CHARACTER BOUNDS FOR FINITE GROUPS OF LIE TYPE

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ABSTRACT. We establish new bounds on character values and character ratios for finite groups G of Lie type, which are considerably stronger than previously known bounds, and which are best possible in many cases. These bounds have the form $|\chi(g)| \le c\chi(1)^{\alpha_g}$, and give rise to a variety of applications, for example to covering numbers and mixing times of random walks on such groups. In particular we deduce that, if G is a classical group in dimension n, then, under some conditions on G and $g \in G$, the mixing time of the random walk on G with the conjugacy class of g as a generating set is (up to a small multiplicative constant) n/s, where s is the support of g.

CONTENTS

1. Introduction

For a finite group G, a *character ratio* is a complex number of the form $\frac{\chi(g)}{\chi(1)}$, where $g \in G$ and χ is an irreducible character of G. Upper bounds for absolute values of character values and character ratios have long been of interest, for various reasons; these include applications to random generation, covering numbers, mixing times of random walks, the study of word maps, representation varieties and other areas. For example, character ratios are connected with the well-known formula

$$\frac{\prod_{i=1}^k |C_i|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(c_1) \cdots \chi(c_k)\chi(g^{-1})}{\chi(1)^{k-1}}$$

expressing the number of ways of writing an element $g \in G$ as a product $x_1 x_2 \cdots x_k$ of elements $x_i \in C_i$, where $C_i = c_i^G$ are G-conjugacy classes of elements c_i, $1 \le i \le k$, and the sum is over the set $\mathrm{Irr}(G)$ of all irreducible characters of G (see [2, 10.1]). This connection is sometimes a starting point for such applications; it has been particularly exploited for almost simple (or quasisimple) groups G.

Another classical formula involving character ratios goes back to Frobenius in 1896 [10]. It asserts that, for any finite group G, the number $N(g)$ of ways to express an element $g \in G$ as a commutator $[x, y]$ $(x, y \in G)$ satisfies

$$N(g) = |G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)}.$$

This formula is widely used, and served (together with character bounds) as an important tool in the proof of Ore's conjecture [30].

We are particularly interested in so-called *exponential character bounds*, namely bounds of the form

$$|\chi(g)| \le \chi(1)^{\alpha_g},$$

sometimes with a multiplicative constant, holding for *all* characters $\chi \in \mathrm{Irr}(G)$, where $0 \le \alpha_g \le 1$ depends on the group element $g \in G$. Obviously, if g is central in G, then we must have $\alpha_g = 1$, but for most elements g we aim to find $\alpha_g < 1$ which is as small (and explicit) as possible. One advantage of exponential character bounds is that they imply the inequality $|\frac{\chi(g)}{\chi(1)}| \le \chi(1)^{-(1-\alpha_g)}$, so the upper bound on the character ratio becomes smaller as the character degree grows.

The first exponential character bound was established in 1995 for symmetric groups S_n by Fomin and Lulov [9]. They show that, for permutations $g \in \mathsf{S}_n$ which are products of n/m cycles of length m and for all characters $\chi \in \mathrm{Irr}(\mathsf{S}_n)$ we have

$$|\chi(g)| \le c(m) n^{\frac{1}{2} - \frac{1}{2m}} \chi(1)^{\frac{1}{m}}, \tag{1.1}$$

for a suitable function $c : \mathbb{N} \to \mathbb{N}$.

In [36] this bound and some extensions of it were applied in various contexts, including the theory of Fuchsian groups. Subsequently, exponential character bounds which hold for *all* permutations $g \in \mathsf{S}_n$ and which are essentially best possible were established in 2008 in [25], with applications to a range of problems: mixing times of random walks, covering by powers of conjugacy classes, as well as probabilistic and combinatorial properties of word maps.

Can we find good exponential character bounds for groups of Lie type? This problem has turned out to be quite formidable; it has been considered by various researchers over the past two decades, and various approaches have been attempted, but it is only in this paper that strong (essentially best possible) such bounds are established.

The first significant bound on character ratios for groups of Lie type was obtained in 1993 by Gluck [13], who showed that $\frac{|\chi(g)|}{\chi(1)} \leq Cq^{-1/2}$ for any non-central element $g \in G(q)$, a group of Lie type over $\mathbb{F}_q$, and any non-linear irreducible character χ of $G(q)$, where C is an absolute constant. In [14], he proved a bound of the form

$$\frac{|\chi(g)|}{\chi(1)} \leq \chi(1)^{-\gamma/n},$$

when $G(q)$ is a classical group with natural module $V = \mathbb{F}_q^n$ of dimension n, and $\gamma = \gamma(q, d)$ is a positive real number depending on q and on $d = \dim[V, g]$, the dimension of the commutator space of g on V. While this result provides an exponential character bound $|\chi(g)| \leq \chi(1)^{\alpha_g}$, the exponent $\alpha_g = 1 - \gamma/n$ is not explicit, and in the general case we have $\gamma(q, d) \leq 0.001$, so $\alpha_g \geq 1 - \frac{1}{1000n}$, which is very close to 1.

An explicit character bound for finite classical groups, with natural module $V = \mathbb{F}_q^n$, in terms of the *support* $\mathrm{supp}(g)$ of the element g was obtained in [26, 4.3.6]: namely,

$$\frac{|\chi(g)|}{\chi(1)} < q^{-\sqrt{\mathrm{supp}(g)}/481}, \tag{1.2}$$

where $\mathrm{supp}(g)$ is the codimension of the largest eigenspace of g on $V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. These results have applications to covering numbers, mixing times and word maps.

In this paper we obtain asymptotically much stronger bounds for character ratios of finite groups of Lie type in good characteristic (this restriction comes from the fact that our proof relies on certain results in the Deligne-Lusztig theory, which currently are only known to hold in good characteristics). In fact we provide the first explicit exponential character bounds for groups of Lie type, and show that these bounds are asymptotically optimal in many cases.

These character bounds lead to several new results on random walks and covering by products of conjugacy classes that are far stronger than previously known such results. Further applications to the theory of representation varieties of Fuchsian groups and probabilistic generation of groups of Lie type will be given in a sequel to this paper [32].

We also prove the first bounds on character ratios for Brauer characters, for the groups $\mathrm{SL}_n(q)$ and $\mathrm{GL}_n(q)$, and in characteristics coprime to q.

We now describe our results. Throughout the paper, let $\mathbb{K}$ be an algebraically closed field of characteristic p, $\mathcal{G}$ a connected reductive algebraic group over $\mathbb{K}$, $F : \mathcal{G} \to \mathcal{G}$ a Frobenius endomorphism, and $G = \mathcal{G}^F$. For a subgroup X of $\mathcal{G}$ write X_{unip} for the set of *non-identity* unipotent elements of X. For a fixed F, a Levi subgroup $\mathcal{L}$ of $\mathcal{G}$ will be called *split*, if it is an F-stable Levi subgroup of an F-stable parabolic subgroup of $\mathcal{G}$. For an F-stable Levi subgroup $\mathcal{L}$ of $\mathcal{G}$ and $L = \mathcal{L}^F$, we define

$$\alpha(L) := \max_{u \in L_{\mathrm{unip}}} \frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}}, \quad \alpha(\mathcal{L}) := \max_{u \in \mathcal{L}_{\mathrm{unip}}} \frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}}$$

if $\mathcal{L}$ is not a torus, and $\alpha(L) = \alpha(\mathcal{L}) := 0$ otherwise.

Theorem 1.1. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that the following statement holds. Let $\mathcal{G}$ be a connected reductive algebraic group such that $[\mathcal{G}, \mathcal{G}]$ is simple of rank r over a field of good characteristic $p > 0$. Let $G := \mathcal{G}^F$ for a Frobenius endomorphism $F : \mathcal{G} \to \mathcal{G}$. Let $g \in G$ be any element such that $\mathbf{C}_G(g) \leq L := \mathcal{L}^F$, where $\mathcal{L}$ is a proper split Levi subgroup of $\mathcal{G}$. Then, for any character $\chi \in \mathrm{Irr}(G)$ and $\alpha := \alpha(L)$, we have*

$$|\chi(g)| \leq f(r)\chi(1)^{\alpha}.$$

Remark 1.2. (i) The α-bound in Theorem 1.1 is sharp in several cases – see Example 2.8. In fact, this α-bound is *always* sharp in the case of $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$, by Theorem 1.3.

(ii) If $r \geq 9$ and $q \geq r^2 + 1$, then the function $f(r)$ in Theorem 1.1 can be chosen to be $2^{2r+\sqrt{2r}+3} \cdot (r!)^2$ (with the main term being the square of the largest order of the Weyl group of a simple algebraic group of rank r) – see Proposition 2.7. Moreover, $\alpha \lesssim_r 1 - 1/r$ by Theorem 1.6 and $\chi(1) \geq q^r/3$ if $\chi(1) > 1$ by [24], hence Theorem 1.1 yields $|\chi(g)| \lesssim_r \chi(1)^{\alpha+1/2r} \lesssim_r \chi(1)^{1-1/2r}$ if $q > r^{4r}$; in fact, $\chi(1) \geq q^{r^2/2}$ for most of $\chi \in \mathrm{Irr}(G)$, for which the bound becomes $|\chi(g)| \lesssim_r \chi(1)^{\alpha+1/3r} \lesssim_r \chi(1)^{1-2/3r}$ if $q > r^{12}$. (Here, we say that $f_1(x) \lesssim_x f_2(x)$ for two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ if $\limsup_{x\to\infty} f_1(x)/f_2(x) \leq 1$.)

(iii) Although the aforementioned choice of $f(r)$ in Theorem 1.1 can be improved, Example 2.8(vi) shows that $f(r)$ should be at least the largest degree of complex irreducible characters of the Weyl group $W(\mathcal{G})$ of $\mathcal{G}$, which can be quite close to $|W(\mathcal{G})|^{1/2}$. In particular, choosing $\mathcal{G}$ of type A_r and applying [38], [53], we get $f(r) > e^{-1.283\sqrt{r+1}}\sqrt{(r+1)!}$.

Note that Theorem 1.1 and its various consequences also apply for finite twisted groups of Lie type (in good characteristic). The next result shows that the exponent α in Theorem 1.1 is optimal.

Theorem 1.3. *In the notation of Theorem 1.1, there is a constant $C_n > 0$ depending only on n such that the following statement holds. For $G = \mathcal{G}^F = \mathrm{GL}_n(q)$ with $q \geq C_n$ and for any proper split Levi subgroup $\mathcal{L}$ of $\mathcal{G}$, there is a semisimple element $g \in G$ and a unipotent character $\chi \in \mathrm{Irr}(G)$ such that $\mathbf{C}_G(g) = L = \mathcal{L}^F$ and*

$$|\chi(g)| \geq \frac{1}{4}\chi(1)^{\alpha(L)}.$$

The same conclusion holds for $\mathrm{SL}_n(q)$, if for instance we choose q so that $q-1$ is also divisible by $(n!)^n$.

In the case of $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$ we can also prove a version of Theorem 1.1 for Brauer characters in cross-characteristic.

Theorem 1.4. *There exists a function $h : \mathbb{N} \to \mathbb{N}$ such that the following statement holds. Let $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$ or $\mathrm{SL}_n(\mathbb{K})$ be an algebraic group over a field of characteristic $p > 0$ and $F : \mathcal{G} \to \mathcal{G}$ a Frobenius endomorphism, such that $G = \mathcal{G}^F \cong \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$. Let $\ell = 0$ or a prime not dividing q. Let $g \in G$ be any ℓ'-element such that $\mathbf{C}_G(g) \leq L := \mathcal{L}^F$, where $\mathcal{L}$ is a proper split Levi subgroup of $\mathcal{G}$. Then for any irreducible ℓ-Brauer character φ of G and $\alpha := \alpha(L)$, we have*

$$|\varphi(g)| \leq h(n)\varphi(1)^{\alpha}.$$

The above results do not cover, for instance, the case where $g \in \mathcal{G}^F$ is a unipotent element. In fact, if $G = \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$, then Theorem 1.1 leaves out precisely those elements g with semisimple part s such that $\mathbf{C}_{\mathrm{GL}_n(q)}(s) \cong \mathrm{GL}_{n/a}(q^a)$, an extension-field subgroup. In other classical groups, there are further elements left out by Theorem 1.1, including those whose semisimple parts s are quasi-isolated (i.e. $\mathbf{C}_{\mathcal{G}}(s)$ is not contained in any proper Levi subgroup of $\mathcal{G}$). However, we have been able to obtain a complete result covering all elements in $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$:

Theorem 1.5. *There is a function $h : \mathbb{N} \to \mathbb{N}$ such that the following statement holds. For any $n \geq 5$, any prime power q, any irreducible complex character χ of $H := \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$, and any non-central element $g \in H$,*

$$|\chi(g)| \leq h(n) \cdot \chi(1)^{1-\frac{1}{2n}}.$$

For the remaining groups of Lie type, character bounds of a somewhat different kind, which work for arbitrary elements $g \in \mathcal{G}^F$, and are *weaker* than the one in Theorem 1.1 but asymptotically *stronger* than the ones in [14] and [26], will be proved in a sequel to this paper. We also note [16], which explores a completely different approach and establishes

exponential character bounds for $|\chi(g)|$, where either g is an element of a finite group of Lie type G with not-too-large centralizer $\mathbf{C}_G(g)$ or $\chi \in \mathrm{Irr}(G)$ has not-too-large degree.

To be able to apply Theorem 1.1 we need information on the values of $\alpha(L) \leq \alpha(\mathcal{L})$. For classical groups, we prove the following upper bound.

Theorem 1.6. *If $\mathcal{G}$ is a classical algebraic group over $\mathbb{K}$ in good characteristic, and $\mathcal{L}$ is a Levi subgroup of $\mathcal{G}$, then*

$$\alpha(\mathcal{L}) \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right).$$

For exceptional types we obtain fairly complete information.

Theorem 1.7. *If $\mathcal{G}$ is an exceptional algebraic group in good characteristic, the values of $\alpha(\mathcal{L})$ for (proper, non-toral) Levi subgroups $\mathcal{L}$ are as in Table 1.*

In Table 1, for $\mathcal{G} = F_4$ or G_2 the symbols $\tilde{A}_1, \tilde{A}_2$ refer to Levi subsystems consisting of short roots. For $\mathcal{G} = E_7$, there are two Levi subgroups A_5 and A_5': using the notation for the fundamental roots α_i $(1 \leq i \leq 7)$ as in [3], these are the Levi subgroups with fundamental roots $\{\alpha_i : i = 1, 3, 4, 5, 6\}$ and $\{\alpha_i : i = 2, 4, 5, 6, 7\}$ respectively. The notation $\triangleright A_4$, for instance, means that $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$ has a simple factor of type A_4. Furthermore, in this table and elsewhere, we will also sometimes denote a simple algebraic group by the type of its Dynkin diagram, like A_4, C_3, etc., and use T_m to denote an m-dimensional torus.

TABLE 1. α-values for exceptional groups

$\mathcal{G} = E_8$, $\mathcal{L}' =$	E_7	D_7	$\mathcal{L}' \triangleright E_6$	D_6	A_7	$\triangleright D_5$	$\triangleright A_6$	$\triangleright A_5$	$\triangleright D_4$	rest
$\alpha(\mathcal{L}) =$	$\frac{17}{29}$	$\frac{9}{23}$	$\frac{11}{29}$	$\frac{9}{29}$	$\frac{15}{56}$	$\frac{7}{29}$	$\frac{5}{23}$	$\frac{4}{23}$	$\frac{5}{29}$	$\leq \frac{1}{6}$
$\mathcal{G} = E_7$, $\mathcal{L}' =$	E_6	D_6	$\mathcal{L}' \triangleright D_5$	A_6	A_5	$\triangleright A_5'$	$\triangleright D_4$	$\triangleright A_4$	$\triangleright A_3$	rest
$\alpha(\mathcal{L}) =$	$\frac{11}{17}$	$\frac{5}{9}$	$\frac{7}{17}$	$\frac{5}{13}$	$\frac{4}{13}$	$\frac{1}{3}$	$\frac{5}{17}$	$\leq \frac{1}{4}$	$\leq \frac{1}{5}$	$\leq \frac{1}{6}$
$\mathcal{G} = E_6$, $\mathcal{L}' =$	D_5	A_5	D_4	$\mathcal{L}' \triangleright A_4$	$\triangleright A_3$	$\triangleright A_2$	A_1^k			
$\alpha(\mathcal{L}) =$	$\frac{7}{11}$	$\frac{1}{2}$	$\frac{5}{11}$	$\frac{3}{8}$	$\frac{3}{11}$	$\leq \frac{7}{27}$	$\leq \frac{3}{20}$			
$\mathcal{G} = F_4$, $\mathcal{L}' =$	B_3	C_3	$A_2\tilde{A}_1, A_2$	$\tilde{A}_2 A_1$	$\tilde{A}_2$	$A_1\tilde{A}_1$	A_1	$\tilde{A}_1$		
$\alpha(\mathcal{L}) =$	$\frac{1}{2}$	$\frac{7}{15}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{11}$		
$\mathcal{G} = G_2$, $\mathcal{L}' =$	A_1	$\tilde{A}_1$								
$\alpha(\mathcal{L}) =$	$\frac{1}{3}$	$\frac{1}{4}$								

We can now easily deduce the following.

Corollary 1.8. *Let $\mathcal{G}$, $G = \mathcal{G}^F$, and f be as in Theorem 1.1. Suppose $y \in G$ is a (semisimple) element such that $\mathbf{C}_G(y) = \mathcal{L}^F$, where $\mathcal{L}$ is a proper split Levi subgroup of $\mathcal{G}$. Then for any non-linear $\chi \in \mathrm{Irr}(G)$,*

$$|\chi(y)| \leq f(r)\,\chi(1)^{1 - \frac{1}{2}\frac{\dim y^{\mathcal{G}}}{\dim \mathcal{G}}}.$$

Next, we establish a new bound on character ratios of certain elements, in terms of the support (which is defined right after (1.2)) of its semisimple part.

Theorem 1.9. *Assume $\mathcal{G} = \mathrm{SL}_n(\mathbb{K})$ with $n \geq 2$, $\mathrm{Sp}_n(\mathbb{K})$ with $n \geq 4$, or $\mathrm{Spin}_n(\mathbb{K})$ with $n \geq 7$, all in good characteristic, and define*

$$c := c(\mathcal{G}) = \begin{cases} (r+1)/(2r+4), & \mathcal{G} = \mathrm{SL}_{r+1}, \\ r/(4r+2), & \mathcal{G} = \mathrm{Sp}_{2r}, \\ r/(4r-2), & \mathcal{G} = \mathrm{Spin}_{2r}, \\ 1/4, & \mathcal{G} = \mathrm{Spin}_{2r+1}. \end{cases}$$

Let $G = \mathcal{G}^F = G(q)$ be defined over $\mathbb{F}_q$ and f be as in Theorem 1.1, and let $g \in G$ be any element such that its semisimple part y has centralizer $\mathbf{C}_G(y) = \mathcal{L}^F$, where $\mathcal{L}$ is a proper split Levi subgroup of $\mathcal{G}$. Then, for any non-linear $\chi \in \mathrm{Irr}(G)$,

$$\frac{|\chi(g)|}{\chi(1)} \leq 3f(r)\, q^{-c \cdot \mathrm{supp}(y)}.$$

In particular it follows that $\frac{|\chi(y)|}{\chi(1)} \leq 3f(r)\, \chi(1)^{-c \cdot \mathrm{supp}(y)}$, and that for any $\epsilon > 0$, $r \geq r(\epsilon)$ and q larger than a suitable function of r, we have

$$\frac{|\chi(y)|}{\chi(1)} \leq q^{-(b-\epsilon) \cdot \mathrm{supp}(y)},$$

where $b = 1/2$ in the SL_{r+1} case and $b = 1/4$ in the other cases.

Theorem 1.9 and its consequences considerably improve the bound (1.2) from [26, 4.3.6] for elements as above.

We also obtain more precise character bounds for $\mathrm{GL}_n(q)$. To state them we need some notation. For positive integers $n_1, \ldots, n_m$ define

$$\beta(n_1, \ldots, n_m) := \max \frac{\sum_{i=1}^m (n_i^2 - \sum_{j=1}^n a_{ij}^2)}{n^2 - \sum_{j=1}^n (\sum_{i=1}^m a_{ij})^2},$$

where $n = n_1 + \ldots + n_m$ and the maximum is taken over all non-negative integers a_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$ satisfying

$$\sum_{j=1}^n a_{ij} = n_i, \ a_{i1} \geq a_{i2} \geq \ldots \geq a_{in}, \ 1 \leq i \leq m, \ \max_{1 \leq i \leq m} a_{i2} > 0,$$

if $\max_{1 \leq i \leq m} n_i \geq 2$, and let $\beta(1, 1, \ldots, 1) = 0$.

Theorem 1.10. *Let $G = \mathrm{GL}_n(q)$ and let $L \leq G$ be a Levi subgroup of the form $L = \mathrm{GL}_{n_1}(q) \times \cdots \times \mathrm{GL}_{n_m}(q)$, where $n_i \geq 1$ and $\sum_{i=1}^m n_i = n$. Let $n_{i_0} = \max_{1 \leq i \leq m} n_i$. Then*

$$\frac{n_{i_0} - 1}{n - t} \leq \alpha(L) = \beta(n_1, \ldots, n_m) \leq \frac{n_{i_0}}{n}$$

if $n_{i_0} \geq 2$ and t is the number of $1 \leq j \leq m$ such that $n_j = n_{i_0}$, and

$$\alpha(L) = \beta(n_1, \ldots, n_m) = 0$$

if $n_{i_0} = 1$. Consequently, for every $g \in G$ with $\mathbf{C}_G(g) \leq L$ and every $\chi \in \mathrm{Irr}(G)$ we have

$$|\chi(g)| \leq f(n-1)\chi(1)^{\beta(n_1, \ldots, n_m)},$$

where $f : \mathbb{N} \to \mathbb{N}$ is the function specified in Theorem 1.1.

