective is...
stabilisation of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system ...

... such a level should be achieved within a time frame sufficient to allow ecosystems to adapt naturally to climate change, to ensure food production is not threatened and to enable economic development to proceed in a sustainable manner"

dangerous levels and dangerous rates or

climate tipping points

Climate Models and Forced (Nonautonomous) Systems

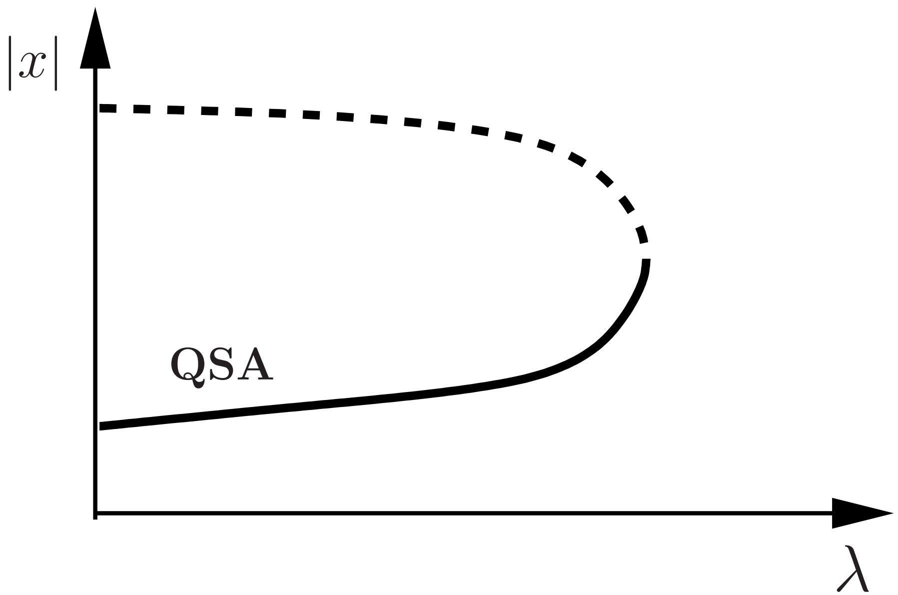


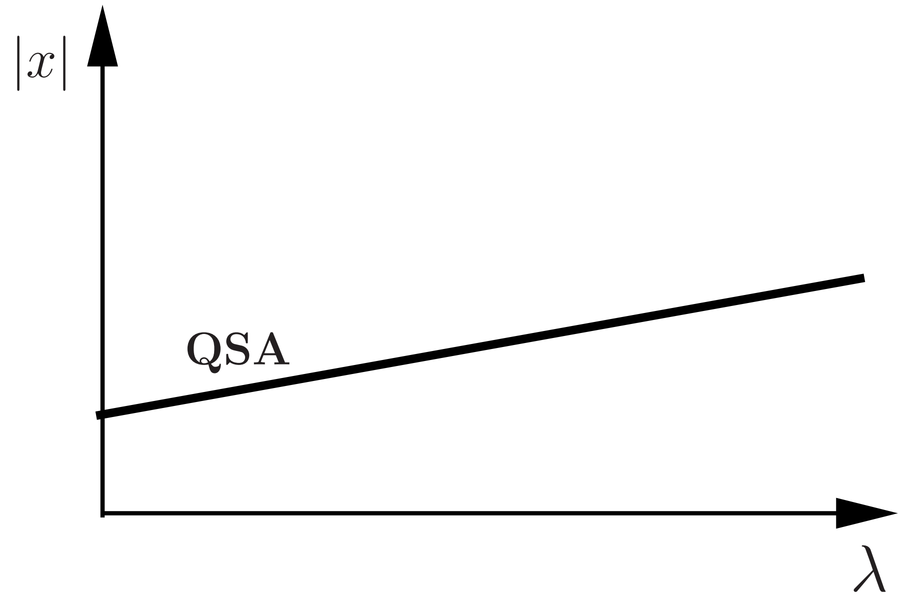
Isaac M. Held, *Bull. Amer. Meteor. Soc.* **86, 1609–1614 (2005)**

- warns of the widening gap between climate simulations and understanding
- encourages the study of model hierarchies to narrow this gap

Two Tipping Types:

