CHIRAL POLYHEDRA AND FINITE SIMPLE GROUPS

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Abstract. We prove that every finite non-abelian simple group acts as the automorphism group of a chiral polyhedron, apart from the groups $PSL_2(q)$, $PSL_3(q)$, $PSU_3(q)$ and A_7.

1. Introduction

Polyhedra and their generalisations to higher ranks, polytopes, are certain ranked partially ordered sets generalising geometric objects that have been studied since the Greeks (see [24, Chapter 1]). Those polytopes whose automorphism group acts transitively on maximal flags are called regular. They have maximum possible rotational and reflectional symmetries. Those that are chiral have maximum rotational symmetries but no reflections. It is already known which finite simple groups are automorphism groups of abstract regular polyhedra (see below for more details). The main purpose of this article is to determine which finite simple groups are automorphism groups of chiral polyhedra.

In order to state the main results we require precise definitions of the above terms. Following [24], an (abstract) *polytope* $(\mathcal{P}, \leq)$ *of rank* n is a partially ordered set with a rank function ranging from -1 to n and satisfying the following properties. The elements of rank i are called the i-faces of $\mathcal{P}$. There exists a unique least face F_{-1} and a unique greatest face F_n. The *flags* are the maximal totally ordered subsets of $\mathcal{P}$ and they must all contain exactly $n+2$ faces of pairwise distinct rank. Two flags Φ and Ψ are called *adjacent* if they differ in exactly one face. They are called i-adjacent if this face is an i-face. The poset $\mathcal{P}$ must be *strongly connected*, that is, every pair of flags must be connected by a path of adjacent flags in $\mathcal{P}$. Finally we require that for any $(i-1)$-face F and any $(i+1)$- face G of $\mathcal{P}$ such that $F \leq G$, there are exactly two i-faces between F and G. If the rank of $\mathcal{P}$ is 3, we call $\mathcal{P}$ a *polyhedron*.

An automorphism of $(\mathcal{P}, \leq)$ is a bijection of the faces of $\mathcal{P}$ that preserves the order $\leq$. The set of all automorphisms of $(\mathcal{P}, \leq)$ with composition forms a group called the *automorphism group* of $(\mathcal{P}, \leq)$ and denoted $\Gamma(\mathcal{P})$. If $\Gamma(\mathcal{P})$ has a unique orbit on the flags of $\mathcal{P}$, we say that $\mathcal{P}$ is *regular*. If $\Gamma(\mathcal{P})$ has

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two orbits such that any two adjacent flags belong to distinct orbits, we say that $\mathcal{P}$ is *chiral*.

As defined for instance in [24], a C-*group* is a group G generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$ which satisfy the following *intersection property*:

$$\forall J, K \subseteq \{0, \ldots, n-1\}, \langle \rho_j \mid j \in J\rangle \cap \langle \rho_k \mid k \in K\rangle = \langle \rho_j \mid j \in J \cap K\rangle.$$

A C-group $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a *string* C-group if its generators satisfy the following relations:

$$(\rho_j \rho_k)^2 = 1 \ \forall j, k \in \{0, \ldots n-1\} \text{ with } \mid j-k \mid \geq 2.$$

In [24] it is shown that string C-groups and abstract regular polytopes are in one-to-one correspondence. Every string C-group gives an abstract regular polytope and, given an abstract regular polytope and one of its base flags, one can construct a set of distinguished generators that, together with the automorphism group of the polytope, give a string C-group. In particular, the automorphism group of an abstract regular polyhedron is generated by three involutions ρ_0, ρ_1, ρ_2, two of which commute (namely, ρ_0, ρ_2). In 1980, it was asked in the Kourovka Notebook (Problem 7.30) which finite simple groups have this property. This was solved by Nuzhin and others in [25, 26, 27, 28, 23]: every non-abelian finite simple group can be generated by three involutions, two of which commute, with the following exceptions:

$$\begin{array}{l} PSL_3(q),\ PSU_3(q),\ PSL_4(2^n),\ PSU_4(2^n), \\ A_6,\ A_7,\ M_{11},\ M_{22},\ M_{23},\ McL. \end{array}$$

The groups $PSU_4(3)$ and $PSU_5(2)$, although mentioned by Nuzhin as being generated by three involutions, two of which commute, have recently been discovered not to have such generating sets by Martin Macaj and Gareth Jones (personal communication). Thus every finite simple group, apart from the above exceptions, is the automorphism group of an abstract regular polyhedron.