Suppose now that m divides n and $n_1 = \ldots = n_m = n/m > 1$. Then we can show that $\beta(n_1, \ldots, n_m) = \frac{1}{m}$, so we immediately obtain the following.

Corollary 1.11. *Let $G = \mathrm{GL}_n(q)$ where q is a prime power. Let $m < n$ be a divisor of n and let $L \leq G$ be a Levi subgroup of the form $L = \mathrm{GL}_{n/m}(q)^m$. Let $g \in G$ with $\mathbf{C}_G(g) \leq L$. Then we have*

$$|\chi(g)| \leq f(n-1)\chi(1)^{\frac{1}{m}}$$

for all characters $\chi \in \mathrm{Irr}(G)$, where $f : \mathbb{N} \to \mathbb{N}$ is the function specified in Theorem 1.1.

Example 2.8 again shows that the exponent $1/m$ in Corollary 1.11 is sharp. In general, Theorem 1.10 determines $\alpha(L)$ up to within $1/n$. It is reasonable to conjecture that, under the hypotheses of Theorem 1.10, $\alpha(L) = (n_{i_0} - 1)/(n - t)$. This conjecture is confirmed in Theorem 4.13 for the case $m = 2$ (as well as in the cases, where either $n \leq 8$, or $m \leq 4$ and $n \leq 13$, by direct calculation).

The bound in Theorem 1.10 and some variations on it have applications to Fuchsian groups (see [32]). Corollary 1.11 may be regarded as a Lie analogue of the Fomin-Lulov character bound (1.1) for S_n mentioned before.

We now present some applications of the above results to the theory of *mixing times* for random walks on finite quasisimple groups of Lie type corresponding to conjugacy classes. Let $G = G(q)$ be such a group, let $y \in G$ be a non-central element, and let $C = y^G$, the conjugacy class of y. Consider the random walk on the corresponding Cayley graph starting at the identity, and at each step moving from a vertex g to a neighbour gs, where $s \in y^G$ is chosen uniformly at random. Let $P^t(g)$ be the probability of reaching the vertex g after t steps. The mixing time of this random walk is defined to be the smallest integer $t = T(G, y)$ such that $||P^t - U||_1 < \frac{1}{e}$, where U is the uniform distribution and $||f||_1 = \sum_{g \in G} |f(g)|$ is the l_1-norm.

Mixing times of such random walks have been extensively studied since the pioneering work of Diaconis and Shashahani [6] on the case $G = \mathsf{S}_n$ and C the class of transpositions in S_n. Additional results on random walks in symmetric and alternating groups have been obtained in various papers, see for instance [43], [54], [40] and [25]. The latter paper obtains essentially optimal results on mixing times in these groups.

However, if we turn from symmetric groups to finite groups G of Lie type, good estimates on mixing times have been obtained only in very few cases. Hildebrand [17] showed that the mixing time for the class of tranvections in $\mathrm{SL}_n(q)$ is of the order of n. In [37] it is shown that if $y \in G$ is a regular element, then the mixing time $T(G, y)$ is 2 when $G \neq \mathrm{PSL}_2(q)$ is large. In [44] it is proved that, if G is any finite simple group, then for a random $y \in G$ we have $T(G, y) = 2$ (namely, the latter equality holds with probability tending to 1 as $|G| \to \infty$). Other than that, the mixing times $T(G, y)$ for groups G of Lie type remain a mystery.

The next result contains bounds for mixing times, and also (in parts (I)(a) and (II)) for the number of steps required so that P^t is close to U in the l_∞-norm, which is stronger than the l_1-norm condition for mixing time (and also implies that the random walks hits all elements of G). Here we define $||f||_\infty = |G| \max_{x \in G} |f(x)|$, and say that $C^t = G$ *almost uniformly pointwise* as $q \to \infty$ if $||P^t - U||_\infty \to 0$ as $q \to \infty$.

We denote by $h := h(\mathcal{G})$, the Coxeter number of $\mathcal{G}$, defined by

$$h(\mathcal{G}) = \frac{\dim \mathcal{G}}{r} - 1,$$

where r is the rank of $\mathcal{G}$. Note that $h \geq 2$ and that $h \to \infty$ as $r \to \infty$.

Theorem 1.12. *Suppose $\mathcal{G}$ is a simple algebraic group in good characteristic, and $G = G(q) = \mathcal{G}^F$ is a finite quasisimple group over $\mathbb{F}_q$. Let $y \in G$ be such that $\mathbf{C}_G(y) \leq L$, where $L = \mathcal{L}^F$ for a proper split Levi subgroup $\mathcal{L}$ of $\mathcal{G}$. Write $C = y^G$.*

(I) *Suppose $\mathcal{G}$ is of classical type.*
  (a) *If $t > (4 + \frac{4}{h}) \frac{\dim \mathcal{G}}{\dim \mathcal{G} - \dim \mathcal{L}}$, then $C^t = G$ almost uniformly pointwise as $q \to \infty$. In particular, $C^t = G$ for sufficiently large q.*
  (b) *The mixing time $T(G, y) \leq \lceil (2 + \frac{2}{h}) \frac{\dim \mathcal{G}}{\dim \mathcal{G} - \dim \mathcal{L}} \rceil$ for large q.*

(II) *Suppose $\mathcal{G}$ is of exceptional type. Then $C^6 = G$ almost uniformly pointwise as $q \to \infty$, and the mixing time $T(G, y) \leq 3$.*

Remarks (i) Note that the multiplicative constants above are very small. For example, $2 + \frac{2}{h} \leq 3$ and it tends to 2 as $r \to \infty$.

(ii) The constant $2 + \frac{2}{h}$ in part I(b) of Theorem 1.12 is best possible for some classes, for example homologies $y = \mathrm{diag}(\mu I_{n-1}, \lambda)$ in $G = \mathrm{SL}_n(q)$ (where $\mu, \lambda \in \mathbb{F}_q^\times$ and $\mu \neq \lambda$ and so $q \geq 3$), for which the bound given by part (I)(b) is $T(G, y) \leq n + 3$ and for which the mixing time is at least n by Lemma 5.2(ii).

(iii) The bound $T(G, y) \leq 3$ for exceptional groups in (II) is best possible for those conjugacy classes y^G with $\mathbf{C}_G(y)$ contained in a proper split Levi subgroup L and $\dim y^{\mathcal{G}} < \frac{1}{2} \dim \mathcal{G}$. For such classes, $|y^G|^2 < |G|/2$ for large q, so the mixing time cannot be 2 by Lemma 5.2(i).

Theorem 1.12(I)(b) implies the following linear bounds for classical groups.

Corollary 1.13. *Let $G = \mathcal{G}^F$ be a quasisimple classical group over $\mathbb{F}_q$, where $\mathcal{G}$ is simple of rank r over $\overline{\mathbb{F}}_q$ in good characteristic, and let $y \in G$ be as in Theorem 1.12. Then for large q,*

(i) *the diameter $\mathrm{diam}(G, y^G) \leq 2r + 4$, and*
(ii) *the mixing time $T(G, y) \leq r + 2$.*

A linear bound for the diameter (of the order of $40r$), which holds for all non-central conjugacy classes, can be found in [27].

Using Theorem 1.5 we can obtain such a bound for *all* conjugacy classes in $\mathrm{SL}_n(q)$:

Corollary 1.14. *Let $G = \mathrm{SL}_n(q)$, let x be an arbitrary non-central element of G and let $C = x^G$.*

(i) *If $t > 4n + 4$, then $C^t = G$ almost uniformly pointwise as $q \to \infty$.*
(ii) *The mixing time $T(G, x) \leq 2n + 3$ for large q.*

Note that [28, Theorem 1] shows that $C^n = G$ for any nontrivial conjugacy class C of $G = \mathrm{PSL}_n(q)$, where $n \geq 3$, $q \geq 4$.

We can also use Theorem 3.3 (or rather its corollary 3.5) to obtain a better bound for unipotent elements of $\mathrm{SL}_n(q)$.

Theorem 1.15. *Let $G = \mathrm{SL}_n(q)$ and let u be a non-identity unipotent element in G. Write $C = u^G$.*

(i) *If $t > 2n$, then $C^t = G$ almost uniformly pointwise as $q \to \infty$. In particular, $C^t = G$ for sufficiently large q.*
(ii) *The mixing time $T(G, u) \leq n$ for sufficiently large q.*

One can compare part (ii) of the above theorem with Hildebrand's result [17] for transvections, where he proves that for n varying, the mixing time for the class of transvections in $\mathrm{SL}_n(q)$ is of the order of n. In our case n may still vary, but q should be much larger than n. The coincidence of values seems striking.

It is interesting to compare the mixing time $T(G, y)$ with the *covering number $cn(G, C)$* of the conjugacy class $C = y^G$, defined as the minimal t for which $C^t = G$. It is known that there is an absolute constant b such that for any conjugacy class $C \neq \{1\}$ of any finite simple group G we have

$$\frac{\log |G|}{\log |C|} \leq cn(G, C) \leq b \frac{\log |G|}{\log |C|}.$$

Indeed the first inequality is trivial, while the second is [35, 1.2].

It is easy to see that, with the above notation,

$$\frac{\log|G| + \log(1 - e^{-1})}{\log|C|} \le T(G, y). \tag{1.3}$$

Indeed, this follows from Lemma 5.2.

It is conjectured in [46, 4.3] that there is an absolute constant c such that for any finite simple group G of Lie type and any non-identity element $y \in G$ we have

$$T(G, y) \le c \frac{\log|G|}{\log|C|}, \tag{1.4}$$

where $C = y^G$.

Note that this statement does not hold for alternating groups G (take $y \in G$ to be a cycle of length around $n/2$ – then $\frac{\log|G|}{\log|C|}$ is bounded, while $T(G, y)$ is of the order of $\log n$).

The above conjecture is related to an older conjecture posed by Lubotzky in [39, p.179]. Lubotzky conjectured that, if G is a finite simple group and C is a non-trivial conjugacy class of G, then the mixing time of the Cayley graph $\Gamma(G, C)$ of G with C as a generating set is linearly bounded above in terms of the diameter of $\Gamma(G, C)$. Since this diameter is exactly the covering number $cn(G, C)$, this conjecture (combined with the more recent upper bound on $cn(G, C)$ mentioned above) implies conjecture (1.4).

Applying Theorem 1.12 we are able to prove the above conjectures in some interesting cases.

Corollary 1.16. *Let $\mathcal{G}$ be a simple algebraic group in good characteristic, and $G = G(q) = \mathcal{G}^F$ a finite quasisimple group over $\mathbb{F}_q$. Suppose q is large (given $\mathcal{G}$). Then Conjecture (1.4) holds for all non-central elements $y \in G$, whenever $\mathbf{C}_G(y) = \mathcal{L}^F$ for a proper split Levi subgroup $\mathcal{L}$ of $\mathcal{G}$.*

Indeed, this readily follows from part I(b) of Theorem 1.12, with a very small constant c (around 3).

Conjecture (1.4) and Corollary 1.16 suggest a distinctive difference between mixing times for S_n as opposed to classical groups $Cl_n(q)$.

Our final result essentially determines the mixing time $T(G, y)$ in terms of the support of y as follows (recall the notation $f_1(x) \lesssim_x f_2(x)$ from Remark 1.2).

Theorem 1.17. *Let $\mathcal{G}$ be a simple, simply connected, classical algebraic group of rank $r \ge 1$ over $\mathbb{K}$ of good characteristic $p > 0$. Let $F : \mathcal{G} \to \mathcal{G}$ be a Frobenius endomorphism so that $G := \mathcal{G}^F$ is defined over $\mathbb{F}_q$ where q is large enough in comparison to r. Also, define*

$$r' := r'(\mathcal{G}) = \begin{cases} r(2r + 4)/(r + 1), & \mathcal{G} = \mathrm{SL}_{r+1}, \\ 4r + 2, & \mathcal{G} = \mathrm{Sp}_{2r}, \ r \ge 2 \\ 4r - 2, & \mathcal{G} = \mathrm{Spin}_{2r}, \ r \ge 4 \\ 4r, & \mathcal{G} = \mathrm{Spin}_{2r+1}, \ r \ge 3. \end{cases}$$

Let $g \in G$ be any element such that its semisimple part y has centralizer $\mathbf{C}_G(y) = \mathcal{L}^F$, where $\mathcal{L}$ is a proper split Levi subgroup of $\mathcal{G}$. Then we have

$$T(G, g) \le \left\lceil (2 + \frac{2}{h}) r'/\mathrm{supp}(y) \right\rceil.$$

Furthermore, we have

$$\frac{1}{2} r'/\mathrm{supp}(y) \lesssim_{|G|} T(G, y) \le \left\lceil (2 + \frac{2}{h}) r'/\mathrm{supp}(y) \right\rceil.$$

Thus, under the above conditions, the mixing time $T(G, y)$ is essentially $r/\mathrm{supp}(y)$ (up to a small multiplicative constant).

2. Character bounds: Proof of Theorem 1.1

Throughout this section, let $\mathcal{G}$ be a connected reductive algebraic group over a field of characteristic $p > 0$, $F : \mathcal{G} \to \mathcal{G}$ a Frobenius endomorphism, and let $G := \mathcal{G}^F$. We will say $\mathcal{G}^F$ is defined over $\mathbb{F}_q$, if q is the common absolute value of eigenvalues of F acting on $X(\mathcal{T}) \otimes \mathbb{R}$, where $X(\mathcal{T})$ is the character group of an F-stable maximal torus $\mathcal{T}$ of $\mathcal{G}$.

First we prove the following statement concerning Harish-Chandra restriction.

Proposition 2.1. *Suppose that $g \in G$ is such that $\mathbf{C}_G(g) \leq \mathcal{L}^F$, where $\mathcal{L}$ is an F-stable Levi subgroup of an F-stable parabolic subgroup $\mathcal{P} = \mathcal{U}\mathcal{L}$ of $\mathcal{G}$ with unipotent radical $\mathcal{U}$. Let $\ell = 0$ or a prime not dividing $p|g|$, $\mathbb{F}$ an algebraically closed field of characteristic ℓ, and let φ be the Brauer character of some $\mathbb{F}G$-module V. Also, let ψ denote the Brauer character of the $\mathcal{L}^F$-module $\mathbf{C}_V(\mathcal{U}^F)$. Then*

$$\varphi(g) = \psi(g).$$

Proof (a) Write $L := \mathcal{L}^F$, $P := \mathcal{P}^F$, and $U := \mathcal{U}^F$. First we handle the case $\ell = 0$. Consider the map $f : U \to U$ given by $f(u) = g^{-1}ugu^{-1}$. Then, for $u, v \in U$ we have that

$$f(u) = f(v) \Leftrightarrow v^{-1}u \in U \cap \mathbf{C}_G(g) \subseteq U \cap L = 1 \Leftrightarrow u = v.$$

Thus the map f is injective, and so bijective. Hence, when u runs over U, ugu^{-1} runs over the elements of gU, each element once:

$$\{ugu^{-1} \mid u \in U\} = gU.$$

Now we decompose $V = \mathbf{C}_V(U) \oplus [V, U]$ as a P-module (note that $P = \mathbf{N}_G(U)$), and let $\Phi = \mathrm{diag}(\Phi_1, \Phi_2)$ denote the representation of P with respect to some basis respecting this decomposition. In particular, no irreducible constituent of $(\Phi_2)|_U$ is trivial, and so $\sum_{u \in U} \Phi_2(u) = 0$. It follows that

$$\sum_{u \in U} \Phi(ugu^{-1}) = \sum_{u \in U} \Phi(gu) = \Phi(g) \sum_{u \in U} \Phi(u) =$$

$$= \mathrm{diag}(\Phi_1(g) \sum_{u \in U} \Phi_1(u), \Phi_2(g) \sum_{u \in U} \Phi_2(u)) = \mathrm{diag}(|U|\Phi_1(g), 0).$$

Taking the trace of both sides, we obtain $|U|\varphi(g) = |U|\psi(g)$, as stated.

(b) For the modular case $\ell > 0$, let χ° denote the restriction of any complex character χ of G or P to ℓ'-elements. It is well known, see e.g [20, Theorem 15.14], that any Brauer character of G is a $\mathbb{Z}$-combination of χ° with $\chi \in \mathrm{Irr}(G)$. It follows that (in the Grothendieck group of $\mathbb{F}G$-modules) we can write $V = V_1 - V_2$, where V_1 and V_2 are some reductions modulo ℓ of $\mathbb{C}G$-modules W_1 and W_2 affording complex characters χ_1 and χ_2. Since $\ell \neq p$, $\mathbf{C}_V(U) = \mathbf{C}_{V_1}(U) - \mathbf{C}_{V_2}(U)$ in the Grothendieck group of $\mathbb{F}P$-modules. Now $g \in P$, $\varphi(g) = \chi_1(g) - \chi_2(g)$, and the statement follows by applying the results of (a) to W_1 and W_2. ∎

Recall that the complex irreducible characters of $G = \mathcal{G}^F$ can be partitioned into *Harish-Chandra series*, see [5, Chapter 9]. We refer to [5] and [8] for basic facts on *Harish-Chandra restriction* ${}^*R_L^G$ and *Harish-Chandra induction* R_L^G. We will also need the following fact (which is well-known to the experts, but the proof of which is given for the reader's convenience).

Proposition 2.2. *There is a constant $A = A(r)$ depending only on the semisimple rank r of $\mathcal{G}$ with the following property. Suppose that $\chi \in \mathrm{Irr}(G)$ is such that ${}^*R_L^G(\chi) \neq 0$ for $L = \mathcal{L}^F$, where $\mathcal{L}$ is a proper split Levi subgroup of $\mathcal{G}$. Then the total number of irreducible constituents of the L-character ${}^*R_L^G(\chi)$ (with counting multiplicities) is at most A. In fact, if $[\mathcal{G}, \mathcal{G}]$ is simple then one can choose $A = W(r)^2$, where $W(r)$ denotes the largest order of the Weyl group of a simple algebraic group of rank r.*

Proof Since ${}^*R_L^G(\chi) \neq 0$, χ is not cuspidal. By [5, Proposition 9.3.1], we may assume that $\mathcal{L}$ is a standard F-stable Levi subgroup of a standard F-stable parabolic subgroup $\mathcal{P} = \mathcal{U}\mathcal{L}$ of $\mathcal{G}$. Suppose that χ belongs to the Harish-Chandra series labeled by a standard Levi subgroup L_1 and a cuspidal character $\psi \in \mathrm{Irr}(L_1)$. Here, $L_1 = \mathcal{L}_1^F$, where $\mathcal{L}_1$ is a proper split Levi subgroup of $\mathcal{G}$, and χ is an irreducible constituent of $R_{L_1}^G(\psi)$.

Suppose now that η is any irreducible constituent of ${}^*R_L^G(\chi)$, and let η belongs to the Harish-Chandra series labeled by a standard Levi subgroup L_2 (of L) and a cuspidal character $\delta \in \mathrm{Irr}(L_2)$. Then η is an irreducible constituent of $R_{L_2}^L(\delta)$. Then by the adjointness of the Harish-Chandra induction and restriction and their transitivity [8, Proposition 4.7], we have that

$$\begin{aligned}
0 < c_\eta := [{}^*R_L^G(\chi), \eta]_L \quad &= [\chi, R_L^G(\eta)]_G \\
&\leq [\chi, R_L^G(R_{L_2}^L(\delta))]_G \quad = [\chi, R_{L_2}^G(\delta)]_G \leq [R_{L_1}^G(\psi), R_{L_2}^G(\delta)]_G.
\end{aligned}$$

Since $\psi \in \mathrm{Irr}(L_1)$ and $\delta \in \mathrm{Irr}(L_2)$ are cuspidal, it follows by [5, Proposition 9.1.5] that the pair (L_1, ψ) is G-conjugate to the pair (L_2, δ) and $R_{L_1}^G(\psi) = R_{L_2}^G(\delta)$. Hence, with no loss of generality we may replace (L_1, ψ) by (L_2, δ). Furthermore, by [5, Proposition 9.2.4], $[R_{L_1}^G(\psi), R_{L_1}^G(\psi)]_G$ can be bounded by the order of the Weyl group $W(\mathcal{G})$ of $\mathcal{G}$ and so in terms of the semisimple rank r as well. Thus we can bound c_η in terms of r. The same is true for $[R_{L_1}^L(\psi), R_{L_1}^L(\psi)]_L$, and so for the number of possibilities for η. In particular, if $[\mathcal{G}, \mathcal{G}]$ is simple, then $|W(\mathcal{L})| \leq |W(\mathcal{G})| \leq W(r)$ and so we can choose $A(r) = W(r)^2$. ∎

From now on we assume that p is a good prime for $\mathcal{G}$ (and $\mathbb{K} = \overline{\mathbb{K}}$ is a field of characteristic p). Then a theory of *generalized Gelfand-Graev representations* (GGGRs) was developed by Kawanaka [22]: for each unipotent element $u \in G = \mathcal{G}^F$ one can associate a GGGR with character Γ_u (which depends only the conjugacy class of u in G).