B-tipping and R-tipping



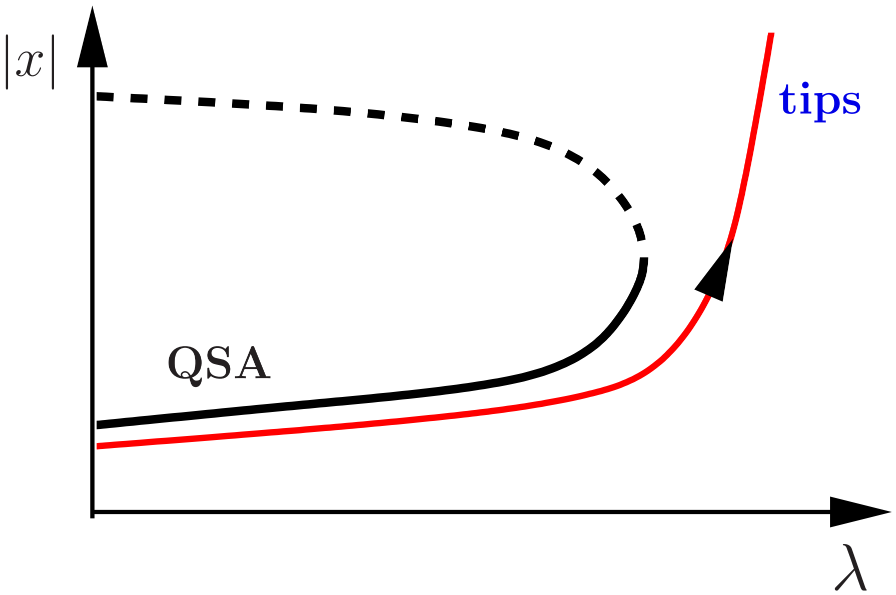


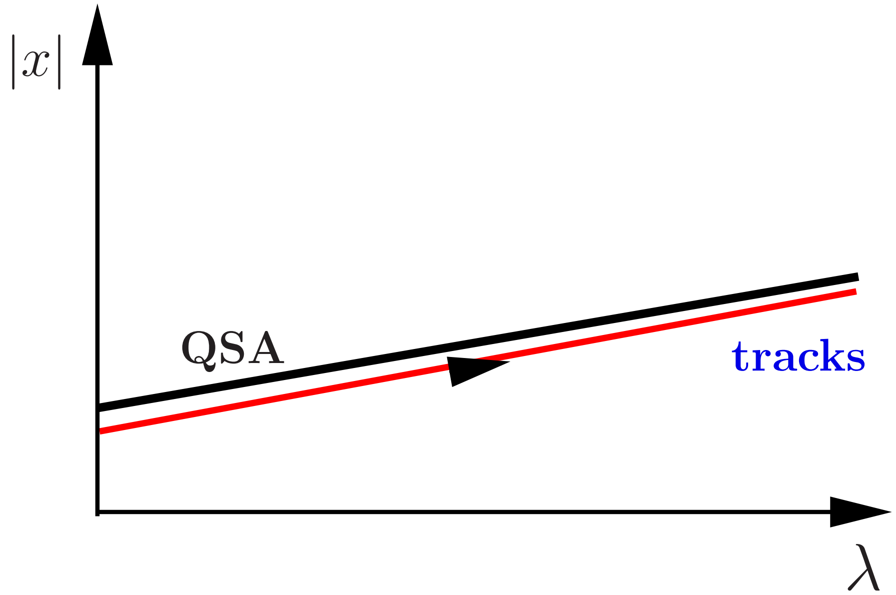
Stability of the unforced system

$$\frac{d}{dt}x = f(x, \lambda)$$

for different but fixed λ: there is an <u>attractor</u> QSA(λ).