Similarly, in [29], it is shown that for a finite group G, the chiral polyhedra having G as automorphism group are in bijective correspondence with pairs $x, t \in G$ satisfying the following conditions:

(i) $G = \langle x, t\rangle$;
(ii) t is an involution;
(iii) there is no involution $\alpha \in \mathrm{Aut}(G)$ such that $x^\alpha = x^{-1}$, $t^\alpha = t$.

Our main result classifies those finite simple groups G possessing such generators:

Theorem 1.1. *Let G be a non-abelian finite simple group, not A_7, $PSL_2(q)$, $PSL_3(q)$ or $PSU_3(q)$. Then there exist $x, t \in G$ such that the following hold:*

(i) *$G = \langle x, t\rangle$;*
(ii) *t is an involution;*
(iii) *there is no involution $\alpha \in \mathrm{Aut}(G)$ such that $x^\alpha = x^{-1}$, $t^\alpha = t$.*

As a consequence, for every nonabelian finite simple group G except A_7, $PSL_2(q)$, $PSL_3(q)$ or $PSU_3(q)$, there exists an abstract chiral polyhedron having automorphism group G. This result was already known to be true for the Suzuki groups [16, 13], the small Ree groups [14], the alternating groups (see Section 5 for details) and some small sporadic groups [12].

We now discuss the exceptions in Theorem 1.1. The groups $PSL_2(q)$ do not have pairs of elements x, t satisfying (i)–(iii) of the theorem; this a consequence of a result of Macbeath [22], as observed by Singerman [31, Theorem 3]. For the group A_7, an exhaustive computer search shows that no pair of elements x, t satisfying (i)–(iii) of Theorem 1.1 exists – see [12].

Thus it remains to consider the groups $PSL_3(q)$ and $PSU_3(q)$; it will be shown that these also do not possess generators x, t satisfying conditions (i)–(iii) in a forthcoming paper [19].

The rest of the paper is devoted to the proof of Theorem 1.1. This is divided into four cases: namely, the case where G is of exceptional Lie type (Section 3), the case where G is classical (Section 4), the case where G is an alternating group (Section 5), and the case where G is a sporadic group (Section 6).

Observe that chiral polyhedra (and regular polyhedra) may be seen also as regular maps. In a recent paper [15], Gareth Jones has studied much further the link between automorphism groups of edge-transitive maps and finite simple groups.

2. Preliminaries

In this section we prove some lemmas needed for the proof of Theorem 1.1, The first two lemmas are straightforward hence we omit their proofs.

Lemma 2.1. *Let G be a finite group, and suppose $x, t \in G$ satisfy $x^t = x^{-1}$. Then the set $\{y \in G : x^y = x^{-1}\}$ is contained in the coset $C_G(x)t$.*

Lemma 2.2. *Let G be a finite group with a subgroup A. Suppose M is a maximal subgroup of G containing A, such that any two G-conjugates of A that are contained in M are M-conjugate. Then the number of G-conjugates of M containing A is $|N_G(A) : N_M(A)|$.*

For a finite group G and a positive integer r, denote by $I_r(G)$ the set of elements of order r in G, and let $i_r(G) = |I_r(G)|$.

The next three lemmas provide upper and lower bounds for the numbers of involutions $i_2(G)$ in groups of Lie type. Recall that a simple group of Lie type over a finite field $\mathbb{F}_q$ can be written as $(\bar{G}^F)'$, the derived group of the fixed point group of a Frobenius endomorphism F of the corresponding simple adjoint algebraic group $\bar{G}$ over $\bar{\mathbb{F}}_q$.

We use the notation $L_n^\epsilon(q)$ (where $\epsilon = \pm$) to denote $PSL_n(q)$ when $\epsilon = +$, and $PSU_n(q)$ when $\epsilon = -$. Similarly, $E_6^\epsilon(q)$ denotes $E_6(q)$ when $\epsilon = +$ and ${}^2E_6(q)$ when $\epsilon = -$.

Lemma 2.3. *Let G be a finite simple group of Lie type over $\mathbb{F}_q$, and write $G = (\bar{G}^F)'$ as above. Let r be the rank of $\bar{G}$, and N the number of positive roots in the root system of $\bar{G}$. Define*

$$M_G = \begin{cases} \frac{1}{2}(N+r), & \text{if } G \text{ is of type } {}^2F_4,\ {}^2G_2 \text{ or } {}^2B_2, \\ N+r, & \text{otherwise.} \end{cases}$$

Then $i_2(\mathrm{Aut}(G)) < 2\left(q^{M_G} + q^{M_G-1}\right)$. Moreover, $i_2(L_2(q)) < q^2$.