Suppose now that $\mathcal{O} = u^{\mathcal{G}}$ is an F-stable unipotent conjugacy class in $\mathcal{G}$. By the Lang-Steinberg theorem, since $\mathcal{G}$ is connected we may assume that $u \in G$. Then $\mathcal{O}$ is called a *unipotent support* for a given $\rho \in \mathrm{Irr}(G)$ if

(i) $\sum_{g \in \mathcal{O}^F} \rho(g) \neq 0$;
(ii) If $\mathcal{O}'$ is any F-stable unipotent class of $\mathcal{G}$ with $\sum_{g \in \mathcal{O}'^F} \rho(g) \neq 0$, then $\dim \mathcal{O}' \leq \dim \mathcal{O}$.

As p is a good prime for $\mathcal{G}$, each $\rho \in \mathrm{Irr}(\mathcal{G}^F)$ has a unique unipotent support $\mathcal{O}_\rho$ [11, Theorem 1.4].

Next, $\mathcal{O} \cap G$ is a disjoint union $\cup_{i=1}^r u_i^G$ of, say, r conjugacy classes in G. If $A(x) = \mathbf{C}_{\mathcal{G}}(x)/\mathbf{C}_{\mathcal{G}}(x)^\circ$ is the component group of the centralizer of $x \in \mathcal{G}$, then one defines

$$\tilde{\Gamma}_u := \sum_{i=1}^r [A(u_i) : A(u_i)^F] \Gamma_{u_i}.$$

Then $\mathcal{O}$ is called a *wave front set* for a given $\rho \in \mathrm{Irr}(G)$ if

(i) $[\tilde{\Gamma}_u, \rho]_G \neq 0$;
(ii) If $\mathcal{O}' = v^{\mathcal{G}}$ is a unipotent class of $\mathcal{G}$ with $v \in \mathcal{G}^F$ such that $[\tilde{\Gamma}_v, \chi]_G \neq 0$, then $\dim \mathcal{O}' \leq \dim \mathcal{O}$.

Work of Lusztig [41] and subsequently [50, Theorem 14.10] show that each $\rho \in \mathrm{Irr}(G)$ has a unique wave front set $\mathcal{O}_\rho^*$. Moreover, if $\mathbf{Z}(\mathcal{G})$ is connected, then $\mathcal{O}_\rho^*$ is the unipotent class denoted by $\xi(\rho)$ in [41, (13.4.3)], and, if G is defined over $\mathbb{F}_q$, then as a polynomial in q with rational coefficients, the degree of ρ is

$$\rho(1) = \frac{1}{n_\rho} q^{(\dim \mathcal{O}_\rho^*)/2} + \text{ lower powers of } q, \tag{2.1}$$

for some positive integer n_ρ dividing $|A(u)|$ if $u \in \mathcal{O}_\rho$. Furthermore, if D_G denotes the *Alvis-Curtis duality* (cf. [8, Chapter 8]), and $\rho^* = \pm D_G(\rho) \in \mathrm{Irr}(G)$ for $\rho \in \mathrm{Irr}(G)$, then

$$\mathcal{O}_{\rho^*} = \mathcal{O}_\rho^*, \tag{2.2}$$

(see e.g. [50, §1.5]).

The next two lemmas are well known to the experts. In particular, they have similar conclusions and proofs to Theorems 4.1(ii) and 1.7 of [37]. However, for application to bounding the function $f(r)$ in Theorem 1.1 (see Proposition 2.7), we need the extra detail in the lemmas concerning polynomials being products of cyclotomic polynomials, which is not made explicit in [37]. We omit their proofs.

Lemma 2.3. *There is a constant $N = N(r)$ depending only on r and a collection of N monic polynomials, each being a product of cyclotomic polynomials, such that the following statement holds. If $\mathcal{G}$ is a connected reductive group of semisimple rank $\leq r$ in characteristic p, $\mathcal{G}^F$ is defined over $\mathbb{F}_q$, and $s \in \mathcal{G}^F$ is semisimple, then*

$$[\mathcal{G}^F : (\mathbf{C}_\mathcal{G}(s)^\circ)^F]_{p'} = f(q),$$

where f is one of the chosen polynomials.

In what follows, with a slight abuse of language, we also view t as a cyclotomic polynomial in variable t.

Lemma 2.4. *There are constants $B_1 = B_1(r)$ and $B_2 = B_2(r)$ depending only on r, and B_2 monic polynomials, each being a product of cyclotomic polynomials in one variable t, such that the following statement holds for any connected reductive algebraic group $\mathcal{G}$ of semisimple rank $\leq r$ with connected center in good characteristic. When $\mathcal{G}^F$ is defined over $\mathbb{F}_q$ and $\chi \in \mathrm{Irr}(\mathcal{G}^F)$, then*

$$\chi(1) = (1/n_\chi)\mathrm{Deg}_\chi^*(q),$$

where Deg_χ^* is one of the chosen monic polynomials, $n_\chi \in \mathbb{N}$, $1 \leq n_\chi \leq B_1$. In fact, if $[\mathcal{G}, \mathcal{G}]$ is simple, then one can take B_1 to be the largest order of the component group $\mathbf{C}_\mathcal{H}(u)/\mathbf{C}_\mathcal{H}(u)^\circ$, where $\mathcal{H}$ is any simple algebraic group of rank r and $u \in \mathcal{H}$ any unipotent element.

Recall that the set of unipotent classes in $\mathcal{G}$ admit the partial order $\leq$, where $u^\mathcal{G} \leq v^\mathcal{G}$ if and only if $u^\mathcal{G} \subseteq \overline{v^\mathcal{G}}$.

Proposition 2.5. *Let p be a good prime for $\mathcal{G}$, $G = \mathcal{G}^F$, and let $u \in G$ be a unipotent element. Then the following statements hold.*

(i) *$D_G(\Gamma_u)$ is unipotently supported, i.e. is zero on all non-unipotent elements of G.*
(ii) *Suppose that $D_G(\Gamma_u)(v) \neq 0$ for some unipotent element $v \in G$. If $\mathbf{Z}(\mathcal{G})$ is disconnected, assume in addition that q is large enough compared to the semisimple rank of $\mathcal{G}$. Then $u^\mathcal{G} \leq v^\mathcal{G}$.*

Proof (i) is well known, and (ii) is [7, Scholium 2.3]. (Even though [7] assumes p is large enough, in fact the proof of [7, Scholium 2.3] needs only that p is a good prime. As pointed out to the authors by J. Michel and J. Taylor, the proof in [7] relies on the validity

of the results in [42], which were shown to hold under the indicated hypotheses by Shoji [47], cf. [48, Theorem 4.2].)

Proposition 2.6. *Let* $\mathcal{G}/\mathbf{Z}(\mathcal{G})$ *be simple,* $\mathbf{Z}(\mathcal{G})$ *be connected,* p *be a good prime for* $\mathcal{G}$, *and let* $G = \mathcal{G}^F$. *Suppose that* $\chi \in \mathrm{Irr}(G)$ *is such that* ${}^*R_L^G(\chi) \neq 0$ *for* $L = \mathcal{L}^F$, *where* $\mathcal{L}$ *is a proper split Levi subgroup of* $\mathcal{G}$, *and let* $\eta \in \mathrm{Irr}(L)$ *be an irreducible constituent of* ${}^*R_L^G(\chi)$. *Let* $\mathcal{O}_\chi^* = v^{\mathcal{G}}$ *and* $\mathcal{O}_\eta^* = u^{\mathcal{L}}$. *Then* $\dim u^{\mathcal{G}} \leq \dim v^{\mathcal{G}}$.

Proof (i) To distinguish between GGGRs for G and L, we will add the relevant superscript to their notation, e.g. Γ_u^L is the GGGR of L labeled by u. First we show that if $R_L^G(D_L\Gamma_u^L)(w) \neq 0$ for $w \in G$, then w is unipotent and $u^{\mathcal{G}} \leq w^{\mathcal{G}}$. Indeed, by Proposition 2.5(i), the generalized character $D_L\Gamma_u^L$ is unipotently supported, whence $R_L^G(D_L\Gamma_u^L)$ is also unipotently supported by [8, Proposition 12.2]. In particular, w is unipotent. Recall that $\mathcal{L}$ is a Levi subgroup of an F-stable parabolic subgroup $\mathcal{P}$ with unipotent radical $\mathcal{U}$. The condition on w now implies that some G-conjugate of w is $w' = xy$ where $x \in \mathcal{U}^F$, $y \in L$, and $D_L\Gamma_u^L(y) \neq 0$. By Proposition 2.5 applied to $D_L\Gamma_u^L$, y is unipotent and $u^{\mathcal{L}} \leq y^{\mathcal{L}}$. It then follows by [12, Lemma 5.2] (which is true for any connected reductive group $\mathcal{G}$) that

$$u^{\mathcal{G}} \leq y^{\mathcal{G}} \leq (xy)^{\mathcal{G}} = w^{\mathcal{G}},$$

as stated.

(ii) By the assumption, we may assume that $u \in L$ and η is an irreducible constituent of the GGGR Γ_u^L. It follows that

$$0 < [{}^*R_L^G(\chi), \eta]_L \leq [{}^*R_L^G(\chi), \Gamma_u^L]_L = [\chi, R_L^G(\Gamma_u^L)]_G =$$
$$= [D_G(\chi), D_G(R_L^G(\Gamma_u^L))]_G = [D_G(\chi), R_L^G(D_L\Gamma_u^L)]_G.$$

Here we use the self-adjointness of D_G and the intertwining property of D_G with R_L^G (see [8, Proposition 8.10, Theorem 8.11]). In particular, there must exist some $w \in G$ such that

$$D_G(\chi)(w) \neq 0, \ R_L^G(D_L\Gamma_u^L)(w) \neq 0.$$

Let $\chi^* = \pm D_G(\chi) \in \mathrm{Irr}(G)$ so that $\mathcal{O}_{\chi^*} = \mathcal{O}_\chi^* = v^{\mathcal{G}}$ with $v \in G$. By (i), the condition $R_L^G(D_L\Gamma_u^L)(w) \neq 0$ implies that w is unipotent and

$$u^{\mathcal{G}} \leq w^{\mathcal{G}}.$$

Now we can apply [1, Theorem 8.1] (which uses only the assumption that $\mathbf{Z}(\mathcal{G})$ is connected and $\mathcal{G}/\mathbf{Z}(\mathcal{G})$ is simple; cf. also [50, Corollary 13.6]) to obtain from $\chi^*(w) \neq 0$ that

$$\dim w^{\mathcal{G}} \leq \dim v^{\mathcal{G}}.$$

It follows that

$$\dim u^{\mathcal{G}} \leq \dim \overline{w^{\mathcal{G}}} = \dim w^{\mathcal{G}} \leq \dim v^{\mathcal{G}},$$

as desired. ∎

Proof of Theorem 1.1.

(i) Denoting $\rho = {}^*R_L^G(\chi)$, we have by Proposition 2.1 that $|\chi(g)| = |\rho(g)| \leq \rho(1)$. Hence, it suffices to bound $\rho(1)$ in terms of $\chi(1)$. Fix the semisimple rank r of $\mathcal{G}$. First we handle the case where $\mathbf{Z}(\mathcal{G})$ is connected. Note that $\mathcal{H} := \mathcal{G}/\mathbf{Z}(\mathcal{G})$ is simple (of rank r) as $[\mathcal{G}, \mathcal{G}]$ is simple. Consider any irreducible constituent η of ρ and let $\mathcal{O}_\eta^* = u^{\mathcal{L}}$ for some $u \in L$ and $\mathcal{O}_\chi^* = v^{\mathcal{G}}$ for some $v \in G$. By Proposition 2.6 we have $\dim u^{\mathcal{G}} \leq \dim v^{\mathcal{G}}$. On the other hand, $\dim u^{\mathcal{L}} \leq \alpha(\dim u^{\mathcal{G}})$ by the choice of α, and so

$$\dim u^{\mathcal{L}} \leq \alpha(\dim v^{\mathcal{G}}). \tag{2.3}$$

Now (2.1) and Lemma 2.4 imply that

$$\eta(1) \leq (q+1)^{(\dim u^{\mathcal{L}})/2}, \ B_1 \chi(1) \geq (q-1)^{(\dim v^{\mathcal{G}})/2}.$$

Let $D = D(r)$ denote the largest dimension of unipotent classes in simple algebraic groups of rank r. Using (2.3) and noting that $\dim v^{\mathcal{G}} = \dim v^{\mathcal{H}} \leq D(r)$, we then get

$$\eta(1) \leq \left(\frac{q+1}{q-1}\right)^{\alpha D/2} \cdot B_1^{\alpha} \chi(1)^{\alpha}.$$

Setting $C := 3^{D/2}$ and applying Proposition 2.2, we now obtain

$$\rho(1) \leq A(\max_{\eta} \eta(1)) \leq AB_1 C \chi(1)^{\alpha},$$

and we are done in this case.

(ii) Next we handle the general case, where $\mathbf{Z}(\mathcal{G})$ may be disconnected. Consider a regular embedding of $\mathcal{G}$ into $\tilde{\mathcal{G}}$ with connected center and with compatible Frobenius map $F : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$, and set $\tilde{G} := \tilde{\mathcal{G}}^F$, $\tilde{\mathcal{Z}} := \mathbf{Z}(\tilde{\mathcal{G}})$. As $\tilde{\mathcal{G}} = \mathcal{Z}[\mathcal{G}, \mathcal{G}]$, $\tilde{\mathcal{G}}$ and $\mathcal{G}$ have the same semisimple rank. Also, if $\mathcal{L}$ is a Levi subgroup of an F-stable parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$, then we can embed $\mathcal{P}$ in the F-stable parabolic subgroup $\tilde{\mathcal{P}} = \mathcal{U}\tilde{\mathcal{L}} = \mathbf{N}_{\tilde{\mathcal{G}}}(\mathcal{U})$, with the same unipotent radical $\mathcal{U}$ as of $\mathcal{P}$ and with $\tilde{\mathcal{L}} = \mathcal{Z}\mathcal{L}$. Now, set $\tilde{L} := \tilde{\mathcal{L}}^F$ and note that

$$\tilde{G} = G\tilde{L}. \tag{2.4}$$

Consider any $\chi \in \mathrm{Irr}(G)$ and some $\tilde{\chi} \in \mathrm{Irr}(\tilde{G})$ lying above χ, and denote

$$\rho := {}^*R_L^G(\chi), \ \tilde{\rho} := {}^*R_{\tilde{L}}^{\tilde{G}}(\tilde{\chi}).$$

Note that $\tilde{\mathcal{P}}^F = U\tilde{L}$, and by (2.4) we can choose a set of representatives of G-cosets in $\tilde{G}$ that is contained in $\tilde{L}$. Hence, by Clifford's theorem we can write

$$\tilde{\chi}|_G = \sum_{i=1}^{t} \chi^{x_i},$$

where $1 = x_1, \ldots, x_t \in \tilde{L}$. As $\tilde{L}$ normalizes U, we see that the Harish-Chandra restrictions ρ_i of χ^{x_i} to the Levi subgroup L all have the same dimension, equal to $[\chi|_U, 1_U]_U$. Thus

$$\rho(1) = \tilde{\rho}(1)/t, \ \chi(1) = \tilde{\chi}(1)/t.$$

Now, any unipotent element $v \in \tilde{L}$ is contained in $\tilde{L} \cap \mathcal{G} = L$, and $v^{\tilde{\mathcal{G}}} = v^{\mathcal{G}}$ and similarly $v^{\tilde{\mathcal{L}}} = v^{\mathcal{Z}\mathcal{L}} = v^{\mathcal{L}}$. Thus the constants α for L and for $\tilde{L}$ as defined in Theorem 1.1 are the same. Applying Lemma 2.1 to χ and the result of (i) to $\tilde{\chi}$, we now have

$$|\chi(g)| = |\rho(g)| \leq \rho(1) = \frac{1}{t}\tilde{\rho}(1) \leq \frac{1}{t} f(r)\tilde{\chi}(1)^{\alpha} \leq f(r)\chi(1)^{\alpha}.$$

This completes the proof of Theorem 1.1. $\blacksquare$

Remark In the case of $\mathrm{GL}_n(q)$, it is possible to give an alternate proof of Theorem 1.1 which does not use recent results on unipotent supports and wave front sets; we do not give this here, but a sketch can be found in the last section of [29].

The next result provides a bound for the function f in Theorem 1.1.

Proposition 2.7. *Under the assumptions of Theorem 1.1, suppose that $q \geq q_0 \geq 2$. Then $f(r)$ can be chosen to be*

$$W(r)^2 \cdot B(r) \cdot \left(\frac{q_0 + 1}{q_0 - 1}\right)^{(d(r)-r)/2},$$

where $W(r)$ is the largest order of the Weyl group of $\mathcal{H}$, $B(r)$ is the largest order of $A(u)$ for unipotent elements $u \in \mathcal{H}$, and $d(r)$ is the largest dimension of $\mathcal{H}$, when $\mathcal{H}$ runs over simple algebraic groups of rank r. In particular, if $r \geq 9$ and $q \geq r^2 + 1$, one can take

$$f(r) = 2^{2r+\sqrt{2r}+3}(r!)^2.$$

Proof By the proof of Theorem 1.1 we may choose $f(r) = AB_1C_1$, with $C_1 = (\frac{q_0+1}{q_0-1})^{(d(r)-r)/2}$ (because $D = d(r) - r$). Next, $A \leq W(r)^2$ by Proposition 2.2 and $B_1 \leq B(r)$ by Lemma 2.4. Now assume that $r \geq 9$ and $q_0 \geq r^2 + 1$. Then $W(r) = 2^r \cdot r!$, $d(r) = 2r^2 + r$ and so

$$\left(\frac{q_0 + 1}{q_0 - 1}\right)^{(d(r)-r)/2} \leq \left(1 + \frac{2}{r^2}\right)^{r^2}.$$

It remains to bound $B(r)$. If $\mathcal{H} = \mathrm{Spin}_n(\mathbb{K})$ (with $n = 2r$ or $2r + 1$) and $u = \sum_i J_i^{r_i}$ is a unipotent element in $\mathcal{H}$ with r_i Jordan blocks of size $i \geq 1$, then, according to [33, §3.3.5], $|A(u)| \leq \max(2, 2^k)$, where k is the number of odd i with $r_i > 0$. Note that

$$2r + 1 \geq \sum_{j=1}^{k}(2j - 1) = k^2,$$

and so $|A(u)| \leq 2^{\sqrt{2r+1}}$. Other simple groups of rank r can be analyzed similarly using [33, Theorem 3.1] and yield smaller bound on $|A(u)|$. Hence we can take $B(r) = 2^{\sqrt{2r+1}}$ and complete the proof by observing that

$$\left(1 + \frac{2}{r^2}\right)^{r^2} \cdot 2^{\sqrt{2r+1}} < 2^{\sqrt{2r}+3}.$$

∎

We conclude the section with some examples illustrating the sharpness of the α-bound in Theorem 1.1.

Example 2.8. (i) Let $G := \mathrm{GL}_n(q)$ with $q > 2$, and let $g = \mathrm{diag}(\epsilon, I_{n-1}) \in G$ for some $1 \neq \epsilon \in \mathbb{F}_q^\times$. Then $L := \mathbf{C}_G(g) = \mathrm{GL}_1(q) \times \mathrm{GL}_{n-1}(q)$ is a proper split Levi subgroup of G.

Let $\chi = \rho_n$ denote the unipotent character of $\mathrm{GL}_n(q)$ labeled by the partition $(n - 1, 1)$. Then $\rho_n(1) = (q^n - q)/(q - 1)$. A computation inside the Weyl group of G (using the Comparison Theorem [18, Theorem 5.9]) shows that

$$^*R_L^G(\rho_n) = 1_{\mathrm{GL}_1(q)} \otimes (\rho_{n-1} + 1_{\mathrm{GL}_{n-1}(q)}).$$

Now Proposition 2.1 implies that

$$|\chi(g)| = \rho_{n-1}(1) + 1 = \frac{q^{n-1} - 1}{q - 1} \approx \chi(1)^{\frac{n-2}{n-1}}$$

if q is large enough. For this Levi subgroup L, the value of α in Theorem 1.1 is precisely $\frac{n-2}{n-1}$ (see Proposition 4.3), so the α-bound is perfectly sharp in this example.

(ii) The Steinberg character St of a group $G = \mathcal{G}^F$ as in Theorem 1.1 provides a good source of examples, since its values are easily calculated (see [5, 6.4.7]): for a semisimple element $g \in G$,

$$|\mathsf{St}(g)| = |\mathbf{C}_G(g)|_p,$$

(where p is the underlying characteristic).

As a first example, let $G = \mathrm{GL}_n(q)$ and let $g = \mathrm{diag}(\epsilon, I_{n-1})$ as in the previous example. Then

$$\mathsf{St}(g) = |\mathrm{GL}_{n-1}(q)|_p = q^{\frac{1}{2}(n-1)(n-2)} = \mathsf{St}(1)^{\frac{n-2}{n}},$$

while $\alpha = \frac{n-2}{n-1}$ for the Levi subgroup $\mathbf{C}_G(g)$, as observed above.