B-tipping and R-tipping

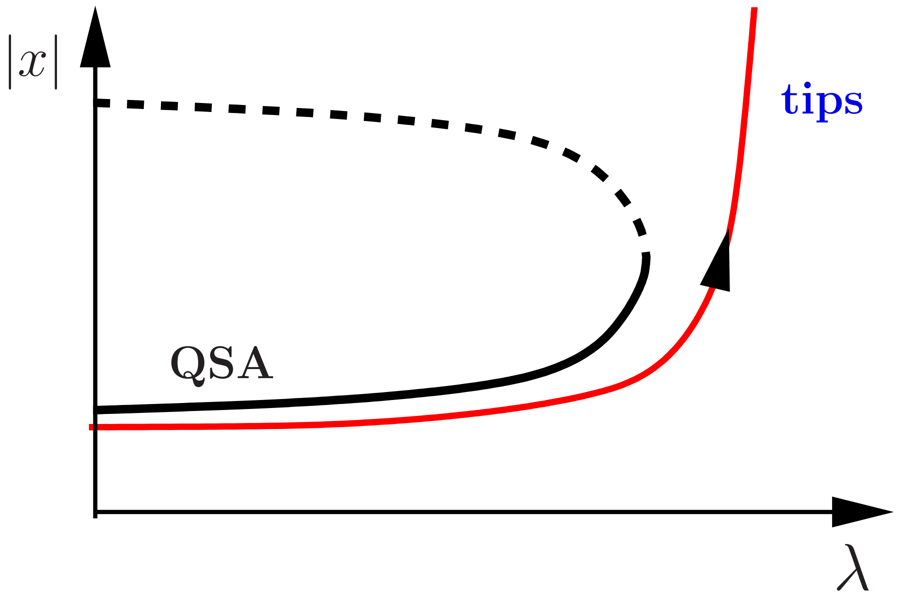


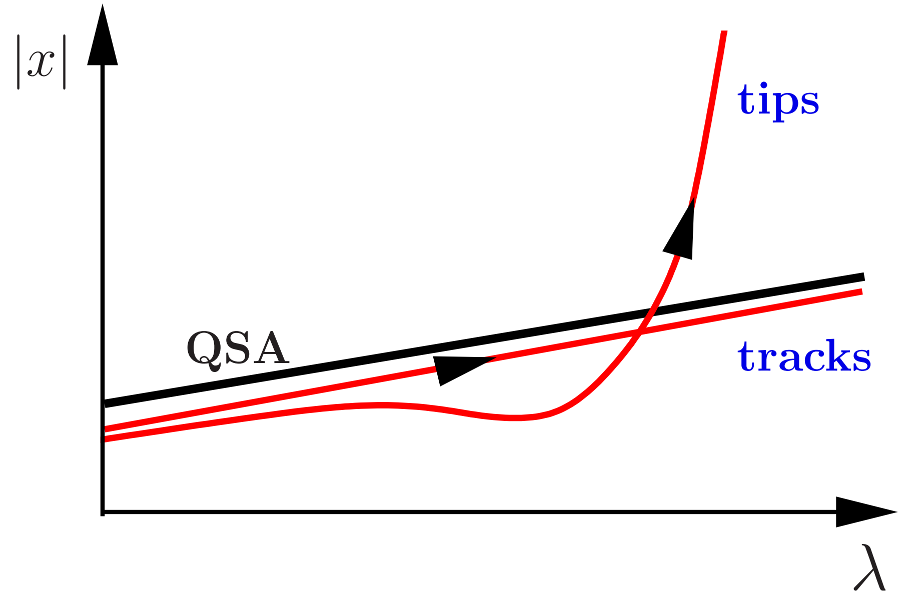


tips for any rate of change

tracks the QSE

B-tipping and R-tipping





tips for any rate of change

tracks the QSE below some critical rate

tips above some critical rate!

[**C.M. Luke and P. Cox,** *Eur. J. Soil Sci.*, **62, 5-12 (2011)**]

[**S.Wieczorek, P. Ashwin, C.M. Luke, P. Cox,** *Proc. Roy. Soc. A*, **May 8 (2011)**]

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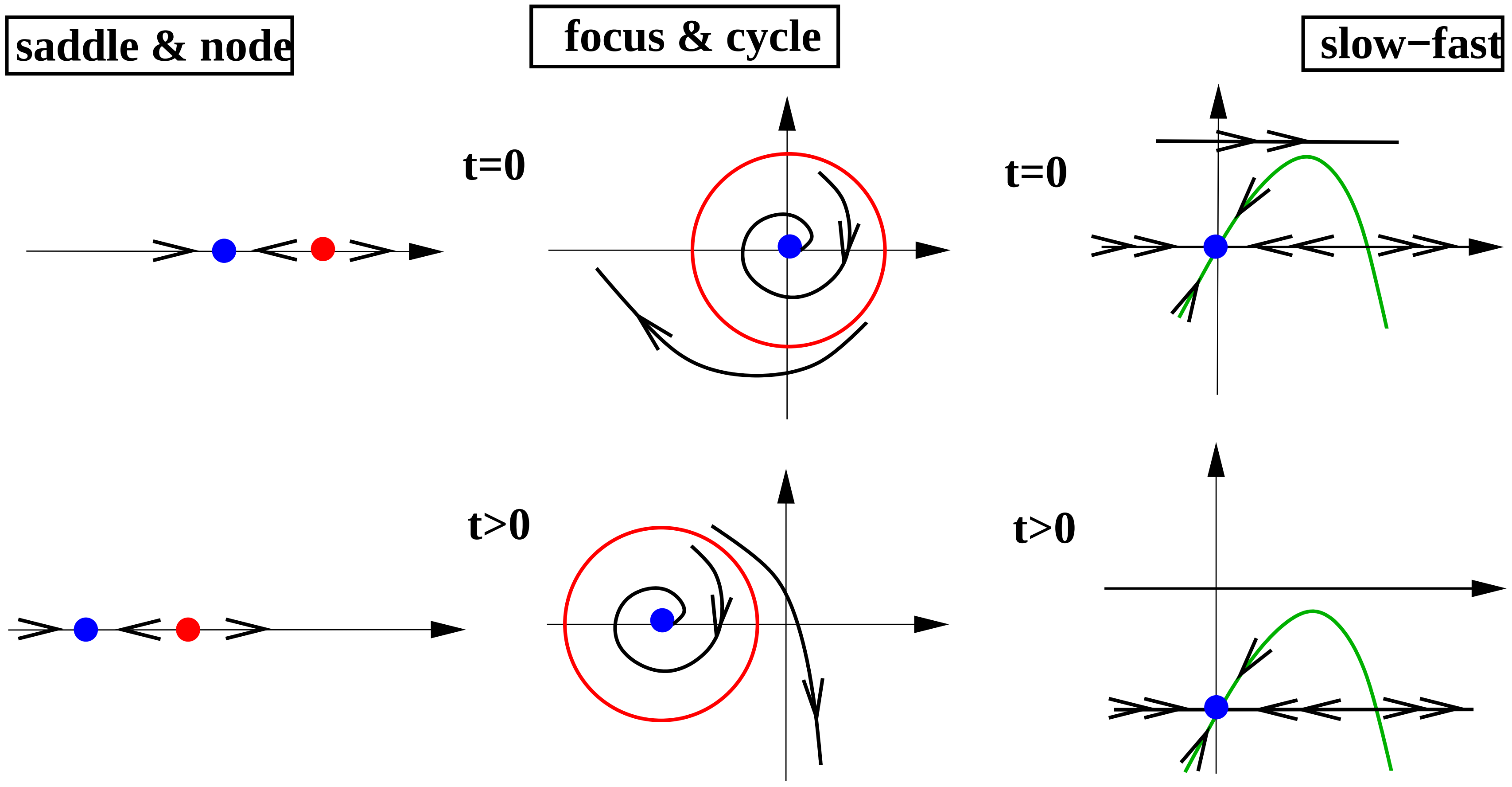
Three Basic Examples of R-tipping