Proof. The first assertion is [18, Prop. 1.3]. The second follows from the well known fact that $L_2(q)$ has one class of involutions, of size q^2-1 for q even, and of size$\frac{1}{2}q(q\pm 1)$ for q odd. □

For the lower bounds we need the following elementary observation, which is proved exactly as in the proof of [18, Lemma 1.2(i)].

Lemma 2.4. *Suppose $\{a_1,\dots,a_l\}$ and $b_1,\dots,b_m\}$ are two sets of distinct integers, all at least 2.*

(i) *For $q \ge 2$ and $\epsilon = \pm 1$,*

$$\frac{\prod_1^l (q^{a_i}-\epsilon)}{\prod_1^m (q^{b_i}-\epsilon)} > \frac{1}{2} q^{\sum a_i - \sum b_i}.$$

(ii) *For $q \ge 3$,*

$$\frac{\prod_1^l (q^{a_i}-1)}{\prod_1^m (q^{b_i}+1)} > \frac{2}{3} q^{\sum a_i - \sum b_i}.$$

Lemma 2.5. *Let G be a finite simple group of Lie type over $\mathbb{F}_q$.*

(i) *If G is of classical type, then $i_2(G) > \frac{1}{4}q^{N_G}$, where N_G is defined as follows:*

G	N_G
$L_n^\epsilon(q)$ $(\epsilon = \pm)$	$[\frac{1}{2}n^2]$
$PSp_{2m}(q)$, $P\Omega_{2m+1}(q)$	m^2+m
$P\Omega_{2m}^\epsilon(q)$ $(\epsilon = \pm)$	m^2-1

(ii) *If G is of exceptional type, then with one exception $i_2(G) > \frac{1}{2}q^{N_G}$, where N_G is defined below; the exception is $G = E_7(q)$, in which case $i_2(G) > \frac{1}{4}q^{N_G}$.*

G	$E_8(q)$	$E_7(q)$	$E_6^\epsilon(q)$	$F_4(q)$	$G_2(q)$	${}^2F_4(q)$	${}^2G_2(q)$	${}^2B_2(q)$	${}^3D_4(q)$
N_G	128	70	40	28	8	14	4	3	16

Proof. An asymptotic version of this result is proved in [21, Props. 4.1, 4.3]. To verify the lemma we need to keep track of the constants in the proofs of those results. To do this for (i), we record in Table 1 some lower bounds for $i_2(G)$ given by the proof of [21, Prop. 4.1]. We then obtain the inequality $i_2(G) > \frac{1}{4}q^{N_G}$ using Lemma 2.4. The proof of part (ii) is similar, using the list of involution centralizers given in the proof of [21, Prop. 4.3]; note that the exceptional bound for $G = E_7(q)$ arises because when q is odd we only have a bound $i_2(G) \ge |E_7(q) : A_7^\epsilon(q).2|$ in this case. □

Table 1. Lower bounds for $i_2(G)$, G classical

G	conditions	$i_2(G) \geq$
$L^{\epsilon}_{2m}(q)$	q odd	$\lvert GL^{\epsilon}_{2m}(q) : GL_m(q^2).2\rvert$ or $\lvert GL^{\epsilon}_{2m}(q) : GL^{\epsilon}_m(q)^2.2\rvert$
	q even	$\lvert GL^{\epsilon}_{2m}(q) : q^{m^2} GL^{\epsilon}_m(q)\rvert$
$L^{\epsilon}_{2m+1}(q)$	q odd	$\lvert GL^{\epsilon}_{2m+1}(q) : GL^{\epsilon}_{m+1}(q) \times GL^{\epsilon}_m(q)\rvert$
	q even	$\lvert GL^{\epsilon}_{2m+1}(q) : q^{m^2+2m} GL^{\epsilon}_m(q) GL^{\epsilon}_1(q)\rvert$
$PSp_{2m}(q)$	q odd	$\lvert Sp_{2m}(q) : GL^{\epsilon}_m(q).2\rvert$
	q even	$\lvert Sp_{2m}(q) : q^{(m^2+m)/2} Sp_{m-1}(q)\rvert$ or $\lvert Sp_{2m}(q) : q^{(m^2+3m-2)/2} Sp_{m-2}(q)\rvert$
$\Omega_{2m+1}(q)$	q odd	$\lvert O_{2m+1}(q) : O^{\epsilon}_{m+1}(q) \times O^{\delta}_m(q)\rvert$
$P\Omega^{\epsilon}_{2m}(q)$	q odd, m odd	$\lvert O^{\epsilon}_{2m}(q) : O^{\delta}_{m+1}(q) \times O^{\nu}_{m-1}(q)\rvert$
	q odd, m even, $\epsilon = -$	$\lvert O^{-}_{2m}(q) : O^{\delta}_m(q) \times O^{-\delta}_m(q)\rvert$
	q odd, m even, $\epsilon = +$	$\lvert O^{+}_{2m}(q) : (O^{\delta}_m(q))^2.2\rvert$
	q even	$\lvert \Omega^{\epsilon}_{2m}(q) : q^{(m^2-m+2)/2} Sp_{m-1}(q)\rvert$ or $\lvert \Omega^{\epsilon}_{2m}(q) : q^{(m^2+m-2)/2} Sp_{m-2}(q)\rvert$