As another example, let $G = \mathrm{GL}_n(q)$ and suppose $n = mk$, where $2 \leq m \leq q-1$ and $k > 1$. Let $\lambda_1, \ldots, \lambda_m$ be distinct elements of $\mathbb{F}_q^\times$, and define

$$g = \mathrm{diag}(\lambda_1 I_k, \cdots, \lambda_m I_k) \in G.$$

Let $L = \mathbf{C}_G(g) = \mathrm{GL}_k(q)^m$. By Corollary 1.11, $\alpha(L) = \frac{1}{m}$. On the other hand,

$$\mathsf{St}(g) = q^{\frac{1}{2}mk(k-1)} = \mathsf{St}(1)^{\frac{k-1}{mk-1}},$$

and the exponent $\frac{k-1}{mk-1}$ is close to $\alpha = \frac{1}{m}$ for k large and m fixed.

Similar examples showing the sharpness of Theorem 1.1 for the Steinberg character of other classical groups can be constructed using [32, Lemma 3.4].

(iii) Fix $m \geq 2$ and consider $G = \mathrm{GL}_{2m}(q)$ with q large enough (compared to m). Again let $\lambda_1, \ldots, \lambda_m$ be distinct elements of $\mathbb{F}_q^\times$, and define

$$g = \mathrm{diag}(\lambda_1 I_2, \cdots, \lambda_m I_2) \in G.$$

Then $L = \mathbf{C}_G(g) = \mathrm{GL}_2(q)^m$, and $\alpha = \alpha(L) = 1/m$ as mentioned above. Consider the unipotent characters $\chi^{(2m-j,j)}$ of G labeled by the partition $(2m-j, j)$, $0 \leq j \leq m$. Then $\sum_{i=0}^{j} \chi^{(2m-i,i)}$ is the permutation character of G acting on the set of j-dimensional subspaces of the natural module $V = \mathbb{F}_q^{2m}$. Note that g fixes $mq^{m-1}(1 + O(q^{-1}))$ $(m-1)$-dimensional subspaces of V, and $(q+1)^m(1+O(q^{-1}))$ m-dimensional subspaces of V. It follows that for $\chi := \chi^{(m,m)}$ we have

$$\chi(g) = q^m(1 + O(q^{-1})),$$

whereas $\chi(1) = q^{m^2}(1 + O(1/q))$. Thus $\chi(g) \approx \chi(1)^\alpha$.

(iv) More generally, fix $k, m \geq 2$ and consider $G = \mathrm{GL}_{mk}(q)$ with q large enough (compared to $\max(m,k)$). Again let $\lambda_1, \ldots, \lambda_m$ be distinct elements of $\mathbb{F}_q^\times$, and define

$$g = \mathrm{diag}(\lambda_1 I_k, \cdots, \lambda_m I_k) \in G.$$

Then $L = \mathbf{C}_G(g) = \mathrm{GL}_k(q)^m$, and $\alpha = \alpha(L) = 1/m$ as mentioned above. Consider the unipotent character $\chi := \chi^\mu$ of G labeled by the partition $\mu := (m^k) \vdash mk$. Observe that ${}^*R_L^G(\chi)$ contains the Steinberg character St_L of L. (Indeed, by [16, Proposition 5.3], the Alvis-Curtis duality functor D_G sends χ to $\pm\chi^\nu$, where $\nu = \mu' = (k^m) \vdash mk$, whereas $D_L(1_L) = \mathsf{St}_L$. Now, by [8, Corollary 8.13] we have

$$
\begin{aligned}
[{}^*R_L^G(\chi), \mathsf{St}_L]_L &= [{}^*R_L^G(\pm D_G(\chi^\nu)), \mathsf{St}_L]_L &= \pm[D_L({}^*R_L^G(\chi^\nu)), D_L(1_L)]_L \\
&= \pm[{}^*R_L^G(\chi^\nu), 1_L]_L &= \pm[\chi^\nu, R_L^G(1_L)]_G.
\end{aligned}
$$

But note that L is a Levi subgroup of a rational parabolic subgroup of type ν of $\mathrm{GL}_{mk}(\mathbb{K})$, whence χ^ν is an irreducible constituent of $R_L^G(1_L)$, and the claim follows.) Since χ is a unipotent character and the Harish-Chandra restriction preserves rational series, every irreducible constituent of ${}^*R_L^G(\chi)$ is a unipotent character of L and so contains $g \in \mathbf{Z}(L)$ in its kernel. It now follows from Proposition 2.1 that

$$\chi(g) = {}^*R_L^G(\chi)(g) = {}^*R_L^G(\chi)(1) \geq \mathsf{St}_L(1) = q^{mk(k-1)/2}.$$

On the other hand, the degree formula [5, §13.8] implies that

$$\chi(1) = q^{m^2 k(k-1)/2}(1 + O(q^{-1})),$$

and we again obtain that $\chi(g) \gtrsim \chi(1)^\alpha$.

(v) As far as the exceptional groups of Lie type are concerned, it is again interesting to use the Steinberg character to test the sharpness of Theorem 1.1. For example, let $G = E_8(q)$, and suppose $g \in G$ is a semisimple element with centralizer a Levi subgroup of type E_7. Then

$$\mathsf{St}(g) = |E_7(q)|_p = q^{63} = \mathsf{St}(1)^\beta,$$

where $\beta = \frac{21}{40}$, while the α-value of this Levi is $\frac{17}{29}$, by Theorem 1.7. One can calculate such β-values for all the Levi subgroups in Table 1 of Theorem 1.7; it is never the case that $\beta = \alpha$, but in some cases the values of β and α are quite close.

(vi) We offer one more example with $G = \mathcal{G}^F = \mathrm{SL}_n(q)$, with $q \geq n + 2$, and

$$g = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \in G,$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{F}_q^\times$ are pairwise distinct. Then $T = \mathbf{C}_G(g)$ is a maximally split maximal torus. Let $\mu \vdash n$ be such that the irreducible character S^μ of the Weyl group $W(\mathcal{G}) \cong \mathsf{S}_n$ labeled by μ has the largest possible degree, and let $\chi := \chi^\mu$ denote the unipotent character of G labeled by μ. As in (iv), every irreducible constituent of ${}^*R_T^G(\chi)$ contains g in its kernel. A computation in $W(\mathcal{G})$ and Proposition 2.1 show that

$$\chi(g) = {}^*R_T^G(\chi)(g) = {}^*R_T^G(\chi)(1) = S^\mu(1),$$

whereas $\alpha(T) = 0$. Thus for the function f in Theorem 1.1 we have

$$f(n-1) \geq S^\mu(1) \geq e^{-1.283\sqrt{n}}\sqrt{n!},$$

with the latter following from the main result of [38], [53].

3. General and special linear groups

In this section we prove Theorems 1.3, 1.4 and 1.5. Along the way we establish character bounds for unipotent elements of $\mathrm{GL}_n(q)$ (see Theorem 3.3), and also for elements $g = su = us$ with extension-field centralizers for their semisimple parts s (Theorem 3.2).

3.1. **Proof of Theorem 1.4.** We will keep the notation of §2.

(i) First we consider the case $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$. In this case, the centralizer of any element in $\mathcal{G}$ is connected and one can check (e.g. using [15]) that $n_\rho = 1$ in (2.1). Let φ be an irreducible ℓ-Brauer character of $G = \mathcal{G}^F = \mathrm{GL}_n(q)$ and $g \in G$ as in Theorem 1.4. By Proposition 2.1, $|\varphi(g)| = |\psi(g)|$ for $\psi := {}^*R_L^G(\varphi)$. According to [4, Theorem B], one can label complex and ℓ-Brauer characters of G and find a complex character $\chi \in \mathrm{Irr}(G)$ with the same label as of φ such that both the generic degree of χ and the lower bound (given in [4, Theorem B]) are monic polynomials in q of same degree say N_χ. Using (2.1) and the equality $n_\chi = 1$, we have

$$N_\chi := (\dim \mathcal{O}_\chi^*)/2.$$

As $\chi(1)$ is a product of cyclotomic polynomials in q, we also have that

$$\chi(1) \leq (q+1)^{N_\chi}.$$

Furthermore, one can easily check that the lower bound in [4, Theorem B] satisfies

$$\varphi(1) \geq (q-1)^{N_\chi}.$$

As $N_\chi \leq \dim \mathcal{G} = n^2$, there is a constant $D = D(n) \leq 3^{n^2}$ such that

$$\frac{\chi(1)}{\varphi(1)} \leq D. \tag{3.1}$$

Since χ and φ have the same labeling, φ is a constituent of the restriction χ° of χ to ℓ'-elements of G. Let $\mathcal{P} = \mathcal{U}\mathcal{L}$ be an F-stable parabolic subgroup of $\mathcal{G}$ with Levi subgroup $\mathcal{L}$ and unipotent radical $\mathcal{U}$. As $U := \mathcal{U}^F$ is an ℓ'-group, we also have that

$$\psi(1) = [\varphi|_U, 1_U]_U \leq [\chi|_U, 1_U]_U = \rho(1),$$

where $\rho := {}^*R_L^G(\chi)$. The proof of Theorem 1.1 yields a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\rho(1) \leq f(n)\chi(1)^\alpha.$$

Choosing h such that $h(n) \geq f(n)D$ and applying (3.1), we obtain

$$|\varphi(g)| = |\psi(g)| \leq \psi(1) \leq \rho(1) \leq h(n)\varphi(1)^{\alpha},$$

as desired.

(ii) Embed $\mathrm{SL}_n(q)$ in $\mathrm{GL}_n(q)$ and argue as in part (ii) of the proof of Theorem 1.1. ∎

3.2. **Proof of Theorem 1.3.** (i) First we consider the case $G = \mathrm{GL}_n(q)$. If $\mathcal{L}$ is a torus, we can choose a regular semisimple element $g \in L$ and take $\chi = 1_G$. Assume now that $\mathcal{L}$ is not a torus, and choose $u \in L_{\mathrm{unip}}$ such that

$$\alpha(L) = (\dim u^{\mathcal{L}})/(\dim u^{\mathcal{G}}). \tag{3.2}$$

We may assume that

$$L = \mathrm{GL}_{n_1}(q) \times \mathrm{GL}_{n_2}(q) \times \cdots \times \mathrm{GL}_{n_r}(q), \ \ u = \mathrm{diag}(u_1, u_2, \ldots, u_r) \tag{3.3}$$

where $n_1 \geq n_2 \geq \ldots \geq n_r \geq 1$; furthermore, $u_i \in \mathrm{GL}_{n_i}(q)$ is a unipotent element, the sizes of whose Jordan blocks form a partition $\lambda_i \vdash n_i$. Let $\mu_i \vdash n_i$ be the partition conjugate to λ_i and let χ^{μ_i} be the unipotent character of $\mathrm{GL}_{n_i}(q)$ labeled by μ_i. Now Green's formula for the degree of χ^{μ_i} (see the discussion before [4, Theorem A]) implies that $\chi^{\mu_i}(1)$ is a monic polynomial in q of degree $(1/2) \dim u_i^{\mathcal{L}}$. Hence, if we choose C_n large enough, then using Lemma 2.3 we see that

$$\gamma(1) \geq (1/2)q^{(1/2)\dim u^{\mathcal{L}}}, \tag{3.4}$$

for $\gamma := \chi^{\mu_1} \otimes \chi^{\mu_2} \otimes \ldots \otimes \chi^{\mu_r} \in \mathrm{Irr}(L)$ whenever $q \geq C_n$.

Next, let $\mu := \mu_1 + \mu_2 + \ldots + \mu_r$, where we have added zero parts to μ_i so that $\mu_1, \ldots, \mu_r$ have the same number of parts, and then take the ith part of μ to be the sum of all the ith parts of $\mu_1, \ldots, \mu_r$. Again using Green's formula, we then see that the unipotent character $\chi = \chi^{\mu}$ of G labeled by μ is a monic polynomial in q of degree $(1/2) \dim^{u^{\mathcal{G}}}$, whence

$$\chi(1) \leq 2q^{(1/2)\dim u^{\mathcal{G}}} \tag{3.5}$$

if $q \geq C_n$.

For any $\nu \vdash m$, let S^{ν} denote the irreducible character of S_m labeled by ν. An application of the Littlewood-Richardson formula [21, 2.8.14] shows that the restriction of S^{μ} to $\mathsf{S}_{n_1} \times \mathsf{S}_{n_2} \times \ldots \times \mathsf{S}_{n_r}$ contains $S^{\mu_1} \otimes S^{\mu_2} \otimes \ldots \otimes S^{\mu_r}$. This computation in the Weyl groups of $\mathcal{G}$ and $\mathcal{L}$ implies that γ is an irreducible constituent of ${}^*R_L^G(\chi)$. Now, if $q \geq n+1$, then we can choose a semisimple element $g \in \mathbf{Z}(L)$ such that $\mathbf{C}_G(g) = L$. (Indeed, with L given in (3.3) we can fix $m \leq n < q$ distinct elements $\lambda_1, \ldots, \lambda_m \in \mathbb{F}_q^{\times}$ and choose $g \in G$ that has λ_i as an eigenvalue with multiplicity n_i.)

As in Example 2.8(iv), we have that every irreducible constituent of ${}^*R_L^G(\chi)$ has g in its kernel, and so by Proposition 2.1,

$$\chi(g) = {}^*R_L^G(\chi)(g) = {}^*R_L^G(\chi)(1) \geq \gamma(1).$$

Hence the statement follows from the choice (3.2) of u and the bounds (3.4), (3.5).

(ii) To handle the case of $\mathrm{SL}_n(q)$, first we recall that any unipotent character of G remains irreducible over $\mathrm{SL}_n(q)$. Furthermore, as mentioned in the proof of Theorem 1.1, for any proper split Levi subgroup $\mathcal{L}$ of $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$ we have $\mathcal{G} = [\mathcal{G}, \mathcal{G}]\mathcal{Z}$ and $\mathcal{L} = (\mathcal{L} \cap [\mathcal{G}, \mathcal{G}])\mathcal{Z}$, where $\mathcal{Z} := \mathbf{Z}(\mathcal{G})$. It follows that

$$\dim u^{\mathcal{G}} = \dim u^{[\mathcal{G},\mathcal{G}]}, \ \ \dim u^{\mathcal{L}} = \dim u^{\mathcal{L} \cap [\mathcal{G},\mathcal{G}]}$$

for any element $u \in \mathcal{L}_{\mathrm{unip}} = (\mathcal{L} \cap [\mathcal{G}, \mathcal{G}])_{\mathrm{unip}}$. Finally, the condition $(n!)^n$ divides $q - 1$ implies by the next Lemma 3.1 that, for any non-toral Levi subgroup L given in (3.3) we can find an element $g \in \mathrm{SL}_n(q)$ with $\mathbf{C}_G(g) = L$. Now the statement for $\mathrm{SL}_n(q)$ follows from (i). ∎

Lemma 3.1. *Let* $1 \leq n_1 \leq n_2 \leq \ldots \leq n_r$ *with* $r, n_r \geq 2$, $n = \sum_{i=1}^{r} n_i$, *and let* q *be a prime power such that* $N := n_r \cdot \prod_{i=1}^{r-1}(n_i + 1)$ *divides* $q - 1$. *Then, for*

$$L := \mathrm{GL}_{n_1}(q) \times \mathrm{GL}_{n_2}(q) \times \cdots \times \mathrm{GL}_{n_r}(q) < G := \mathrm{GL}_n(q),$$

there exists a semisimple element $s \in \mathrm{SL}_n(q)$ *such that* $\mathbf{C}_G(s) = L$.

Proof   Choose $\zeta \in \mathbb{F}_q^{\times}$ of order N, and for any $d | N$ let $\zeta_d := \zeta^{N/d}$. Define

$$h_i = \zeta_{(n_1+1)(n_2+1)\ldots(n_i+1)}^{-1} I_{n_i} \in \mathrm{GL}_{n_i}(q),\ 1 \leq i \leq r-1,\ h_r = \zeta I_{n_r} \in \mathrm{GL}_{n_r}(q).$$

We prove by induction on $1 \leq i \leq r-1$ that $\prod_{j=1}^{i} \det(h_i) = \zeta_{(n_1+1)(n_2+1)\ldots(n_i+1)}$. The induction base $i = 1$ is obvious. For the induction step from $i-1$ to $i \geq 2$, we have

$$\prod_{j=1}^{i} \det(h_i) = \zeta_{(n_1+1)(n_2+1)\ldots(n_{i-1}+1)} \zeta_{(n_1+1)(n_2+1)\ldots(n_i+1)}^{-n_i} = \zeta_{(n_1+1)(n_2+1)\ldots(n_i+1)}.$$

Hence, for $s := \mathrm{diag}(h_1, h_2, \ldots, h_r) \in \mathbf{Z}(L)$ we have

$$\det(s) = \zeta_{(n_1+1)(n_2+1)\ldots(n_{r-1}+1)} \zeta^{-n_r} = 1.$$

The construction of s and the condition $n_r \geq 2$ ensure that $\mathbf{C}_G(s) = L$. ∎

3.3. Elements with extension-field centralizers.

Theorem 3.2. *Let* $G = \mathrm{GL}_n(q)$ *with* $n \geq 2$ *and* $q \geq 8$, *and let* $g = su = us$ *with* $s \in G$ *semisimple and* $u \in G$ *unipotent. Suppose that* $\mathbf{C}_G(s) \cong \mathrm{GL}_{n/k}(q^k)$ *for some* $1 < k \mid n$. *Then*

$$\frac{|\chi(g)|}{\chi(1)} \leq \frac{f(n)}{q^{n/3}}$$

for any $\chi \in \mathrm{Irr}(G)$ *with* $\chi(1) > 1$ *and* $f(n) = (11/7)^n - 13/10$. *In particular, if* $q \geq 227$ *then* $|\chi(g)| \leq \chi(1)^{1-1/2n}$ *for all* $\chi \in \mathrm{Irr}(G)$.

Proof   We proceed by induction on $n \geq 2$. Let $L := \mathrm{SL}_n(q)$ and let $W = \mathbb{F}_q^n$ denote the natural G-module. Since $\chi(1) > 1$, all irreducible constituents of χ_L are non-trivial. In particular, if $n \geq 3$ then $\chi(1) \geq (q^n - q)/(q-1) > q^{n-1}$ by [52, Theorem 1.1].

(i) First we consider the case $k = n$. Since $|\mathbf{C}_G(g)| \leq |\mathbf{C}_G(s)| = q^n - 1$, we have that $|\chi(g)| \leq \sqrt{|\mathbf{C}_G(g)|} < q^{n/2}$; in particular,

$$|\chi(g)|/\chi(1) \leq q^{1-n/2} \leq q^{-n/3} \tag{3.6}$$

if $n \geq 6$, or if $n = 5$ and $\chi(1) \geq q^5$. The condition $\mathbf{C}_G(s) \cong \mathrm{GL}_1(q^n)$ also implies that no eigenvalue of g on W can belong to $\mathbb{F}_q$.

Assume now that $n = 5$ and $\chi(1) < q^5$. By [52, Theorem 3.1], every irreducible constituent of χ_L is one of $q - 1$ Weil characters, of degree $(q^n - 1)/(q-1) - \delta$ with $\delta = 0$ or 1. Since Weil characters of L extend to Weil characters of G, χ is a Weil character. Since no eigenvalue of g on W belongs to $\mathbb{F}_q$, using the well-known character formula for Weil characters of G, see e.g. [51, (1.1)], we now see that $|\chi(g)| \leq q + 1$ and so

$$|\chi(g)|/\chi(1) < (q^2 - 1)/(q^5 - q) < q^{-n/3}.$$

Consider the case $n = 4$. If $\chi(1) \geq (q-1)(q^3-1)/2$, then

$$\frac{|\chi(g)|}{\chi(1)} \leq \frac{(q^4-1)^{1/2}}{(q^3-1)(q-1)/2} < q^{-n/3}.$$

as $q \geq 8$. Assume now that $\chi(1) < (q^3 - 1)(q - 1)/2$. By [52, Theorem 3.1], every irreducible constituent of χ_L is one of $q - 1$ Weil characters, all of which extend to Weil characters of G. Arguing as in the previous case, we see that

$$|\chi(g)|/\chi(1) < (q^2 - 1)/(q^4 - q) < q^{-n/3}.$$

If $n = 3$, then inspecting the character table of G [49] we get

$$\frac{|\chi(g)|}{\chi(1)} \leq \max\left(\frac{1}{q(q + 1)}, \frac{3}{(q^2 - 1)(q - 1)}\right) < q^{-n/3}.$$

Similarly, for $n = 2$ we have $|\chi(g)|/\chi(1) \leq 2/(q - 1) < (1.15)q^{-n/3}$ as $q \geq 8$.

Note that $f(n) > 1.16$ for all $n \geq 2$. Hence, to complete the induction base $2 \leq n \leq 5$, it remains to consider the case $(n, k) = (4, 2)$. Again inspecting the character table of G [49], we see that

$$|\chi(g)|/\chi(1) < 1/(q - 1)^2 < q^{-n/3}.$$

(ii) From now on we may assume that $n \geq 6$ and $2 \leq k < n$. Consider the action of u on the natural module $W' = \mathbb{F}_{q^k}^{n/k}$ of $\mathbf{C}_G(s)$. If this action induces an element with only one Jordan block, then $|\mathbf{C}_G(g)| = |\mathbf{C}_{\mathrm{GL}_{n/k}(q^k)}(u)| < q^n$ and again (3.6) holds. Thus we may assume that the $\langle g \rangle$-module W' is decomposable as a direct sum of two $\langle g \rangle$-submodules: $W' = W'_1 \oplus W'_2$, with

$$\dim_{\mathbb{F}_{q^k}} W'_1 = a \geq n/2k, \ \dim_{\mathbb{F}_{q^k}} W'_2 = b \geq 1.$$

Viewing W'_i as vector spaces over $\mathbb{F}_q$, we get a g-invariant decomposition $W = W_1 \oplus W_2$, with $\dim W_1 = ak \geq n/2$ and $\dim W_2 = bk \geq 2$. Writing $g = \mathrm{diag}(g_1, g_2)$ with $g_i \in G_i := \mathrm{GL}(W_i)$ and let s_i denote the semisimple part of g_i, we have

$$\mathbf{C}_{G_1}(s_1) = \mathrm{GL}(W'_1) \cong \mathrm{GL}_a(q^k), \ \mathbf{C}_{G_2}(s_2) = \mathrm{GL}(W'_2) \cong \mathrm{GL}_b(q^k).$$

In particular, the induction hypothesis applies to the elements $g_i \in G_i$.