Understanding:

- identify the tipping mechanism
- calculate the critical rate (analytically)

Subcritical Hopf Normal Form with Steady Drift

Consider

$$\frac{d}{dt}\,z = (-1+i\omega)(z-\lambda) + |z-\lambda|^2(z-\lambda), \qquad z\in\mathbb{C}, \quad \lambda = r\,t \in \mathbb{R}$$

$$\frac{d}{dt}\,\lambda = r$$

Reduce to a co-moving system for $s = z - \lambda$

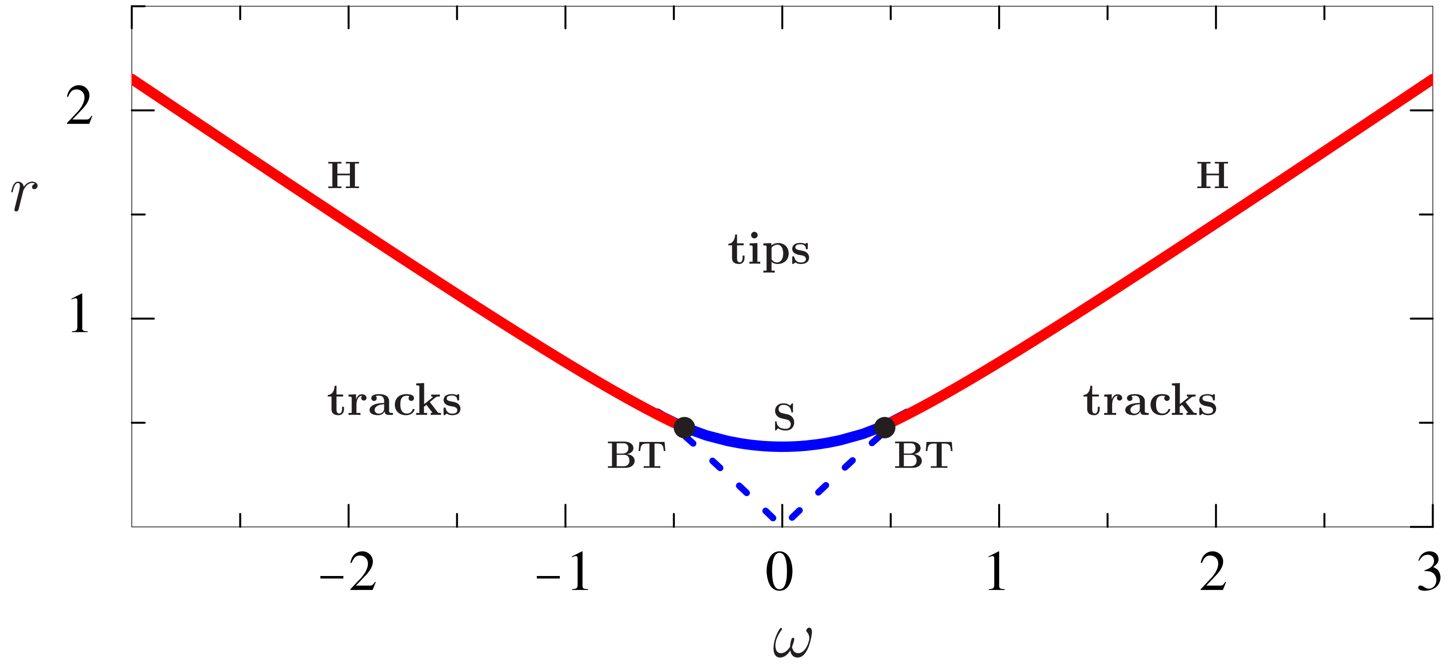
$$\frac{d}{dt}\,s = (-1+i\omega)s + |s|^2 s - r, \qquad s\in\mathbb{C}$$

Tracking the QSE in the forced syst. $\Rightarrow$ **Stable equilib. in the co-moving syst.**

The critical rate problem reduces to a bifurcation problem:
Find $r = r_c$ **where the stable equilib. of the co-moving system disappers/destabilises.**

[**P. Ashwin, S.Wieczorek, R. Vitolo, P. Cox,** *Phil. Trans. Roy. Soc. A*, **(2012)**]