Note that the exponents M_G and N_G in Lemmas 2.3 and 2.5 are equal, except in cases where G possesses an involutory graph automorphism (types $L^{\epsilon}_n, P\Omega^{\epsilon}_{2m}, E^{\epsilon}_6$).

3. Proof of Theorem 1.1 for G of exceptional type

Let G be a finite simple group of exceptional Lie type over $\mathbb{F}_q$ (i.e. of type E_8, E_7, E^{ϵ}_6, F_4, G_2, ${}^2F_4$, ${}^2G_2$, ${}^2B_2$ or ${}^3D_4$). Assume that $q > 2$ when G is of type G_2 or ${}^2F_4$, and $q > 3$ for type ${}^2G_2$ (for the excluded groups, $G_2(2)' \cong U_3(3)$ and ${}^2G_2(3)' \cong L_2(8)$ and these will be dealt with as classical groups in the next section).

Write $G = (\bar{G}^F)'$, the derived group of the fixed point group of a Frobenius endomorphism F of the corresponding simple adjoint algebraic group $\bar{G}$ over $\bar{\mathbb{F}}_q$. Let $d = |\bar{G}^F : G|$, so that

$$d = \begin{cases} (2, q-1), & \text{if } G = E_7(q), \\ (3, q-\epsilon), & \text{if } G = E^{\epsilon}_6(q), \\ 1, & \text{otherwise.} \end{cases}$$

By [17, Section 2], there is a cyclic maximal torus $\langle x\rangle$ of G of order as given in Table 2. In the table, $\Phi_n(q)$ denotes the n^{th} cyclotomic polynomial evaluated at q. Morover, $T = C_{\bar{G}^F}(x)$ is a maximal torus of $\bar{G}^F$ of order $d|\langle x\rangle|$, and also $C_{Aut(G)}(x) = T$ (see [30, 2.8(iii)]).

Let w_0 be the longest element of the Weyl group $W(\bar{G})$. Suppose that $w_0 = -1$ (i.e. $\bar{G} \neq E_6$). Then the involutions in $N_{\bar{G}}(T)$ that invert x are conjugates of t_0, a preimage in $N(T)$ of w_0. Now $C_{\bar{G}}(t_0)$ has dimension equal to the number of positive roots in the root system of $\bar{G}$. Hence we see from the lists of possibilities for involution centralizers in G, in [10, Section 4.5] for q odd, and in [1] for q even, that $C_G(t_0)$ is as given in Table 2.

Similarly, if $\bar{G} = E_6$, the involutions in $\mathrm{Aut}(G)$ inverting x are conjugate to t_0, a preimage in $N_{G\langle\tau\rangle}(T)$ of $w_0\tau = -1$, where τ is an involutory graph automorphism of G, and again $C_G(t_0)$ is as in Table 2.

TABLE 2. Torus $\langle x\rangle$ and centralizer $C_G(t_0)$, G exceptional

G	$\lvert\langle x\rangle\rvert$	$C_G(t_0)$
$E_8(q)$	$\Phi_{30}(q)$	$D_8(q)$, q odd $[q^{84}].B_4(q)$, q even
$E_7(q)$	$\frac{1}{d}\Phi_{18}(q)\Phi_2(q)$, $q \geq 3$ 129, $q = 2$	$A_7^\epsilon(q).2$, $q \equiv \epsilon \bmod 4$ $[q^{42}].B_3(q)$, q even
$E_6^\epsilon(q)$	$\frac{1}{d}\Phi_9(q)$, $\epsilon = +$ $\frac{1}{d}\Phi_{18}(q)$, $\epsilon = -$	$C_4(q)$, q odd $[q^{15}].C_3(q)$, q even
$F_4(q)$	$\Phi_{12}(q)$, $q \geq 3$ 17, $q = 2$	$(A_1(q)C_3(q)).2$, q odd $[q^{18}].A_1(q)^2$, q even
$G_2(q)$	$\Phi_6(q)$, $q \geq 4$ 13, $q = 3$	$(A_1(q)A_1(q)).2$, q odd $[q^3].A_1(q)$, q even
${}^3D_4(q)$	$\Phi_{12}(q)$	$(A_1(q)A_1(q^3)).2$, q odd $[q^9].A_1(q)$, q even
${}^2F_4(q)$	$q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$	$[q^9].A_1(q)$
${}^2G_2(q)$	$q + \sqrt{3q} + 1$	$A_1(q) \times 2$
${}^2B_2(q)$	$q + \sqrt{2q} + 1$	$[q^2]$