(iii) Let V be a $\mathbb{C}G$-module affording the character χ, and denote $L_i := [G_i, G_i]$ for $i = 1, 2$. We decompose the $G_1 \times G_2$-module V as

$$V = V_1 \oplus V_2 \oplus V_3,$$

where $V_1 := \mathbf{C}_V(L_1)$, every irreducible constituent of V_2 is trivial on L_2 but *not* on L_1, and every irreducible constituent of V_3 is nontrivial on L_1 and on L_2. Let χ_j denote the $G_1 \times G_2$-character afforded by V_j, for $1 \leq j \leq 3$.

If $\alpha \otimes \beta$ is any irreducible constituent of χ_3, then $\alpha(1), \beta(1) > 1$ by the construction of V_3, whence

$$|\alpha(g_1)|/\alpha(1) \leq f(ak)q^{-ak/3}, \ |\beta(g_2)|/\beta(1) \leq f(bk)q^{-bk/3}$$

by the induction hypothesis applied to $g_1 \in G_1$ and $g_2 \in G_2$. It follows that

$$|\chi_3(g)|/\chi_3(1) \leq f(ak)f(bk)q^{-n/3}. \tag{3.7}$$

Next, let $\alpha \otimes \beta$ be any irreducible constituent of χ_2. Then $\alpha(1) > 1$ and $\beta(1) = 1$ by the construction of V_2, whence

$$|\alpha(g_1)|/\alpha(1) \leq f(ak)q^{-ak/3}, \ |\beta(g_2)| = 1$$

by the induction hypothesis applied to $g_1 \in G_1$. It follows that

$$|\chi_2(g)|/\chi_2(1) \leq f(ak)q^{-ak/3}. \tag{3.8}$$

(iv) We will now estimate $\chi_j(1)/\chi(1)$ for $j = 1, 2$. Let $d(X)$ denote the smallest degree of a nontrivial complex representation of a finite group X, and let

$$a_{m,q} := \frac{\sqrt{q-1}}{d(\mathrm{SL}_m(q))} + \frac{1}{d(\mathrm{SL}_{m-1}(q))}, \quad b_{m,q} := \sum_{i=m+1}^{\infty} a_{i,q}.$$

The proof of [26, Proposition 4.2.3] shows that, if U is any nontrivial irreducible $\mathbb{C}\mathrm{SL}_n(q)$-module for $n > m \geq 3$ and $\mathrm{SL}_m(q)$ is embedded naturally in $\mathrm{SL}_n(q)$, then

$$\dim \mathbf{C}_U(\mathrm{SL}_m(q)) \leq b_{m,q} \dim U. \tag{3.9}$$

By [52, Theorem 1.1], for $m \geq 4$ we have $a_{m,q} < (q + \sqrt{q-1})/q^{m-2}(q+1)$. It follows that

$$b_{m,q} < \frac{q + \sqrt{q-1}}{q+1} \sum_{i=m-1}^{\infty} q^{-i} = \frac{q(q + \sqrt{q-1})}{q^{m-1}(q^2-1)} < \frac{1.36}{q^{m-1}} \tag{3.10}$$

if $m \geq 3$ and $q \geq 8$. As $a_{3,q} = \sqrt{q-1}/(q^2+q) + e/(q-1)$ with $e := 3 - \gcd(q,2)$ and $q \geq 8$, we then have

$$b_{2,q} = a_{3,q} + b_{3,q} < 1.3q^{-2/3}. \tag{3.11}$$

Now, since $ak \geq n/2 \geq 3$, we have $ak - 1 \geq n/3$. Applying (3.9) and (3.10), we get

$$\chi_1(1)/\chi(1) \leq b_{ak}(q) < 1.36q^{-n/3}. \tag{3.12}$$

Similarly,

$$\chi_2(1)/\chi(1) \leq b_{bk}(q) < 1.36q^{-(bk-1)} \leq 0.17q^{-bk/3}. \tag{3.13}$$

if $bk \geq 3$, and

$$\chi_2(1)/\chi(1) \leq b_2(q) < 1.3q^{-2/3}. \tag{3.14}$$

if $bk \geq 2$ (using (3.11) instead of (3.10)). Note that in the case $bk = 2$, we must have $6 \leq n = ak + 2$, and so $ak \geq 4$, $ak - 1 \geq 1 + n/3$, whence instead of (3.12) we have

$$\chi_1(1)/\chi(1) \leq b_{ak}(q) < 0.17q^{-n/3}. \tag{3.15}$$

(v) Now, if $bk \geq 3$, then putting (3.7), (3.8), (3.12), (3.13) together, we obtain

$$|\chi(g)| \leq |\chi_1(1)| + |\chi_2(g)| + |\chi_3(g)| \leq \frac{\chi(1)}{q^{n/3}} \cdot (1.36 + 0.17f(ak) + f(ak)f(bk)).$$

If $bk = 2$, then (3.7), (3.8), (3.14), (3.15) altogether imply

$$|\chi(g)| \leq |\chi_1(1)| + |\chi_2(g)| + |\chi_3(g)| \leq \frac{\chi(1)}{q^{n/3}} \cdot (0.17 + 1.3f(ak) + f(ak)f(bk)).$$

The choice $f(n) = (11/7)^n - 1.3$ ensures that

$$f(n) = f(ak)f(bk) + 1.3f(ak) + 1.3f(bk) + 0.39 > f(ak)f(bk) + 0.17f(ak) + 1.36,$$

whence $|\chi(g)|/\chi(1) \leq f(n)q^{-n/3}$, completing the induction step of the proof. The last statement then follows, since $f(n) < q^{n/12}$ when $q \geq 227$ and $\chi(1) < q^{n^2/2}$. $\blacksquare$

3.4. Unipotent elements in general linear groups.

Theorem 3.3. *There is a function $g : \mathbb{N} \to \mathbb{N}$ such that the following statement holds. For any $n \geq 2$, any prime power q, $\ell = 0$ or any prime not dividing q, any irreducible ℓ-Brauer character φ of $G := \mathrm{GL}_n(q)$, and any unipotent element $1 \neq u \in G$,*

$$|\varphi(u)| \leq g(n) \cdot \varphi(1)^{\frac{n-2}{n-1}}.$$

Proof Note that the statement holds when $n = 2$ (choosing $g(2) = 1$) as in this case we have $|\varphi(u)| \leq 1$. So in what follows we may assume $n \geq 3$.

Recall the partial order $\leq$ on the set of unipotent classes of $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$: $x^{\mathcal{G}} \leq y^{\mathcal{G}}$ precisely when $x^{\mathcal{G}} \subseteq \overline{y^{\mathcal{G}}}$, and we consider $G = \mathcal{G}^F$ for a suitable Frobenius endomorphism F. Note that the unipotent classes in $\mathcal{G}$ are parametrized by partitions of n. We will prove by induction using the partial order $\leq$ that, if u is parametrized by a partition $\lambda \vdash n$ then

$$|\varphi(g)| \leq g_\lambda(n) \cdot \varphi(1)^{\frac{n-2}{n-1}}$$

for some positive constant $g_\lambda(n)$ depending only on λ. Then the statement follows by taking

$$g(n) := \max_{\lambda \vdash n} g_\lambda(n).$$

Observe that u is a *Richardson* unipotent element, that is, we can find an F-stable parabolic subgroup $\mathcal{P}$ with unipotent radical $\mathcal{U}$ such that $u^{\mathcal{G}} \cap \mathcal{U}$ is an open dense subset of $\mathcal{U}$ that forms a single $\mathcal{P}$-orbit. Indeed, as shown in [19, §5.5], if $\mu = (\mu_1, \ldots, \mu_l) \vdash n$ is the conjugate partition associated to λ, then one can just take $\mathcal{P}$ to be the standard parabolic subgroup generated by the upper-triangular Borel subgroup together with matrices in the block-diagonal form, with block sizes $\mu_1 \times \mu_1, \ldots, \mu_l \times \mu_l$. Furthermore, $\mathbf{C}_{\mathcal{G}}(u)$ is connected (as $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$), of dimension equal to $\dim \mathcal{P} - \dim \mathcal{U}$, and contained in $\mathcal{P}$, see [5, Corollary 5.2.2]. Since $u^{\mathcal{G}} \cap \mathcal{U}$ is a single $\mathcal{P}$-orbit and $\mathbf{C}_{\mathcal{G}}(u) = \mathbf{C}_{\mathcal{G}}(u)^{\circ} = \mathbf{C}_{\mathcal{P}}(u)$ is connected, by the Lang-Steinberg theorem, $u^{\mathcal{G}} \cap \mathcal{U}$ contains an F-stable element u', i.e. $u' \in u^{\mathcal{G}} \cap U$ for $U := \mathcal{U}^F$. The connectedness of $\mathbf{C}_{\mathcal{G}}(u)$ implies by the Lang-Steinberg theorem that $u, u' \in u^{\mathcal{G}} \cap U$ are G-conjugate. Replacing u by u', we may assume that $u \in u^{\mathcal{G}} \cap U$. Then, again applying the Lang-Steinberg theorem, we see that any element $w \in u^{\mathcal{G}} \cap U$ can be written as huh^{-1} for some $h \in P := \mathcal{P}^F$. Conversely, the P-orbit of u is contained in $u^{\mathcal{G}} \cap U$. Thus $u^{\mathcal{G}} \cap U$ is a single P-orbit, and so

$$|u^{\mathcal{G}} \cap U| = [P : C],$$

where $C := \mathbf{C}_P(u) = \mathbf{C}_{\mathcal{P}}(u)^F = \mathbf{C}_{\mathcal{G}}(u)^F$. The structure of $\mathbf{C}_{\mathcal{G}}(u)$ is given in [33, Theorem 3.1]. As $\dim \mathbf{C}_{\mathcal{G}}(u) = \dim \mathcal{P} - \dim \mathcal{U}$, there is a constant $A(n)$ depending only on n such that

$$|u^{\mathcal{G}} \cap U| \geq \frac{2}{3}|U| \tag{3.16}$$

for all $q \geq A(n)$ and all $\lambda \vdash n$. By taking $g(n)$ large enough, say

$$g(n) \geq \max_{q' = p^r < A(n)} \left\{ \frac{|\psi(w)|}{\psi(1)^{\frac{n-2}{n-1}}} \mid 1 \neq w \in \mathrm{GL}_n(q'), \ w \text{ unipotent}, \ \psi \in \mathrm{IBr}_\ell(\mathrm{GL}_n(q')) \right\}, \tag{3.17}$$

we may assume that the condition $q \geq A(n)$ is indeed satisfied.

Let $1 \neq v \in U \smallsetminus u^{\mathcal{G}}$ be labeled by $\nu \vdash n$. Then

$$v \in \mathcal{U} = \overline{u^{\mathcal{G}} \cap \mathcal{U}},$$

and so $v^{\mathcal{G}} \leq u^{\mathcal{G}}$. In particular, if $u^{\mathcal{G}}$ is minimal with respect to $\leq$, then no such v exists. If $u^{\mathcal{G}}$ is not minimal, then by the induction hypothesis applied to $v^{\mathcal{G}}$ we have

$$|\varphi(v)| \leq g_\nu(n) \cdot \varphi(1)^{\frac{n-2}{n-1}} \tag{3.18}$$

for some positive constant $g_\nu(n)$ depending only on ν. We will let $g'_\lambda(n)$ be the largest among all $g_\nu(n)$ when ν runs over the partitions for all such v.

Let $\rho := {}^*R_L^G(\varphi)$, where L is a Levi subgroup of P. Then

$$\rho(1) = [\varphi_U, 1_U]_U = \frac{1}{|U|} \left(\varphi(1) + \sum_{1 \neq v \in U \smallsetminus u^{\mathcal{G}}} \varphi(v) + \sum_{w \in u^{\mathcal{G}} \cap U} \varphi(w) \right),$$

and so
$$|u^{\mathcal{G}} \cap U| \cdot |\varphi(u)| \leq |U|\rho(1) + \sum_{1 \neq v \in U \smallsetminus u^{\mathcal{G}}} |\varphi(v)| + \varphi(1).$$

It now follows from (3.16) and (3.18) that
$$|\varphi(u)| \leq \frac{3}{2}\rho(1) + \frac{1}{2}g'_\lambda(n)\varphi(1)^{\frac{n-2}{n-1}} + \frac{3}{2|U|}\varphi(1).$$

The proof of Theorem 1.4 and the bound $\alpha \leq \frac{n-2}{n-1}$ in Proposition 4.3 imply that
$$\rho(1) \leq h(n)\varphi(1)^{\frac{n-2}{n-1}}.$$

On the other hand, $|U| \geq q^{n-1}$ and $\varphi(1) < q^{n^2/2}$, whence for $n \geq 4$ we have
$$\frac{\varphi(1)}{|U|} < \varphi(1)^{\frac{n-2}{n-1}}.$$

The same conclusion holds for $n = 3$ since $\varphi(1) < q^4$ in this case. Hence the statement follows for u by taking
$$g_\lambda(n) := \frac{3}{2}h(n) + \frac{1}{2}g'_\lambda(n) + \frac{3}{2}. \qquad \blacksquare$$

3.5. Special linear groups.

Proposition 3.4. *Let $n \geq 3$ and let $\mathbb{F}$ be an algebraically closed field of characteristic ℓ, where either $\ell = 0$ or $\ell \nmid q$. Let V be an irreducible $\mathbb{F}\mathrm{GL}_n(q)$-module which is reducible over $\mathrm{SL}_n(q)$. Then one of the following holds:*

(i) $\dim(V) > q^{(n^2+n)/4}(q-1)$.
(ii) $n = 3$ and $\dim(V) \geq (q-1)(q^2-1)$.
(iii) $n = 4$ and $\dim(V) \geq (q-1)(q^2-1)(q^3-1)$.
(iv) $2|n$, and $\dim(V) = \prod_{j=1}^{n/2}(q^{2j-1}-1)$ or $\prod_{j=1}^{n/2}(q^{n/2+j}-1)/(q^j-1)$. *Furthermore, V is as described in [23, Proposition 5.10(ii), (iii)], and $V|_{\mathrm{SL}_n(q)}$ is a sum of two irreducible constituents.*

Proof Repeat the same proof of [23, Proposition 5.10], but for all n. $\qquad \blacksquare$

3.6. Proof of Theorem 1.5.
(i) In this proof, let $G := \mathrm{GL}_n(q) = \mathcal{G}^F$ in the notation of §2, and let $S := [G, G]$. Write $g = su = us$ with $s \in G$ semisimple and $u \in G$ unipotent. By Theorem 1.4 and Proposition 4.3, we may assume that there is no proper split Levi subgroup $\mathcal{L}$ of $\mathcal{G}$ such that $\mathbf{C}_G(s) \leq \mathcal{L}^F$, equivalently, $\mathbf{C}_G(s) \cong \mathrm{GL}_{n/k}(q^k)$ for some $k|n$. For a fixed n, by choosing $h(n)$ large enough (similarly to the choice (3.17) of $g(n)$), we may assume that $q \geq 227$. Hence, we are done by Theorems 3.2 (when $k > 1$) and 3.3 (when $k = 1$) if $H = G$.

From now on we will assume that $H = S$, and let $\tilde{\chi} \in \mathrm{Irr}(G)$ be lying above χ. Applying the result for G, we are done if $\chi = \tilde{\chi}|_S$. Hence, we may assume that $\tilde{\chi}|_S$ is reducible, and so either
$$\chi(1) \geq \tilde{\chi}(1)/[G:S] > q^{(n^2+n)/4} \tag{3.19}$$
or case (iv) of Proposition 3.4 holds for a $\mathbb{C}G$-module V affording $\tilde{\chi}$. Since
$$\frac{1}{2}\prod_{j=1}^{n/2}(q^{2j-1}-1) > q^{n^2/4}(1 - \sum_{j=1}^{n/2}q^{1-2j})/2 > q^{(n^2-1)/4}$$

when $2|n$ and $q \geq 227$, we now have that
$$\chi(1) > q^{(n^2-1)/4}. \tag{3.20}$$

Assume in addition that $g^S = g^G$. By Clifford's theorem we may write $\tilde{\chi}|_S = \sum_{i=1}^{t} \chi^{x_i}$ for some elements $x_i \in G$. Since $g^S = g^G$, g^{x_i} is S-conjugate to g and so $\chi^{x_i}(g) = \chi(g)$. It follows that

$$|\chi(g)|/\chi(1) = |\sum_{i=1}^{t} \chi^{x_i}(g)|/t\chi(1) = |\tilde{\chi}(g)|/\tilde{\chi}(1),$$

and so we are done again. So we may assume that $g^S \neq g^G$.

(ii) Here we consider the case $k > 1$, and recall that u is a unipotent element in $\mathbf{C}_G(s) = \mathrm{GL}_{n/k}(q^k)$. Note that $\det_{\mathbb{F}_{q^k}}$ maps $\mathbf{C}_G(s)$ onto $\mathbb{F}_{q^k}^\times$, and the norm map $\mathbb{F}_{q^k}^\times \to \mathbb{F}_q^\times$ is surjective. It follows that $s^G = s^S$. Hence, our assumption $g^G \neq g^S$ implies that $u \neq 1$. It is well known that the centralizer of any non-central element in $\mathrm{GL}_m(q)$ has order at most q^{m^2-2m+2}. It follows that

$$|\mathbf{C}_G(g)| = |\mathbf{C}_{\mathbf{C}_G(s)}(u)| \leq q^{n^2/k-2n+2k} \leq q^{n^2/2-2n+4},$$

whence $|\chi(g)| \leq |\mathbf{C}_G(g)|^{1/2} \leq q^{n^2/4-n+2}$. Together with (3.20), this implies

$$|\chi(g)| < \chi(1)^{1-1/2n}.$$

(iii) Now we consider the case $k = 1$, i.e. $s \in \mathbf{Z}(G)$, and prove the stronger bound that

$$|\chi(g)| \leq h(n)\chi(1)^{\frac{n-2}{n-1}}. \tag{3.21}$$

Without loss of generality we may assume that $g = u$. Let r_i denote the number of Jordan blocks of size i in the Jordan canonical form of u for each $i \geq 1$; in particular, $\sum_i ir_i = n$. It is easy to see that $g^G = g^S$ if $\gcd(i \mid r_i \geq 1) = 1$. So the assumption $g^G \neq g^S$ implies

$$\gcd(i \mid r_i \geq 1) > 1, \tag{3.22}$$

in particular, $r_1 = 0$. We claim (for $n \geq 5$) that either

$$|\mathbf{C}_G(g)| \leq q^{(n^2-3n+6)/2} \tag{3.23}$$

or g has type $J_2^{n/2}$ (i.e. $r_2 = n/2$). Indeed, by [33, Theorem 3.1] we have that $|\mathbf{C}_G(g)| < q^N$, where

$$N := \sum_i ir_i^2 + 2\sum_{i<j} ir_ir_j. \tag{3.24}$$

Now, if $r_2 = 0$, then $3N \leq (\sum_i ir_i)^2 = n^2$ and so (3.23) holds for $n \geq 6$. If $r_2 = 0$ and $n = 5$, then (3.22) implies that $r_5 = 1$, again yielding (3.23). Suppose now that $n/2 > r := r_2 > 0$, whence $r_3 = 0$ by (3.22) and $n - 2r = t := \sum_{j \geq 4} jr_j \geq 4$. Then

$$N = 2r^2 + 4r\sum_{j \geq 4} r_j + \sum_{j \geq 4} jr_j^2 + 2\sum_{4 \leq j < j'} jr_jr_{j'} \leq 2r^2 + rt + t^2/4 \leq (n^2 - 4n + 8)/2.$$

In the case of (3.23), $|\chi(g)| \leq q^{(n^2-3n+6)/4}$ and so (3.21) holds because of (3.20).

It remains to consider the case $g = J_2^{n/2}$. Let $W = \mathbb{F}_q^n = \langle e_1, \ldots, e_n \rangle_{\mathbb{F}_q}$ denote the natural module for G, with $g(e_1) = e_1$. Here, $|\mathbf{C}_G(g)| < q^{n^2/2}$ by (3.24), whence

$$|\chi(g)| \leq q^{n^2/4}. \tag{3.25}$$

Suppose first that

$$\chi(1) > q^{(n-1)(n-2)/2}. \tag{3.26}$$

If $n \geq 8$, then (3.25) and (3.26) immediately imply (3.21). In the remaining case we have $n = 6$. An application of Clifford's theorem to the normal subgroup $\mathrm{SL}_6(q)\mathbf{Z}(\mathrm{GL}_6(q))$ of $\mathrm{GL}_6(q)$ yields $2 \leq \tilde{\chi}(1)/\chi(1) \leq 6$. In particular, in the case of Proposition 3.4(iv) we have

$$\chi(1) \leq \frac{(q^4-1)(q^5-1)(q^6-1)}{2(q-1)(q^2-1)(q^3-1)} < q^{10},$$

contrary to (3.26). Thus Proposition 3.4(i) must hold, whence

$$\chi(1) \geq \frac{1}{6} q^{21/2}(q-1) \geq q^{45/4}$$

(for $q \geq 1301$, which can be guaranteed by taking $h(6)$ large enough). The latter, together with (3.25), implies (3.21).