The Tipping Diagram



Critical rate:

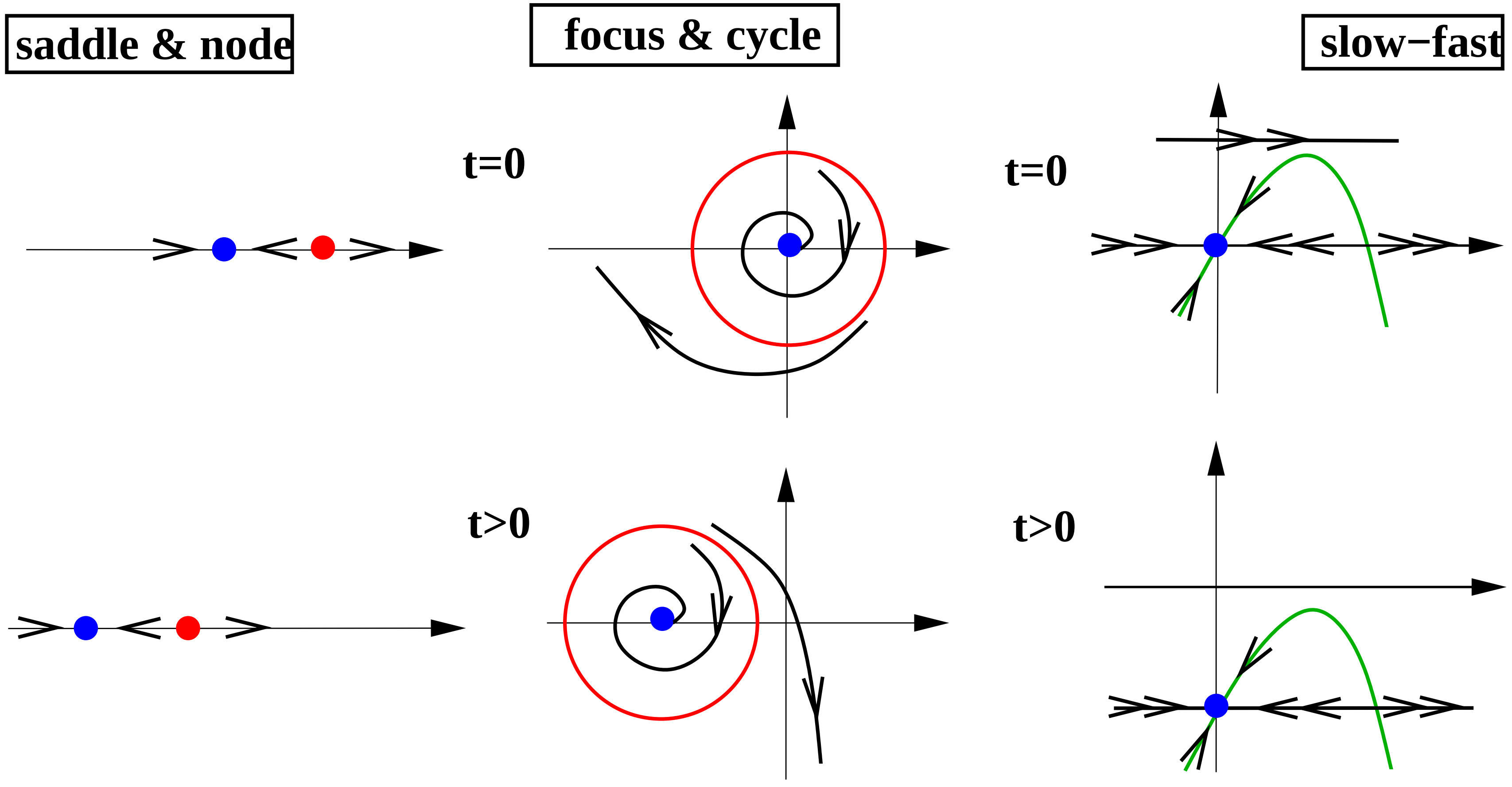
$$r_c(\omega) = \sqrt{(1+4\omega^2)/8} \qquad \textbf{if} \quad \omega^2 \geq 1/4$$

$$r_c(\omega) = \sqrt{|s_e|^6 - 2|s_e|^4 + (1+\omega^2)|s_e|^2} \qquad \textbf{if} \quad \omega^2 < 1/4$$

where

$$|s_e|^2 = \frac{2}{3}\left[1 \pm \sqrt{1 - \frac{3}{4}(1+\omega^2)}\right]$$

Three Basic Examples of R-tipping



Critical Rates in Slow-Fast Systems

- (Existence) A forced system with folded slow (critical) manifold:

$$\frac{d}{dt}\,x = f(x,z,\lambda,\epsilon), \quad \epsilon\,\frac{d}{dt}\,z = g(x,z,\lambda,\epsilon), \quad \frac{d}{dt}\,\lambda = r,$$

  that preserves a stable equilibrium has a critical rate r_C above which it tips, if the reduced system:

$$\frac{d}{dt}\,z = -\frac{\partial g/\partial x|_S\; f|_S + r\;\partial g/\partial \lambda|_S}{\partial g/\partial z|_S}, \quad \frac{d}{dt}\,\lambda = r,$$

  has a folded singularity for some $r > 0$.

- (Computing) General condition for critical rate r_C with dependence on system parameters and initial conditions.

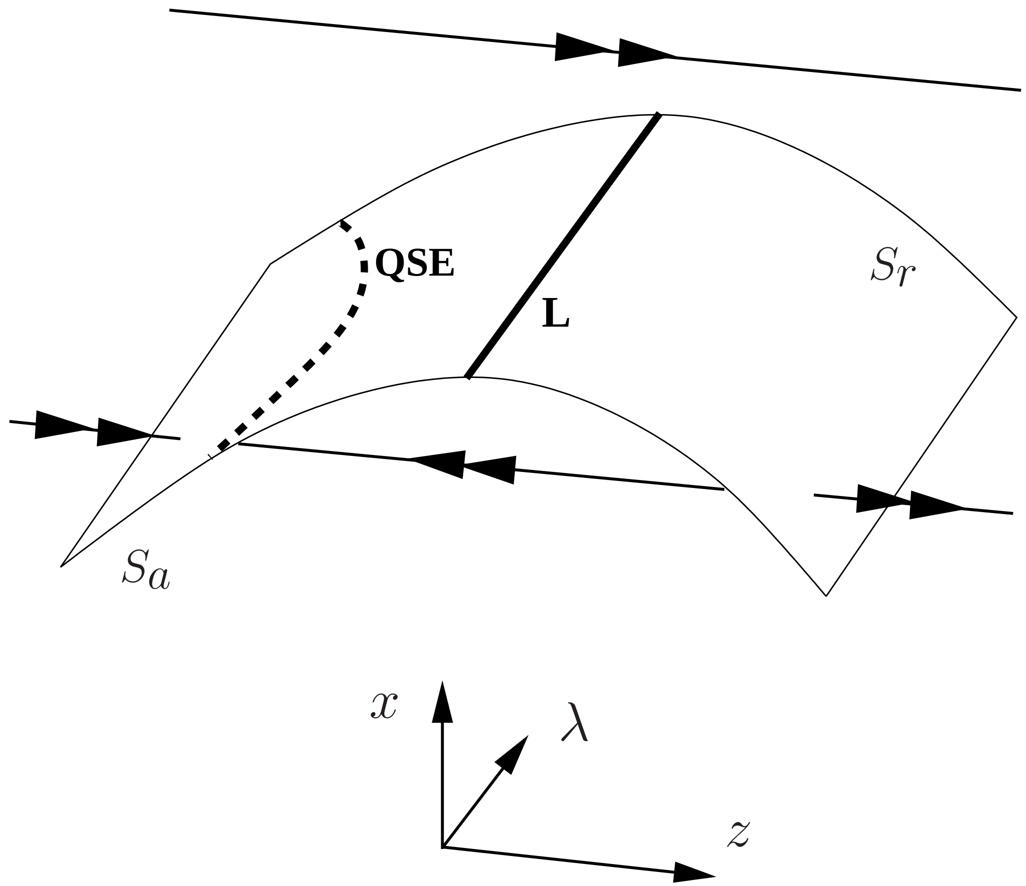
[S.Wieczorek, P. Ashwin, C.M. Luke, P. Cox, *Proc. Roy. Soc. A*, May 8 (2011)]

The Phase Space of the Slow-Fast Problem

Slow (critical) manifold $S = S_a \cup L \cup S_r$

Fold L **is given by** $\partial g/\partial z|_S = 0$

No equilibrium points because $r > 0$



The reduced system:

$$\frac{d}{dt}\, z = -\frac{\partial g/\partial x|_S \; f|_S + r\; \partial g/\partial \lambda|_S}{\partial g/\partial z|_S},$$

$$\frac{d}{dt}\, \lambda = \; r$$

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Example: Climate-Carbon Cycle Models

Carbon Facts:

Peatland soils contain 400–1000 billion tones of carbon.

Question:

How will peatlands respond to global warming?

Photo: Peatland fires in Russia in Summer 2010

Simple Soil-Carbon & Temperature Model

$$\frac{d}{dt}\,C = \Pi - C\,r_0\,e^{\alpha T},$$

$$\epsilon\;\frac{d}{dt}\,T = -\frac{\lambda}{A}\,(T - T_a) + C\,r_0\,e^{\alpha T}, \qquad \textbf{where} \quad \epsilon = \frac{\mu}{A} = 0.064$$

$$\frac{d}{dt}\,T_a = r.$$

C–**soil carbon content,** T–**soil temperature,** T_a–**atmospheric temperature**