It remains to consider the case where (3.26) does not hold. Let χ be afforded by a $\mathbb{C}S$-module V and let $P := Stab_S(\langle e_1 \rangle_{\mathbb{F}_q}) = UL$. We decompose the P-module V as $\mathbf{C}_V(U) \oplus [U, V]$ and let γ, respectively δ, denote the P-character of $\mathbf{C}_V(U)$, respectively of $[U, V]$. In particular, $\gamma = {}^*R_L^S(\chi)$, and so, arguing as in part (ii) of the proof of Theorem 1.4 we get

$$|\gamma(g)| \leq \gamma(1) \leq f(n)\chi(1)^{\frac{n-2}{n-1}} \tag{3.27}$$

(for some function $f : \mathbb{N} \to \mathbb{N}$). Next, we decompose

$$[U, V] = \sum_{1_U \neq \lambda \in \mathrm{Irr}(U)} V_\lambda,$$

as a direct sum of U-eigenspaces, which are transitively permuted by $L \cong \mathrm{GL}_{n-1}(q)$. Note that g has prime order $p|q$, and it acts on $\mathrm{Irr}(U) \smallsetminus \{1_U\}$ with exactly $q^{n/2} - 1$ fixed points. Certainly, the trace of g in its action on $\sum_{\lambda \in \mathcal{O}} V_\lambda$ for any nontrivial g-orbit $\mathcal{O}$ on $\mathrm{Irr}(U) \smallsetminus \{1_U\}$ is zero. Since $\chi(1) \leq q^{(n-1)(n-2)/2}$, we have that

$$|\delta(g)| \leq (q^{n/2} - 1)\dim(V_\lambda) = (q^{n/2} - 1) \cdot \frac{\dim([U,V])}{q^{n-1} - 1} < \frac{\chi(1)}{q^{n/2-1}} \leq \chi(1)^{\frac{n-2}{n-1}}.$$

Together with (3.27), this completes the proof. ∎

The above proof yields the following analogue of Theorem 3.3:

Corollary 3.5. *Let $S := \mathrm{SL}_n(q) \leq G := \mathrm{GL}_n(q)$, and let $u \in G$ be any nontrivial unipotent element. Assume that either $\ell = 0$, or $\ell \nmid q$ and $u^G = u^S$. Then for any $\varphi \in \mathrm{IBr}_\ell(S)$,*

$$|\varphi(u)| \leq g(n) \cdot \varphi(1)^{\frac{n-2}{n-1}}.$$

Remark 3.6. For any $\varepsilon > 0$, it seems possible to improve the term $q^{n/3}$ in Theorem 3.2 to $q^{n/(2+\varepsilon)}$ at the price of using much bigger $f(n)$, as well as a much bigger lower bound on q. As a consequence, one could perhaps improve the exponent $1 - 1/2n$ in Theorem 1.5 to $1 - 1/((1 + \varepsilon)n)$. But we did not try to pursue it.

4. Bounds for the constant $\alpha(\mathcal{L})$: Proof of Theorems 1.6, 1.7 and 1.10

For the proof of Theorem 1.6, it is convenient to handle the classical types SL, Sp and SO separately. As in the theorem, let $\mathbb{K}$ be an algebraically closed field of good characteristic. Note that by the definition of $\alpha(\mathcal{L})$, this value does not depend on the isogeny type of $\mathcal{G}$.

4.1. **Case $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$ or $\mathrm{SL}_n(\mathbb{K})$.** To prove Theorem 1.6 in this case we use the following lemma, which transfers attention from unipotent to semisimple elements in the analysis of $\alpha(\mathcal{L})$. Denote by J_i a unipotent $i \times i$ Jordan block matrix, and by $\sum_i J_i^{n_i}$ the matrix in $\mathrm{SL}_n(\mathbb{K})$ with n_i diagonal blocks J_i for each i, where $n = \sum in_i$.

Lemma 4.1. *Let $u = \sum_i J_i^{n_i}$ be a unipotent element of $\mathrm{GL}_n(\mathbb{K})$ where $n = \sum in_i$. Let $\lambda_j \ (j \in \mathbb{N})$ be distinct scalars in $\mathbb{K}^\times$, and for each i let $D_i = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_i)$. Define*

$$s := \sum_i D_i^{n_i} \in \mathrm{GL}_n(\mathbb{K}).$$

Then $\dim \mathbf{C}_{\mathrm{GL}_n(\mathbb{K})}(u) = \dim \mathbf{C}_{\mathrm{GL}_n(\mathbb{K})}(s)$.

Proof Observe that

$$
\begin{aligned}
\dim \mathbf{C}_{\mathrm{GL}_n(\mathbb{K})}(s) &= \left(\textstyle\sum_{i\geq 1} n_i\right)^2 + \left(\textstyle\sum_{i\geq 2} n_i\right)^2 + \cdots \\
&= \textstyle\sum_i i n_i^2 + 2\sum_{i<j} i n_i n_j
\end{aligned}
$$

which is equal to $\dim \mathbf{C}_{\mathrm{GL}_n(\mathbb{K})}(u)$ by [33, 3.1]. ■

For a subgroup X of $\mathrm{GL}_n(\mathbb{K})$, define X_{ss} to be the set of semisimple elements of X.

Corollary 4.2. *If $n \geq 2$ and $\mathcal{L}$ is a Levi subgroup of $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$ or $\mathrm{SL}_n(\mathbb{K})$, then*

$$
\alpha(\mathcal{L}) \leq \max_{s \in \mathcal{L}_{\mathrm{ss}} \smallsetminus \mathbf{Z}(\mathcal{G})} \frac{\dim s^{\mathcal{L}}}{\dim s^{\mathcal{G}}}.
$$

Proof We have $\mathcal{L} = \mathcal{G} \cap \prod_{j=1}^r \mathrm{GL}_{a_j}(\mathbb{K})$, where $\sum_{j=1}^r a_j = n$. Let $u \in \mathcal{L}_{\mathrm{unip}}$, so that

$$
u = \sum_{j=1}^r \sum_i J_i^{n_{ij}},
$$

where $\sum_i i n_{ij} = a_j$. The condition $u \neq 1$ means that there are some $i \geq 2$ and $j \geq 1$ such that $n_{ij} > 0$. If we define $s = \sum_{j=1}^r \sum_i D_i^{n_{ij}}$, where D_i is as in the statement of Lemma 4.1 (and the scalars λ_j are chosen so that s has determinant 1 in the case where $\mathcal{G} = \mathrm{SL}_n(\mathbb{K})$), then $s \notin \mathbf{Z}(\mathcal{G})$. Now Lemma 4.1 shows that $\dim \mathbf{C}_{\mathcal{L}}(u) = \dim \mathbf{C}_{\mathcal{L}}(s)$ and $\dim \mathbf{C}_{\mathcal{G}}(u) = \dim \mathbf{C}_{\mathcal{G}}(s)$. ■

Proof of Theorem 1.6 for $\mathrm{GL}_n(\mathbb{K})$, $\mathrm{SL}_n(\mathbb{K})$

We prove the theorem for $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$ and point out the small adjustment needed for $\mathrm{SL}_n(\mathbb{K})$ at the end of the proof. Let $\mathcal{L}$ be a Levi subgroup of $\mathcal{G}$. Adopting an obvious notational convention we take

$$
\mathcal{L} = \mathrm{GL}_a(\mathbb{K}) \times \mathrm{GL}_b(\mathbb{K}) \times \cdots \times \mathrm{GL}_z(\mathbb{K}).
$$

Write $V_n = V_a \oplus V_b \oplus \cdots \oplus V_z$ for the corresponding direct sum decomposition of $V_n = \mathbb{K}^n$.

In view of Corollary 4.2, it suffices to prove that

$$
\max_{s \in \mathcal{L}_{\mathrm{ss}} \smallsetminus \mathbf{Z}(\mathcal{G})} \frac{\dim s^{\mathcal{L}}}{\dim s^{\mathcal{G}}} \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right). \tag{4.1}
$$

Let s be a semisimple element of $\mathcal{L}$, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of s on V_n. Write

$$
s|_{V_a} = \mathrm{diag}\,(\lambda_1^{(a_1)}, \ldots, \lambda_k^{(a_k)}), \ldots, s|_{V_z} = \mathrm{diag}\,(\lambda_1^{(z_1)}, \ldots, \lambda_k^{(z_k)}),
$$

where $\sum_{i=1}^k a_i = a$, and so on (superscripts denote multiplicities). Then

$$
\mathbf{C}_{\mathcal{G}}(s) = \mathrm{GL}_{a_1+b_1+\ldots}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{a_k+b_k+\ldots}(\mathbb{K}),
$$

$$
\mathbf{C}_{\mathcal{L}}(s) = \prod_{i=1}^k \mathrm{GL}_{a_i}(\mathbb{K}) \times \prod_{i=1}^k \mathrm{GL}_{b_i}(\mathbb{K}) \times \cdots .
$$

To prove (4.1) we need to show

$$
\frac{1}{2}\frac{\dim \mathcal{G} - \dim \mathcal{L}}{\dim \mathcal{G}} \leq \frac{\dim s^{\mathcal{G}} - \dim s^{\mathcal{L}}}{\dim s^{\mathcal{G}}}. \tag{4.2}
$$

Now

$$
\frac{1}{2}\frac{\dim \mathcal{G} - \dim \mathcal{L}}{\dim \mathcal{G}} = \frac{ab + ac + bc + \ldots}{(a + b + \ldots)^2},
$$

while
$$\frac{\dim s^{\mathcal{G}} - \dim s^{\mathcal{L}}}{\dim s^{\mathcal{G}}} = \frac{\sum_{i \neq j}(a_i b_j + a_i c_j + b_i c_j + \dots)}{\sum_{i<j}(a_i + b_i + \cdots)(a_j + b_j + \cdots)}.$$

Hence (4.2) is equivalent to the inequality

$$\left(\sum a_i \sum b_i + \sum a_i \sum c_i + \sum b_i \sum c_i + \cdots\right) \cdot \left(\sum_{i<j}(a_i + b_i + \cdots)(a_j + b_j + \cdots)\right)$$
$$\leq \left(\sum a_i + \sum b_i + \cdots\right)^2 \cdot \sum_{i \neq j}(a_i b_j + a_i c_j + b_i c_j + \cdots). \tag{4.3}$$

Now observe that all the terms on the left hand side of this inequality appear with at most the same multiplicity on the right hand side. Hence (4.3) holds, and the proof is complete for $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$.

For the case where $\mathcal{G} = \mathrm{SL}_n(\mathbb{K})$, we need to prove the inequality (4.3) with the first term on the right hand side replaced by $(\sum a_i + \sum b_i + \cdots)^2 - 1$. This remains true, since the terms on the right hand side but not the left hand side of (4.3) include $\sum_{i \neq j}(a_i^3 b_j + a_i^3 c_j + b_i^3 c_j + \cdots)$, which is at least $\sum_{i \neq j}(a_i b_j + a_i c_j + b_i c_j + \cdots) \geq 1$. ∎

We also deduce the following general bound, which was used in the proof of Theorem 3.3 and also in Example 2.8.

Proposition 4.3. *If $\mathcal{L}$ is a Levi subgroup of $\mathcal{G} = \mathrm{GL}_n(\mathbb{K})$, then $\alpha(\mathcal{L}) \leq \frac{n-2}{n-1}$, with equality if and only if $\mathcal{L} = \mathrm{GL}_{n-1}(\mathbb{K}) \times \mathrm{GL}_1(\mathbb{K})$.*

Proof Choose maximal $a \leq \frac{n}{2}$ such that $\mathcal{L} \leq \mathrm{GL}_a(\mathbb{K}) \times \mathrm{GL}_{n-a}(\mathbb{K})$. By Theorem 1.6 for $\mathrm{GL}_n(\mathbb{K})$, proved above, we have

$$\alpha(\mathcal{L}) \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right) \leq \frac{1}{2}\left(1 + \frac{a^2 + (n-a)^2}{n^2}\right).$$

One checks that the right hand side is less than $\frac{n-2}{n-1}$ for $n \geq 2a$, except in the following cases:

(a) $a = 1$, in which case $\mathcal{L} = \mathrm{GL}_1(\mathbb{K}) \times \mathrm{GL}_{n-1}(\mathbb{K})$ (by the maximal choice of a);
(b) $a = 2$, $n \leq 5$.

In case (b) we compute the values of $\alpha(\mathcal{L})$ and find that $\alpha(\mathcal{L}) \leq \frac{1}{2} < \frac{n-2}{n-1}$ (note that $n \geq 2a = 4$ in this case).

Hence it remains to consider case (a). We claim that in this case, $\alpha(\mathcal{L}) = \frac{n-2}{n-1}$. Let u be a nontrivial unipotent element of $\mathcal{L}$, and write $u = \sum J_i^{n_i}$, where $\sum i n_i = n$. Then u projects to the element $J_1^{n_1-1} + \sum_{i \geq 2} J_i^{n_i}$ in the factor $\mathrm{GL}_{n-1}(\mathbb{K})$ of $\mathcal{L}$, so by [33, 3.1], we have

$$\dim \mathbf{C}_{\mathcal{G}}(u) = \sum i n_i^2 + 2\sum_{i<j} i n_i n_j,$$
$$\dim \mathbf{C}_{\mathcal{L}}(u) = 1 + (n_1 - 1)^2 + \sum_{i \geq 2} i n_i^2 + 2(n_1 - 1)\sum_{j \geq 2} n_j + 2\sum_{2 \leq i < j} i n_i n_j.$$

Defining $s := \dim[V, u] = n - \sum n_i$, it follows that

$$(\dim \mathcal{G} - \dim \mathcal{L}) - (\dim \mathbf{C}_{\mathcal{G}}(u) - \dim \mathbf{C}_{\mathcal{L}}(u)) = 2s.$$

Next observe that

$$\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} \leq \frac{n-2}{n-1} \quad \Leftrightarrow \quad (n-1)\left((\dim \mathcal{G} - \dim \mathcal{L}) - (\dim \mathbf{C}_{\mathcal{G}}(u) - \dim \mathbf{C}_{\mathcal{L}}(u))\right) \geq \dim u^{\mathcal{G}}$$
$$\Leftrightarrow \quad 2(n-1)s \geq \dim u^{\mathcal{G}}.$$

By [34, 3.4(i)] and its proof, we have $\dim u^{\mathcal{G}} \leq s(2n-s)$, so the above inequality holds when $s \geq 2$. Finally, when $s = 1$ we have $u = J_2 + J_1^{n-2}$, and we calculate that $\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} = \frac{n-2}{n-1}$. Hence $\alpha(\mathcal{L}) = \frac{n-2}{n-1}$ in case (1), and the proof is complete. ∎

4.2. **Symplectic groups.** Now we prove Theorem 1.6 for symplectic groups. We revert to Lie-theoretic notation, so assume that

$$\mathcal{G} = C_n = \mathrm{Sp}_{2n}(\mathbb{K}) = \mathrm{Sp}(V),$$

where $V = V_{2n}(K)$ is the natural module for $\mathcal{G}$ and $n \geq 2$.

Let $\mathcal{L}$ be a Levi subgroup of $\mathcal{G}$, so that $\mathcal{L}' = C_{n-r} \times \prod A_{r_i} \leq C_{n-r} \times A_{r-1}$, where $1 \leq r \leq n$. The first lemma deals with the case where $r = n$ (and, recall that T_r denotes an r-dimensional torus).

Lemma 4.4. *If $\mathcal{L} \leq A_{n-1}T_1$, then $\alpha(\mathcal{L}) \leq \frac{1}{2}$.*

Proof Assume $\mathcal{L} \leq A_{n-1}T_1 = \mathrm{GL}_n(\mathbb{K})$, and let u be a nontrivial unipotent element of $\mathcal{L}$. Write $u = \sum J_i^{n_i} \in \mathrm{SL}_n(q)$, where $\sum i n_i = n$. As an element of $G = \mathrm{Sp}_{2n}(q)$, u has Jordan form $\sum J_i^{2n_i}$. Hence by [33, 3.1],

$$\dim \mathbf{C}_{\mathrm{GL}_n(\mathbb{K})}(u) = \sum i n_i^2 + 2\sum_{i<j} i n_i n_j := c_u,$$
$$\dim \mathbf{C}_{\mathcal{G}}(u) = 2\sum i n_i^2 + 4\sum_{i<j} i n_i n_j + \sum_{i \text{ odd}} n_i.$$

So $\dim \mathbf{C}_{\mathcal{G}}(u) = 2c_u + s_u$, where $s_u = \sum_{i \text{ odd}} n_i$. It follows that

$$\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} \leq \frac{\dim u^{\mathrm{GL}_n}}{\dim u^{\mathcal{G}}} \leq \frac{n^2 - c_u}{2n^2 + n - 2c_u - s_u} \leq \frac{1}{2},$$

and the conclusion follows.   ∎

Lemma 4.5. *If $\mathcal{L} = C_{n-r}T_r$, then $\alpha(\mathcal{L}) \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right)$.*

Proof Let u be a nontrivial unipotent element of $\mathcal{L}' = C_{n-r} = \mathrm{Sp}_{2n-2r}(\mathbb{K})$, and write $u = \sum J_i^{n_i}$ with $\sum i n_i = 2n - 2r$. In $G = \mathrm{Sp}_{2n}(q)$, u has Jordan form $J_1^{n_1+2r} + \sum_{i \geq 2} J_i^{n_i}$. Using [33, 3.1], we find that

$$\dim \mathbf{C}_{\mathcal{L}}(u) - \dim \mathbf{C}_{\mathcal{G}}(u) = 2r \sum n_i + 2r^2. \tag{4.4}$$

As in [34, p.509], define

$$s := \dim[V, u] = 2n - 2r - \sum n_i.$$

Then (4.4) implies that $\dim u^{\mathcal{G}} - \dim u^{\mathcal{L}} = 2rs$. It also follows from [34, 3.4] and its proof that

$$\dim u^{\mathcal{G}} \leq \frac{1}{2}s(4n - s + 1). \tag{4.5}$$

Now observe that

$$\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right) \quad \Leftrightarrow \quad \dim u^{\mathcal{G}} \leq \frac{2\dim \mathcal{G}(\dim u^{\mathcal{G}} - u^{\mathcal{L}})}{\dim \mathcal{G} - \dim \mathcal{L}}$$
$$\Leftrightarrow \quad \dim u^{\mathcal{G}} \leq \frac{2s(2n^2 + n)}{2n - r}.$$

Clearly (4.5) implies that the last inequality holds, and so we are done.   ∎

Lemma 4.6. *Suppose $\mathcal{L}' \leq C_{n-r} \times A_{r-1} \leq C_{n-r} \times C_r < \mathcal{G}$ with $r > 1$, and let $u = u_1 u_2 \in \mathcal{L}$ be a unipotent element with $u_1 \in C_{n-r}$, $u_2 \in A_{r-1} < C_r$. Then*

$$\dim u^{\mathcal{G}} \geq \dim u_1^{\mathcal{G}} + \dim u_2^{C_r}.$$

Proof Let $u_1 = \sum J_i^{a_i} \in C_{n-r}$ and $u_2 = \sum J_i^{b_i} \in A_{r-1}$, where $\sum i a_i = 2n - 2r$, $\sum i b_i = r$. Then

$$u_1 = J_1^{a_1+2r} + \sum_{i \geq 2} J_i^{a_i} \in \mathcal{G} = \mathrm{Sp}_{2n}(\mathbb{K}),$$
$$u_2 = \sum J_i^{2b_i} \in C_r, \text{ and}$$
$$u = \sum J_i^{a_i + 2b_i} \in \mathcal{G}.$$

Now using [33, 3.1], we compute that

$$
\begin{aligned}
& \dim \mathbf{C}_{\mathcal{G}}(u_1) + \dim \mathbf{C}_{C_r}(u_2) - \dim \mathbf{C}_{\mathcal{G}}(u) \\
= {} & 2r^2 + r + 2r \sum a_i - 2 \sum i a_i b_i - 2 \sum_{i<j} i(a_i b_j + a_j b_i) \\
= {} & 2r^2 + r + 2\left((\sum a_i)(\sum i b_i) - \sum i a_i b_i - \sum_{i<j} i(a_i b_j + a_j b_i) \right) \\
\geq {} & 2r^2 + r \\
= {} & \dim C_r,
\end{aligned}
$$

and the result follows. ■

Proof of Theorem 1.6 for $\mathcal{G} = C_n$

Let $\mathcal{L}$ be a Levi subgroup of $\mathcal{G}$, so $\mathcal{L}' = C_{n-r} \times \prod A_{r_i} \leq C_{n-r} \times A_{r-1} \leq C_{n-r} \times C_r$, where $1 \leq r \leq n$. Let $u = u_1 u_2$ be a nontrivial unipotent element of $\mathcal{L}$, where $u_1 \in C_{n-r}$, $u_2 \in A_{r-1}$. Using Lemma 4.6, we have

$$
\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} \leq \frac{\dim u_1^{C_{n-r}} + \dim u_2^{A_{r-1}}}{\dim u_1^{\mathcal{G}} + \dim u_2^{C_r}}. \tag{4.6}
$$

Also Lemmas 4.4 and 4.5 imply that

$$
\frac{\dim u^{A_{r-1}}}{\dim u^{C_r}} \leq \frac{1}{2}, \quad \text{and} \quad \frac{\dim u^{C_{n-r}}}{\dim u^{\mathcal{G}}} \leq \frac{1}{2}\left(1 + \frac{\dim C_{n-r} T_r}{\dim \mathcal{G}} \right).
$$

Hence (4.6) implies that

$$
\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} \leq \frac{1}{2}\left(1 + \frac{\dim C_{n-r} T_r}{\dim \mathcal{G}} \right) \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}} \right).
$$

This completes the proof of Theorem 1.6 for $\mathcal{G} = C_n$.