From the general condition get the critical rate of global warming

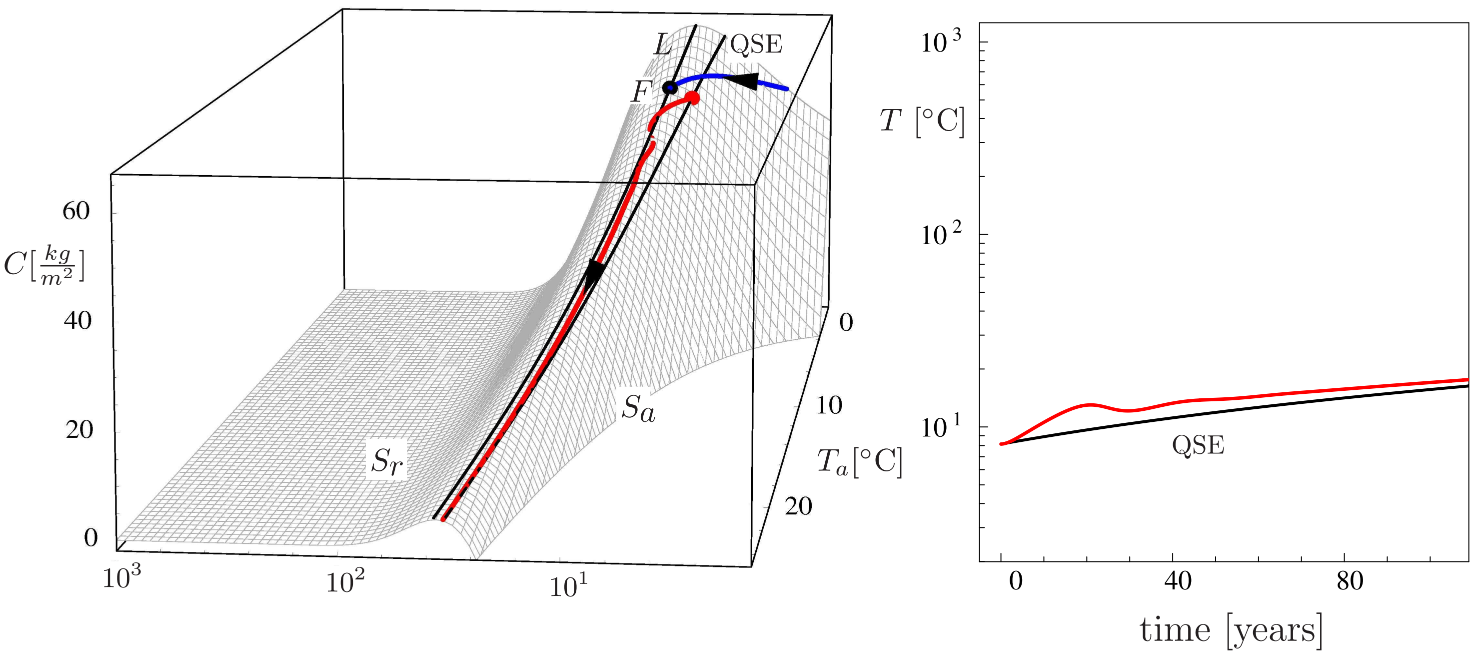
$$r_c = \frac{r_0(1 - \alpha A \Pi/\lambda)}{\alpha}\,\exp\left(\alpha T_a^0 + 1\right) \approx \textbf{8 °C/100 years}$$

[**C.M. Luke and P. Cox,** *Eur. J. Soil Sci.*, **62, 5-12 (2011)**]

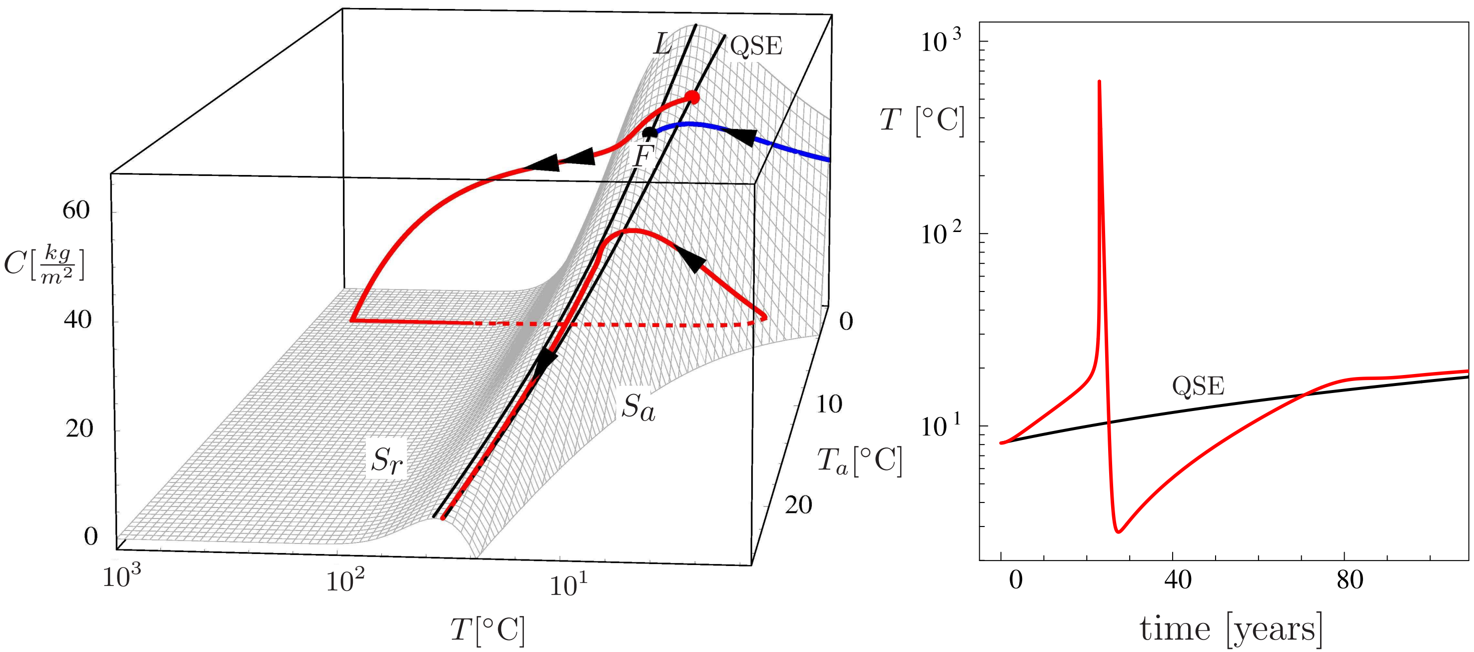
[**S.Wieczorek, P. Ashwin, C.M. Luke, P. Cox,** *Proc. Roy. Soc. A*, **May 8 (2011)**]

The Compost-Bomb Instability

r = 7.5 °C/100 years < r_C



r = 9.0 °C/100 years > r_C



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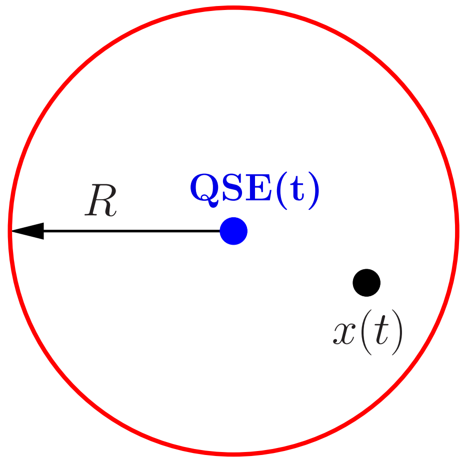
Towards a General Theory of R-tipping

The unforced system has a QSE$(\lambda(t))$ with a tipping radius $R > 0$.

Consider a linear forced system

$$\frac{d}{dt}x = A\left(x - \mathbf{QSE}(t)\right) \quad \textbf{for} \quad |x - \mathbf{QSE}(t)| < R.$$



The system tracks the QSE up to time t if

$$\|A^{-1}\| \cdot |v_{max}(t)| < R, \qquad \textbf{where} \quad v_{max}(t) = \sup_{u \le t}\left|\frac{d\mathbf{QSE}}{dt}\right| = \sup_{u \le t}\left|\frac{d\mathbf{QSE}}{d\lambda}\,\frac{d\lambda}{dt}\right|$$

The system R-tips at time t if

$$\|A\|^{-1} \cdot |v_{max}(t)| = R.$$

[**P. Ashwin, S.Wieczorek, R. Vitolo, P. Cox,** *Phil. Trans. Roy. Soc. A*, **(2012)**]

Towards a General Theory of R-tipping: The question of timescales.

The slowest timescale of the unforced system (leading eigenvalue)

$$\|A^{-1}\|^{-1} \quad \textbf{(units of } \mathrm{s}^{-1}\textbf{)}$$

should be compared with timescale for the motion of the QSE:

$$R^{-1}\left(\left|\frac{d\mathbf{QSE}}{d\lambda}\,\frac{d\lambda}{dt}\right|\right) \quad \textbf{(units of } \mathrm{s}^{-1}\textbf{)}$$

If $R \approx 1$ **and** $\frac{d}{d\lambda}\mathbf{QSE} \approx 1$**, tipping may occur when** $|d\lambda/dt| \approx \|A^{-1}\|^{-1}$

If $R \approx 1$ **and** $\frac{d}{d\lambda}\mathbf{QSE} \approx 1/\epsilon$**, tipping may occur when** $|d\lambda/dt| \approx \epsilon\|A^{-1}\|^{-1}$

Summary of Different Tipping Mechanisms

"B-tipping" where the state of the system changes abruptly or qualitatively due to a bifurcation of a quasi-static attractor
(Thompson & Sieber, Kuehn).

"N-tipping" where noisy fluctuations result in the system departing from a neighbourhood of a quasi-static attractor.
(work on noise-induced escape from attractors)

"R-tipping" where the forced system fails to track a continuously changing quasi-static attractor.

Outlook

• General Theory of R-tipping
(non-autonomous bifurcations, singular perturbation theory)

• A unifying description of the interplay between
different tipping mechanisms