4.3. **Orthogonal groups.** We complete the proof of Theorem 1.6 by handling the orthogonal groups. The proof for $\mathcal{G} = B_n = \mathrm{SO}_{2n+1}(\mathbb{K})$ is very similar to that for $\mathcal{G} = C_n$: one shows that Lemmas 4.4–4.6 also hold in the B_n case (with $\mathcal{L} = B_{n-r} T_r$ in Lemma 4.5 and $\mathcal{L}' \leq B_{n-r} \times A_{r-1} \leq B_{n-r} \times D_r$ in Lemma 4.6), and the theorem follows. Things are a little different in the D_n case, so assume now that

$$
\mathcal{G} = D_n = \mathrm{SO}_{2n}(\mathbb{K}) = \mathrm{SO}(V) \quad (n \geq 4).
$$

Let $\mathcal{L}$ be a Levi subgroup of $\mathcal{G}$. Then $\mathcal{L}' = D_{n-r} \times \prod A_{r_i} \leq D_{n-r} \times A_{r-1}$, where $1 \leq r \leq n$ and $r \neq n - 1$.

Lemma 4.7. *Suppose $\mathcal{L}' \leq A_{n-1}$ and $\mathcal{L}' \neq A_{n-1}$. Then $\alpha(\mathcal{L}) \leq \frac{1}{2}$.*

Proof By assumption, $\mathcal{L} \leq \mathrm{GL}_a(\mathbb{K}) \times \mathrm{GL}_b(\mathbb{K})$ where $a + b = n$ and $a, b \geq 1$. Let $u = u_1 u_2 \in \mathcal{L}$, where $u_1 = \sum J_i^{a_i} \in \mathrm{GL}_a(\mathbb{K})$ and $u_1 = \sum J_i^{b_i} \in \mathrm{GL}_b(\mathbb{K})$ (so $\sum i a_i = a$, $\sum i b_i = b$). Then $u = \sum J_i^{2a_i + 2b_i} \in \mathcal{G}$. By [33, 3.1],

$$
\dim \mathbf{C}_{\mathcal{L}}(u) = \sum i a_i^2 + \sum i b_i^2 + 2 \sum i a_i a_j + 2 \sum i b_i b_j =: c_u,
$$

and

$$
\begin{aligned}
\dim \mathbf{C}_{\mathcal{G}}(u) & = 2 \sum i(a_i + b_i)^2 + 4 \sum_{i<j} i(a_i + b_i)(a_j + b_j) - \sum_{i\ odd}(a_i + b_i) \\
& = 2c_u + 4 \sum i a_i b_i + 4 \sum_{i<j} i(a_i b_j + a_j b_i) - \sum_{i\ odd}(a_i + b_i).
\end{aligned}
$$

Then $\dim u^{\mathcal{L}} = a^2 + b^2 - c_u$, while

$$
\dim u^{\mathcal{G}} = 2(a + b)^2 - (a + b) - \left(2c_u + 4 \sum i a_i b_i + 4 \sum_{i<j} i(a_i b_j + a_j b_i) - \sum_{i\ odd}(a_i + b_i) \right).
$$

To prove the lemma we need to show that $\dim u^{\mathcal{G}} \geq 2(a^2 + b^2 - c_u)$. Using the equations $\sum i a_i = a$, $\sum i b_i = b$, this amounts to showing that

$$4\sum(i^2 - i)a_i b_i + 4\sum_{i<j} i(j-1)(a_i b_j + a_j b_i) + \sum_{i \text{ odd}} (a_i + b_i) \geq \sum i a_i + \sum j b_j. \quad (4.7)$$

Consider a term $k a_k + l b_l$ on the right hand side, with $a_k, b_l \neq 0$. If $k = l = 1$ this occurs in the sum $\sum_{i \text{ odd}}(a_i + b_i)$; if $k = l \geq 2$ it is less then or equal to the term $4(k^2 - k)a_k b_k$ on the left hand side; and if $k < l$ or $l < k$, it is at most $4k(l-1)a_k b_l$ or $4l(k-1)a_k b_l$, respectively. Hence the inequality (4.7) holds, completing the proof of the lemma. ■

The proofs of the next two lemmas are very similar to those of Lemmas 4.5 and 4.6.

Lemma 4.8. *If $\mathcal{L} = D_{n-r}T_r$, then $\alpha(\mathcal{L}) \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right)$.*

Lemma 4.9. *Suppose $\mathcal{L}' \leq D_{n-r} \times A_{r-1} \leq D_{n-r} \times D_r < \mathcal{G}$ with $r > 1$, and let $u = u_1 u_2 \in \mathcal{L}$ be a unipotent element with $u_1 \in D_{n-r}$, $u_2 \in A_{r-1} < D_r$. Then*

$$\dim u^{\mathcal{G}} \geq \dim u_1^{\mathcal{G}} + \dim u_2^{D_r}.$$

Let $\mathcal{L}' = D_{n-r} \times \mathcal{L}_1 \leq D_{n-r} \times A_{r-1}$, where $\mathcal{L}_1 = \prod A_{r_i} \leq A_{r-1}$. If either $\mathcal{L}_1 < A_{r-1}$ or $r = 1$, then Theorem 1.6 follows from Lemmas 4.7–4.9 just as in the argument following (4.6) for the case where $\mathcal{G} = C_n$. Hence it remains to handle the case where $\mathcal{L}' = D_{n-r} \times A_{r-1}$ with $2 \leq r \leq n$, $r \neq n - 1$. We deal with this case in the next two lemmas.

Lemma 4.10. *Suppose $\mathcal{L}' = D_{n-r} \times A_{r-1}$ with $r \geq 3$. Then $\alpha(\mathcal{L}) \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right)$.*

Proof   Let $u = u_1 u_2$ be a unipotent element of $\mathcal{L}$, where $u_1 \in D_{n-r}$, $u_2 \in A_{r-1} < D_r$. We will show that

$$\frac{\dim u_2^{A_{r-1}}}{\dim u_2^{D_r}} \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right). \quad (4.8)$$

Given this, the lemma follows, since by Lemma 4.9 we have

$$\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} \leq \frac{\dim u_1^{D_{n-r}} + \dim u_2^{A_{r-1}}}{\dim u_1^{\mathcal{G}} + \dim u_2^{D_r}},$$

and this is at most $\frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right)$ by Lemma 4.8 and (4.8).

It remains to establish (4.8). Let $u_2 = \sum J_i^{a_i} \in A_{r-1} = \mathrm{SL}_r(\mathbb{K})$, so that u_2 has Jordan form $\sum J_i^{2a_i}$ in D_r. Then

$$\dim \mathbf{C}_{\mathrm{GL}_r(\mathbb{K})}(u_2) = \sum i a_i^2 + 2\sum_{i<j} i a_i a_j,$$
$$\dim \mathbf{C}_{D_r}(u_2) = 2\sum i a_i^2 + 4\sum_{i<j} i a_i a_j - \sum_{i \text{ odd}} a_i.$$

Write $s_2 := \sum_{i \text{ odd}} a_i$. Then

$$\frac{\dim u_2^{A_{r-1}}}{\dim u_2^{D_r}} = \frac{1}{2}(1 + \frac{r - s_2}{\dim u_2^{D_r}}),$$

so to prove (4.8) it suffices to show

$$\frac{r - s_2}{\dim u_2^{D_r}} \leq \frac{\dim \mathcal{L}}{\dim \mathcal{G}} = \frac{2(n-r)^2 - (n-r) + r^2}{2n^2 - n}. \quad (4.9)$$

It is straightforward to see that the right hand side of (4.9) is at least $\frac{1}{3}$, so (4.9) holds if $\dim u_2^{D_r} \geq 3(r - s_2)$. The minimum value of $\dim u_2^{D_r}$ occurs when $u_2 = J_2 + J_1^{r-2} \in \mathrm{SL}_r(q)$, in which case $\dim u_2^{D_r} = 4r - 6$. This shows that (4.9) holds when $r \geq 6$.

It remains to establish (4.9) for $r = 3, 4, 5$. For $r = 5$, the possibilities for $u_2 \in \mathrm{SL}_r(q)$ are as follows:

$u_2 \in \mathrm{SL}_5(q)$	$J_2 + J_1^3$	$J_2^2 + J_1$	$J_3 + J_1^2$	$J_3 + J_2$	$J_4 + J_1$	J_5
$\dim u_2^{D_5}$	14	20	26	28	32	36
s_2	3	1	3	1	1	1

For all these possibilities (4.9) holds. The arguments for $r = 3, 4$ are similar. ∎

Lemma 4.11. *Suppose $\mathcal{L}' = D_{n-2} \times A_1$. Then $\alpha(\mathcal{L}) \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right)$.*

Proof Let $u = u_1 u_2 \in \mathcal{L}$ with $u_1 = \sum J_i^{n_i} \in D_{n-2}$ (so $\sum i n_i = 2n - 4$) and $u_2 \in A_1$. If $u_2 = 1$ then the conclusion follows from Lemma 4.8, so assume $u_2 \neq 1$. Then $\dim u_2^{A_1} = 2$ and $u = J_2^{n_2+2} + \sum_{i \neq 2} J_i^{n_i} \in \mathcal{G} = D_n$. By [33, 3.1],

$$\dim \mathbf{C}_{\mathcal{L}}(u) = \tfrac{1}{2}\sum i n_i^2 + \sum_{i<j} i n_i n_j - \tfrac{1}{2}\sum_{i \text{ odd}} n_i + 2,$$
$$\dim \mathbf{C}_{\mathcal{G}}(u) = \tfrac{1}{2}\sum i n_i^2 + \sum_{i<j} i n_i n_j - \tfrac{1}{2}\sum_{i \text{ odd}} n_i + 2n_1 + 4\sum_{i \geq 2} n_i + 4.$$

Define $s := \dim[V, u] = 2n - \sum n_i - 2$. Then

$$\dim u^{\mathcal{G}} - \dim u^{\mathcal{L}} = 4s + 2n_1 - 8.$$

Also [34, 3.4] gives $\dim u^{\mathcal{G}} \leq \frac{1}{2}s(4n - s + 1)$. Hence we see that the desired inequality $\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} \leq \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right)$ is equivalent to the following

$$\frac{1}{2}s(4n - s + 1) \leq \frac{(2n^2 - n)(4s + 2n_1 - 8)}{4n - 7}. \tag{4.10}$$

Now $2n - 4 = \sum i n_i \geq 2\sum_{i \geq 2} n_i$, and hence

$$s = 2n - 2 - n_1 - \sum_{i \geq 2} n_i \geq 2n - 2 - n_1 - (n - 2) = n - n_1.$$

It follows that $4s + 2n_1 \geq 2n + 2s$, and hence (4.10) holds provided

$$\frac{1}{2}s(4n - s + 1) \leq \frac{(2n^2 - n)(2n + 2s - 8)}{4n - 7},$$

which is true for all s when $n \geq 5$. Finally, when $n = 4$ the conclusion of the lemma is easily checked directly. This completes the proof. ∎

This completes the proof of Theorem 1.6.

4.4. **Exceptional groups: Proof of Theorem 1.7.** Let $\mathcal{G}$ be a simple algebraic group of exceptional type in good characteristic. In principle one can check Theorem 1.7 by going through all possible Levi subgroups $\mathcal{L}$ of $\mathcal{G}$, in each case listing all the unipotent class representatives u in $\mathcal{L}$ and using Theorem 3.1 and Tables 22.1.1–5 of [33] to write down the dimensions of $u^{\mathcal{L}}$ and $u^{\mathcal{G}}$. In fact, this is precisely what we do for the Levi subgroups listed in Table 1, and for the remaining ones (labelled "rest" in Table 1) we need a short argument.

We will give the proof of Theorem 1.7 just for $\mathcal{G} = E_7$ and leave the other entirely similar cases to the reader. First suppose that the Levi subgroup $\mathcal{L}$ is one of those listed for $\mathcal{G} = E_7$ in Table 1. In each case we adopt the above procedure of listing unipotent representatives u in $\mathcal{L}$ and calculating $\dim u^{\mathcal{L}}$ and $\dim u^{\mathcal{G}}$. We illustrate below with the case $\mathcal{L}' = D_6$, listing in the first row the Jordan form of u on the 12-dimensional module for $\mathcal{L}'$ and in the second row the class of u in $\mathcal{G}$ as in [33, Table 22.1.2]:

u in D_6	$(2^2,1^8)$	$(3,1^9)$	$(2^4,1^4)$	(2^6)	$(2^6)'$	$(3,2^2,1^5)$	$(3,2^4,1)$
u in E_7	A_1	A_1^2	A_1^2	$(A_1^3)^{(1)}$	$(A_1^3)^{(2)}$	$(A_1^3)^{(2)}$	A_1^4
$\dim u^{\mathcal{L}}$	18	20	28	30	30	32	36
$\dim u^{\mathcal{G}}$	34	52	52	54	64	64	70

$(3,2,1^6)$	$(3^2,2^2,1^2)$	$(3^3,1^3)$	(3^4)	$(4^2,1^4)$	$(5,1^7)$	$(4^2,2^2)$	$(4^2,2^2)'$
A_2	A_2A_1	$A_2A_1^2$	A_2^2	A_3	A_3	$(A_3A_1)^{(1)}$	$(A_3A_1)^{(2)}$
34	40	42	44	44	36	46	46
66	76	82	84	84	84	86	92

$(5,2^2,1^3)$	$(4^2,3,1)$	$(5,3^2,1)$	$(5^2,1^2)$	(6^2)	$(6^2)'$	$(7,1^5)$	$(5,3,1^4)$
$(A_3A_1)^{(1)}$	$A_3A_1^2$	A_3A_2	A_4	$(A_5)^{(1)}$	$(A_5)^{(2)}$	D_4	$D_4(a_1)$
44	48	50	52	54	54	48	46
86	94	98	100	102	108	96	94

$(7,2^2,1)$	$(5,3,2^2)$	$(9,1^3)$	$(7,3,1^2)$	$(11,1)$	$(9,3,)$	$(7,5)$
D_4A_1	$D_4(a_1)A_1$	D_5	$D_5(a_1)$	D_6	$D_6(a_1)$	$D_6(a_2)$
52	48	56	54	60	58	56
102	96	112	106	118	114	110

To compute the information in the tables, we list the possible Jordan forms for unipotent elements u of D_6, and in each case find a Levi subgroup of D_6 in which u is contained as a regular element; this Levi subgroup then gives the label of u as an element of E_7 in Table 22.1.2 of [33]. For cases where all the Jordan blocks have even size – namely the Jordan forms (2^6), $(4^2,2^2)$ and (6^2) – there are two D_6-classes (see [33, 3.11]), and the corresponding E_7-classes can be worked out by computing the dimension of $\mathbf{C}_{L(E_7)}(u)$ using the restriction $L(E_7) \downarrow D_6$ (see [33, 11.8]), where $L(E_7)$ denotes the Lie algebraa of $\mathcal{G} = E_7$.

Inspecting the tables above, we see that the maximum value of $\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}}$ is equal to $\frac{30}{54}$, and is attained when u has Jordan form (2^6) in D_6 and is in the class $(A_1^3)^{(1)}$ of E_7. Hence for $\mathcal{L}' = D_6$ we have $\alpha(\mathcal{L}) = \frac{5}{9}$, as in Table 1 of Theorem 1.7.

Now suppose $\mathcal{L}$ is not one of the Levi subgroups listed for $\mathcal{G} = E_7$ in Table 1 (i.e. $\mathcal{L}$ does not have a factor E_6, D_r or A_r ($r \geq 3$). Then $\dim \mathcal{L} \leq \dim A_2A_2A_1T_2 = 21$.

Let u be a nontrivial unipotent element in $\mathcal{L}$, and assume for a contradiction that

$$\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} > \frac{1}{6}.$$

Now $\dim u^{\mathcal{L}} \leq \dim \mathcal{L} - 7 \leq 14$, and hence $\dim u^{\mathcal{G}} < 84$. It then follows from Table 22.1.2 of [33] that u is in one of the following E_7-classes :

$$A_1, A_1^2, (A_1^3)^{(1)}, (A_1^3)^{(2)}, A_1^4, A_2, A_2A_1, A_2A_1^2.$$

For these classes the maximum possible value of $\dim u^{\mathcal{L}}$ occurs for $\mathcal{L}' = A_2A_2A_1$ or A_1^4, and is as follows:

u	A_1	A_1^2	$(A_1^3)^{(1)}$	$(A_1^3)^{(2)}$	A_1^4	A_2	A_2A_1	$A_2A_1^2$
max $\dim u^{\mathcal{L}}$, $\mathcal{L}' = A_2A_2A_1$	4	8	–	10	–	6	10	12
max $\dim u^{\mathcal{L}}$, $\mathcal{L}' = A_1^4$	2	4	6	6	8	–	–	–
$\dim u^{\mathcal{G}}$	34	52	54	64	70	66	76	82

In all cases we see that $\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}} < \frac{1}{6}$, which is a contradiction. This completes the proof of Theorem 1.7 for $\mathcal{G} = E_7$. ∎

4.5. **Proof of Corollary 1.8 and Theorem 1.9.** The proof of Corollary 1.8 is immediate, since

$$1 - \frac{1}{2}\frac{\dim y^{\mathcal{G}}}{\dim \mathcal{G}} = \frac{1}{2}\left(1 + \frac{\dim \mathbf{C}_{\mathcal{G}}(y)}{\dim \mathcal{G}}\right) = \frac{1}{2}\left(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}}\right),$$

and the right hand side above is at least $\alpha(\mathcal{L})$ by Theorems 1.6 and 1.7.

∎

To prove Theorem 1.9, note that $\chi(1) \geq q^r/3$ by [24]. Also, $\mathbf{C}_G(g) \leq \mathbf{C}_G(y) = L$, so by Theorem 1.1 and the inequality $\alpha(L) \leq 1 - \frac{1}{2}\frac{\dim y^{\mathcal{G}}}{\dim \mathcal{G}}$ obtained above, we have

$$|\chi(g)| \leq f(r)\,\chi(1)^{1-\frac{1}{2}\frac{\dim y^{\mathcal{G}}}{\dim \mathcal{G}}}.$$

Hence it suffices to prove that $\gamma r \geq cs$, where $\gamma := (\dim y^{\mathcal{G}})/(2\dim \mathcal{G})$ and $s := \operatorname{supp}(y)$. Define $a := 1$ if $\mathcal{G} = \mathrm{SL}_n(\mathbb{K})$ and $a := 1/2$ otherwise.

Lemma 3.4 of [34] relates the support of elements of prime order in G with the size of their conjugacy class. The proof of this lemma only uses the fact that these elements are semisimple or unipotent. Since $y \in \mathbf{Z}(L)$ is semisimple, the lemma applies and shows in particular that $|y^G| \geq c' q^{ans}$, where $c' > 0$ is an absolute constant. This implies that $\dim y^{\mathcal{G}} \geq ans$, and so

$$\gamma r \geq \frac{ans}{2\dim \mathcal{G}}r = \frac{anr}{2\dim \mathcal{G}}s = cs,$$

as needed.

∎

4.6. **Bounds for $\mathrm{GL}_n(q)$: proof of Theorem 1.10 and Corollary 1.11.** Let $\mathbb{K}$ an algebraically closed field of characteristic p, and let $\mathcal{L} = \mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_m}(\mathbb{K})$, so that the Levi subgroup L in Theorem 1.10 can be viewed as $\mathcal{L}^F$ for a suitable Frobenius endomorphism F. Fix n pairwise distinct elements $\lambda_1, \ldots, \lambda_n \in \mathbb{K}^\times$. The statements follow from Theorem 1.1 if $n_{i_0} = 1$, so we will assume that $n_{i_0} \geq 2$.

Any unipotent element $u \in L$ can be written as $\operatorname{diag}(u_1, \ldots, u_m)$, where $u_i \in \mathcal{L}_i := \mathrm{GL}_{n_i}(\mathbb{K})$ is unipotent. Write $u_i = J_{b_{i1}} \oplus \ldots \oplus J_{b_{ir_i}}$ for a partition $\nu_i := (b_{i1} \geq b_{i2} \geq \ldots \geq b_{ir_i} \geq 1)$ of n_i, and define

$$s_i := \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{b_{i1}}, \lambda_1, \lambda_2, \ldots, \lambda_{b_{i2}}, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_{b_{ir_i}}) \in \mathcal{L}_i$$

Note that if a_{ij} is the multiplicity of λ_j as an eigenvalue of s_i, then $(a_{i1} \geq a_{i2} \geq \ldots \geq a_{in} \geq 0)$ is the partition of n_i conjugate to ν_i. Now Lemma 4.1 shows that

$$\dim u_i^{\mathcal{L}_i} = \dim s_i^{\mathcal{L}_i} = n_i^2 - \sum_{j=1}^{n} a_{ij}^2.$$

Similarly, setting $s := \operatorname{diag}(s_1, \ldots, s_m) \in \mathcal{L}$, we then get

$$\dim u^{\mathcal{G}} = \dim s^{\mathcal{G}} = n^2 - \sum_{j=1}^{n}(\sum_{i=1}^{m} a_{ij})^2.$$

Note that $u \neq 1$ precisely when $\max_{1 \leq i \leq m} a_{i2} > 0$. Thus $\alpha(L) = \beta(n_1, \ldots, n_m)$. Now Theorem 1.10 follows immediately from Theorem 1.1 and Theorem 4.13(i) below,

Note by the Cauchy-Schwartz inequality that $(\sum_{i=1}^{m} a_{ij})^2 \leq m \sum_{i=1}^{m} a_{ij}^2$ for each j, with equality attained exactly when $a_{1j} = a_{2j} = \ldots = a_{mj}$. Setting $\Delta := \sum_{i,j} a_{ij}^2$, we have

$$\beta(n_1, \ldots, n_m) \leq \max_{\Delta} \frac{(\sum_{i=1}^{m} n_i^2) - \Delta}{n^2 - m\Delta}. \tag{4.11}$$

Now suppose $n_i = n/m$ for $i = 1, \ldots, m$. Then $\sum_{i=1}^{m} n_i^2/n^2 = 1/m$, and so (4.11) implies that $\beta(n_1, \ldots, n_m) \leq 1/m$. In fact equality holds if we choose $a_{1j} = a_{2j} = \ldots = a_{mj}$ for all j. Thus Corollary 1.11 follows. ∎

In what follows, for any partitions $\alpha = (a_1 \geq a_2 \geq \ldots \geq a_n \geq 0) \vdash A$ and $\beta = (b_1 \geq b_2 \geq \ldots \geq b_n \geq 0) \vdash B$ of $A, B \geq 1$ we set

$$g(\alpha) := A^2 - \sum_{i=1}^{n} a_i^2, \ h(\alpha) := \frac{g(\alpha)}{A}, \ \alpha + \beta := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \vdash (A+B). \quad (4.12)$$

Lemma 4.12. *Let $\alpha = (a_1 \geq a_2 \geq \ldots \geq a_n \geq 0) \vdash A$ and $\beta = (b_1 \geq b_2 \geq \ldots \geq b_n \geq 0) \vdash B$ be two partitions of $A, B \geq 1$. Then $h(\alpha) + h(\beta) \leq h(\alpha + \beta)$.*

Proof   We need to show that

$$\frac{A^2 - \sum_{i=1}^{n} a_i^2}{A} + \frac{B^2 - \sum_{i=1}^{n} b_i^2}{B} \leq \frac{(A+B)^2 - \sum_{i=1}^{n}(a_i + b_i)^2}{A+B},$$

equivalently, $\Gamma \geq 0$, where

$$\begin{aligned}
\Gamma &:= AB \sum_{i \neq j}(a_i + b_i)(a_j + b_j) - (A+B)\left(B \sum_{i \neq j} a_i a_j + A \sum_{i \neq j} b_i b_j\right) \\
&= AB \sum_{i \neq j}(a_i b_j + a_j b_i) - \left(B^2 \sum_{i \neq j} a_i a_j + A^2 \sum_{i \neq j} b_i b_j\right) \\
&= AB\left(2AB - 2\sum_i a_i b_i\right) - B^2\left(A^2 - \sum_i a_i^2\right) - A^2\left(B^2 - \sum_i b_i^2\right) \\
&= B^2 \sum_i a_i^2 + A^2 \sum_i b_i^2 - 2AB \sum_i a_i b_i.
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$2AB \sum_i a_i b_i \leq 2 \cdot B\left(\sum_i a_i^2\right)^{1/2} \cdot A\left(\sum_i b_i^2\right)^{1/2} \leq B^2 \sum_i a_i^2 + A^2 \sum_i b_i^2,$$

and the claim follows. ∎

Theorem 4.13. *In the notation of Theorem 1.10, assume that*

$$n_1 = n_2 = \ldots = n_t > n_{t+1} \geq \ldots \geq n_m \geq 1.$$

Then the following statements hold.

(i) $n_1/n \geq \beta(n_1, \ldots, n_m) \geq (n_1 - 1)/(n - t)$.

(ii) *If $m = 2$, then $\beta(n_1, n_2) = (n_1 - 1)/(n - t)$. Moreover, if*

$$1 \neq u = \mathrm{diag}(u_1, u_2) \in \mathrm{GL}_{n_1}(q) \times \mathrm{GL}_{n_2}(q) = L$$

is a unipotent element, then $(\dim u^{\mathcal{L}})/(\dim u^{\mathcal{G}}) = \alpha(L)$ precisely when one of the following conditions holds.

(a) $n_1 = n_2$, *and u_1 and u_2 have the same Jordan canonical form.*

(b) $n_1 > n_2$, *u_1 is a transvection and $u_2 = 1$.*

(c) $n_1 = n_2 + 1$, *and the sizes of Jordan blocks for u_1 and u_2 are*

$$c_1 \geq \ldots \geq c_{j-1} \geq c_j \geq c_{j+1} \geq \ldots \geq c_s, \ c_1 \geq \ldots \geq c_{j-1} \geq c_j - 1 \geq c_{j+1} \geq \ldots \geq c_s,$$

respectively.

Proof (i) To prove the lower bound for $\beta(n_1, \ldots, n_m)$, we choose $(a_{i1}, \ldots, a_{in})$ to be $(n_1 - 1, 1, 0, \ldots, 0)$ if $1 \leq i \leq t$ and $(n_i, 0, \ldots, 0)$ otherwise. To prove the upper bound, for $1 \leq i \leq m$ consider the partition $\alpha_i := (a_{i1}, a_{i2}, \ldots, a_{in}) \vdash n_i$. By Lemma 4.12 we have

$$\sum_{i=1}^{m} \frac{g(\alpha_i)}{n_i} = h(\alpha_1) + h(\alpha_2) + \ldots + h(\alpha_m) \leq h(\sum_{i=1}^{m} \alpha_i) = \frac{g(\sum_{i=1}^{m} \alpha_i)}{n} \qquad (4.13)$$

in the notation of (4.12). Note that the condition $\max_{1 \leq i \leq m} a_{i2} > 0$ ensures that $g(\sum_{i=1}^{m} \alpha_i) > 0$. Since $n_1 = \max_{1 \leq i \leq m} n_i$, (4.13) now implies that

$$\beta(n_1, n_2, \ldots, n_m) = \frac{\sum_{i=1}^{m} g(\alpha_i)}{g(\sum_{i=1}^{m} \alpha_i)} \leq \frac{n_1 (\sum_{i=1}^{m} g(\alpha_i)/n_i)}{g(\sum_{i=1}^{m} \alpha_i)} \leq \frac{n_1}{n}.$$

(ii) We may assume $A := n_1 > B := n_2$ by Corollary 1.11 and its proof. To ease the notation, also write

$$(a_{11}, a_{12}, \ldots, a_{in}) = (a_1, a_2, \ldots, a_n), \quad (a_{21}, a_{22}, \ldots, a_{2n}) = (b_1, b_2, \ldots, b_n).$$

Then we need to show that

$$\frac{\sum_{1 \leq i \neq j \leq n}(a_i a_j + b_i b_j)}{\sum_{1 \leq i \neq j \leq n}(a_i + b_i)(a_j + b_j)} \leq \frac{A - 1}{A + B - 1},$$

equivalently, $\Sigma \geq 0$, where

$$\begin{aligned}
\Sigma &:= (A - 1) \sum_{i \neq j}(a_i + b_i)(a_j + b_j) - (A + B - 1) \sum_{i \neq j}(a_i a_j + b_i b_j) \\
&= (A - 1) \sum_{i \neq j}(a_i b_j + a_j b_i) - B \sum_{i \neq j}(a_i a_j + b_i b_j) \\
&= (A - 1)(2AB - 2 \sum_i a_i b_i) - B(A^2 + B^2 - \sum_i a_i^2 - \sum_i b_i^2) \\
&= B(A^2 - 2A - B^2) + B(\sum_i a_i^2 + \sum_i b_i^2) - 2(A - 1) \sum_i a_i b_i \\
&= B(A^2 - 2A - B^2) + B \sum_i (a_i - b_i)^2 - 2(A - 1 - B) \sum_i a_i b_i \\
&= (A - 1 - B)((A - 1 + B)B - 2 \sum_i a_i b_i) + B(\sum_i (a_i - b_i)^2 - 1) \\
&= (A - 1 - B)(\sum_i (A - 1 + B - 2a_i)b_i) + B(\sum_i (a_i - b_i)^2 - 1).
\end{aligned}$$

Note that the condition $A \geq B + 1$ implies that $\sum_i (a_i - b_i)^2 \geq 1$, with equality attained exactly when

$$A = B + 1, \quad (a_1, a_2, \ldots, a_n) = (b_1, \ldots, b_{i-1}, b_i + 1, b_{i+1}, \ldots, b_n). \qquad (4.14)$$

First we consider the case when $A - 1 + B \geq 2a_i$ for all i. As $B \geq 1$, we see that $\Sigma \geq 0$, with equality attained exactly when (4.14) holds, which means the corresponding unipotent element satisfies (c).

Suppose now that $A - 1 + B \leq 2a_i - 1$ for some i. Then $a_i \geq \sum_{j \neq i} a_j + B$. As $B \geq 1$ and $a_1 \geq a_2 \geq \ldots \geq a_n$, this can happen only for one index i, and this index i is 1, and so

$$a_1 \geq A' + B, \qquad (4.15)$$

where $A' := \sum_{j \geq 2} a_j$. In particular, $A - 1 + B \geq 2a_j$ for all $j \geq 2$. Now by (4.15) we have

$$\begin{aligned}
\Sigma' :=&(A - 1 - B)((A - 1 + b_1)b_1 - 2a_1 b_1) + b_1(a_1 - b_1)^2 \\
=& b_1((A - 1 - B)(b_1 - 1 + A' - a_1) + (a_1 - b_1)^2) \\
=& b_1((a_1 - b_1)(A' - 1) + (A' - 1 - B')(b_1 - 1 + A' - a_1)) \\
=& b_1((a_1 - b_1)B' + (A' - 1)^2 - B'(A' - 1)) \\
=& b_1((a_1 - A' - b_1 + 1)B' + (A' - 1)^2) \geq 0,
\end{aligned}$$

with equality exactly when

$$B' := \sum_{j \geq 2} b_j = 0, \ A' = 1. \tag{4.16}$$

It follows that, if $\sum_{j \geq 2}(a_j - b_j)^2 \geq 1$, then $\Sigma \geq 0$, with equality exactly when (4.16) holds, which means the corresponding unipotent element satisfies (b).

Assume finally that $\sum_{j \geq 2}(a_j - b_j)^2 \leq 0$. Then $a_j = b_j$ for all $j \geq 2$ and $A' = B'$. As $\max(a_2, b_2) > 0$, we must have $A' = B' \geq 1$, and so by (4.15)

$$\Sigma' = b_1((a_1 - b_1 - 1)A' + 1) \geq 2b_1,$$

yielding $\Sigma \geq b_1 \geq 1$. $\blacksquare$

We note that Theorem 4.13(i) was inspired by some correspondence with M. Fraczyk who is studying the situation in Theorem 1.10 using different methods.

5. Random walks

In this section we prove Theorems 1.12–1.15 concerning random walks and covering numbers.

Proof of Theorem 1.12

Suppose $\mathcal{G}$ is a simple algebraic group of rank r in good characteristic, and $G = G(q) = \mathcal{G}^F$ is a finite quasisimple group over $\mathbb{F}_q$. Let $y \in G$ be such that $\mathbf{C}_G(y) \leq L = \mathcal{L}^F$ for a proper split Levi subgroup $\mathcal{L}$ of $\mathcal{G}$. Write $C = y^G$, and let h be the Coxeter number of $\mathcal{G}$.

For a real number s, define

$$\zeta^G(s) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s}. \tag{5.1}$$

We will need the following result, which is [37, Theorem 1.1].

Lemma 5.1. *If $s > \frac{2}{h}$, then $\zeta^G(s) \to 1$ as $q \to \infty$.*

We first prove part I(a) of Theorem 1.12 together with the first statement of part (II) (the $C^6 = G$ statement). We will prove the mixing time assertions later.

Let t be a positive integer. By a well-known result (see [2, Chapter 1, 10.1]), for $g \in G$ the number of ways of writing g as a product of t conjugates of y is

$$N(g) = \frac{|C|^t}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(y)^t \chi(g^{-1})}{\chi(1)^{t-1}}.$$

Define $P^t(g) = \frac{N(g)}{|C|^t}$, the probability that a random product of t conjugates of y is equal to g, and let $U(g) = \frac{1}{|G|}$, the uniform probability distribution on G. Then

$$|P^t(g) - U(g)| \leq \frac{1}{|G|} \sum_{\chi(1) > 1} \left(\frac{|\chi(y)|}{\chi(1)} \right)^t \chi(1)^2. \tag{5.2}$$

Define
$$||P^t - U||_\infty = |G| \max_{g \in G} |P^t(g) - U(g)|.$$

Write $\alpha = \alpha(\mathcal{L})$. Then Theorem 1.1 gives $\frac{|\chi(y)|}{\chi(1)} \leq f(r)\chi(1)^{\alpha-1}$, and so (5.2) implies

$$\begin{aligned} ||P^t - U||_\infty &\leq f(r)^t \sum_{\chi(1)>1} \chi(1)^{t(\alpha-1)+2} \\ &= f(r)^t \left(\zeta^G(t(1-\alpha) - 2) - 1 \right). \end{aligned}$$

By Lemma 5.1, $\zeta^G(t(1-\alpha) - 2) - 1 \to 0$ as $q \to \infty$ provided

$$t(1-\alpha) - 2 > \frac{2}{h}. \tag{5.3}$$

If $\mathcal{G}$ is of exceptional type G_2, F_4, E_6, E_7 or E_8, then $\frac{2}{h}$ is $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{9}$ or $\frac{1}{15}$ respectively, and Theorem 1.7 shows that (5.3) holds in all cases, provided $t \geq 6$. This proves the first statement of Theorem 1.12(II).

Now suppose $\mathcal{G}$ is of classical type. Then $\alpha \leq \frac{1}{2}(1 + \frac{\dim \mathcal{L}}{\dim \mathcal{G}})$ by Theorem 1.6. This implies that (5.3) holds provided $t > (4 + \frac{4}{h})\frac{\dim \mathcal{G}}{\dim \mathcal{G} - \dim \mathcal{L}}$, proving Theorem 1.12(I)(a).

We now prove the assertions on mixing times in Theorem 1.12. For these we use the Diaconis–Shashahani bound [6]:

$$(||P^t - U||_1)^2 \leq \sum_{\chi \in \mathrm{Irr}(G), \chi \neq 1} \left(\frac{|\chi(y)|}{\chi(1)} \right)^{2t} \chi(1)^2. \tag{5.4}$$

As above, Theorem 1.1 shows that the right hand side of (5.4) is less than $f(r)^{2t} \left(\zeta^G(2t(1-\alpha) - 2) - 1 \right)$, and hence tends to 0 as $q \to \infty$ provided $2t(1-\alpha) - 2 > \frac{2}{h}$. Using Theorems 1.7 and 1.6, we now see as before that this inequality holds provided $t \geq 3$ when $\mathcal{G}$ is of exceptional type, and provided $t > (2 + \frac{2}{h})\frac{\dim \mathcal{G}}{\dim \mathcal{G} - \dim \mathcal{L}}$ when $\mathcal{G}$ is classical. This proves the mixing time assertions, completing the proof of Theorem 1.12. ∎

Proof of Theorem 1.15

This is very similar to the previous proof, using Corollary 3.5 instead of Theorem 1.1. Let $G = \mathrm{SL}_n(q)$ and let $u \in G$ be a non-identity unipotent element. Let $t \in \mathbb{N}$, and for $g \in G$ let $P^t(g)$ be the probability that a random product of t conjugates of u is equal to g, and $U(g) = \frac{1}{|G|}$. As in (5.2),

$$|P^t(g) - U(g)| \leq \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G), \chi(1)>1} \left(\frac{|\chi(u)|}{\chi(1)} \right)^t \chi(1)^2.$$

By Corollary 3.5, $\frac{|\chi(u)|}{\chi(1)} \leq g(n)\,\chi(1)^{-\frac{1}{n-1}}$ for $\chi \in \mathrm{Irr}(G)$, and hence

$$\begin{aligned} ||P^t - U||_\infty &\leq g(n)^t \sum_{\chi(1)>1} \chi(1)^{-\frac{t}{n-1}+2} \\ &= g(n)^t \left(\zeta^G(\frac{t}{n-1} - 2) - 1 \right). \end{aligned}$$

By Lemma 5.1, $\zeta^G(\frac{t}{n-1} - 2) - 1 \to 0$ as $q \to \infty$ provided $\frac{t}{n-1} - 2 > \frac{2}{n}$, which holds provided $t > 2n$. This proves part (i) of Theorem 1.15. Part (ii) is proved in the same way, using the bound (5.4). ∎

Corollaries 1.13 and 1.14

Corollary 1.13 follows immediately from Theorem 1.12(I)(b). Corollary 1.14 is proved exactly as above, using Theorem 1.5.

Next, we use some well-known observations to justify the remarks made after the statement of Theorem 1.12.

Lemma 5.2. (i) *Let G be a finite group, and let S be a generating subset of G that satisfies $|S^N| < |G|(1 - 1/e)$ for some integer $N \geq 1$. Then the mixing time $T(G, S)$ of the random walk on the Cayley graph corresponding to S is at least $N + 1$.*

(ii) *Let $G = \mathrm{SL}_n(q)$ with $n \geq 2$ and $S = y^G$ with $y = \mathrm{diag}(\mu I_{n-1}, \lambda)$, where $\mu, \lambda \in \mathbb{F}_q^\times$ and $\mu \neq \lambda$. Then $T(G, y) \geq n$.*

Proof (i) Define $P(g)$ to be $1/|S|$ if $g \in S$ and 0 otherwise, and let $U(g) = 1/|G|$ for all $g \in G$. Consider any $1 \leq k \leq N$. Note that $|S^k| \leq |S^{k+1}|$ and so $|S^k| \leq |S^N|$, whence

$$||P^k - U||_1 \geq \sum_{g \in G \smallsetminus S^k} |P^k(g) - U(g)| = \sum_{g \in G \smallsetminus S^k} |U(g)| \geq \frac{|G \smallsetminus S^k|}{|G|} > 1/e.$$

It follows that $T(G, S) \geq N + 1$.

(ii) Note that $(y^G)^{n-1}$ is contained in X, the set of elements $x \in G$ that have eigenvalue μ on $V = \mathbb{F}_q^n$. Now if we fix $0 \neq v \in V$ and let $Y := \{x \in G \mid x(v) = \mu v\}$, then it is easy to see that $|\mathbf{N}_G(Y)|/|Y| \geq q - 1 \geq 2$. Hence,

$$|X| = |\cup_{g \in G} gYg^{-1}| \leq |Y| \cdot [G : \mathbf{N}_G(Y)] \leq |G|/2.$$

Now we can apply (i) to $S := y^G$. ∎

We conclude with a proof of our last theorem, connecting the mixing times of random walks on classical groups with the support of certain elements.

Proof of Theorem 1.17

Set $s := \mathrm{supp}(y)$. Then $\mathbf{C}_G(g) \leq \mathbf{C}_G(y) = L$. Theorem 1.12 I(b) gives

$$T(G, g) \leq \left\lceil (2 + \frac{2}{h}) \frac{\dim \mathcal{G}}{\dim \mathcal{G} - \dim \mathcal{L}} \right\rceil$$

for large q. Now, $\dim \mathcal{G} - \dim \mathcal{L} = \dim \mathcal{G} - \dim \mathbf{C}_{\mathcal{G}}(y) = \dim y^{\mathcal{G}} \geq ans$ as shown in the proof of Theorem 1.9. This yields

$$T(G, g) \leq \left\lceil (2 + \frac{2}{h}) \frac{\dim \mathcal{G}}{ans} \right\rceil.$$

Let $c = c(\mathcal{G})$ be as in Theorem 1.9. Then we have $\frac{\dim \mathcal{G}}{an} = \frac{r}{c} = r'$. We obtain

$$T(G, g) \leq \left\lceil (2 + \frac{2}{h}) r'/s \right\rceil,$$

proving the first assertion.

It remains to prove the lower bound on $T(G, y)$. By (1.3) we have

$$T(G, y) \geq \frac{\log |G| + \log(1 - e^{-1})}{\log |y^G|} \gtrsim_{|G|} \frac{\dim \mathcal{G}}{\dim y^{\mathcal{G}}}.$$

It follows from [34, 3.4] and its proof that, for y semisimple, we have $|y^G| \leq 2ans$. Hence $\dim y^{\mathcal{G}} \leq 2ans$, which, combined with the inequality above, implies

$$T(G, y) \gtrsim_{|G|} \frac{\dim \mathcal{G}}{2ans} \geq \frac{\dim \mathcal{G}}{2an}/s = \frac{1}{2} r'/s,$$

as required. ∎

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