TABLE 3. Maximal overgroups of $\langle x\rangle$, G exceptional

G	$\lvert\mathcal{M}(x)\rvert$	Groups in $\mathcal{M}(x)$
$E_8(q)$	0	$-$
$E_7(q)$, $q \geq 3$ $E_7(2)$	1 1	${}^2E_6(q).\frac{q+1}{d}$ ${}^2A_7(2).2$
$E_6^\epsilon(q)$	1	$A_2^\epsilon(q^3).3$
$F_4(q)$, $q \geq 3$ $F_4(2)$	$(2,q)$ 2	${}^3D_4(q).3$ $B_4(2)$
$G_2(q)$, $q \geq 5$ $G_2(4)$ $G_2(3)$	1 2 3	$SU_3(q).2$ $SU_3(4).2$, $L_2(13).2$ $SL_3(3).2$, $SL_3(3).2$, $L_2(13)$
${}^3D_4(q)$, ${}^2F_4(q)$ ${}^2G_2(q)$, ${}^2B_2(q)$	0 0	$-$ $-$

Let $\mathrm{Inv}(x)$ be the set of involutions in $\mathrm{Aut}(G)$ that invert x. By Lemma 2.1 we have $\mathrm{Inv}(x) \subseteq Tt_0$. Define $\mathcal{M}(x)$ to be the set of maximal subgroups of G containing x such that $M \neq N_G(\langle x \rangle)$. It is shown in the proof of [11, Prop. 6.2] that the set $\mathcal{M}(x)$ is as given in Table 3.

Define

$$S = \bigcup_{\alpha \in Inv(x)} I_2(C_G(\alpha)) \cup \bigcup_{M \in \mathcal{M}(x)} I_2(M).$$

We claim that

$$i_2(G) > |S|. \tag{3.1}$$

Given (3.1), there exists an involution $t \in G$ such that

(a) t lies in no maximal subgroup of G containing x, and

(b) t centralizes no involution that inverts x.

It then follows that x, t satisfy conditions (i)-(iii) of Theorem 1.1, completing the proof of the theorem for exceptional groups.

So it remains to prove (3.1). By Lemma 2.1 we have $|\mathrm{Inv}(x)| \leq |T|$, and also from the above discussion we know that $C_G(\alpha)$ is conjugate to $C_G(t_0)$ for all $\alpha \in \mathrm{Inv}(x)$. Hence

$$|S| \leq |T|\, i_2(C_G(t_0)) + \sum_{M \in \mathcal{M}(x)} i_2(M). \tag{3.2}$$

Using Lemma 2.3, we obtain the following upper bounds for $i_2(C_G(t_0))$ and for $\sum_{M\in\mathcal{M}(x)} i_2(M)$:

G	$i_2(C_G(t_0)) <$	$\sum i_2(M) \leq$
$E_8(q)$	$2q^{84}(q^{20}+q^{19})$	0
$E_7(q)$	$2q^{42}(q^{12}+q^{11})$	$2(q+1)(q^{42}+q^{41})$
$E_6^{\epsilon}(q)$	$2q^{15}(q^{12}+q^{11})$	$2(q^{15}+q^{12})$
$F_4(q)$	q^{22}	$2(q^{20}+q^{19})$
$G_2(q)$, $q \geq 5$	q^5	$4(q^5+q^4)$
${}^3D_4(q)$	q^{11}	0
${}^2F_4(q)$, $q > 2$	q^{11}	0
${}^2G_2(q)$, $q > 3$	$2q^2$	0

In all these cases we check that the consequent upper bound for $|S|$ using (3.2) is less than the lower bound for $i_2(G)$ given by Lemma 2.5, proving (3.1).

This leaves the following groups to deal with: $G = {}^2B_2(q)$, $G_2(3)$, $G_2(4)$ and ${}^2F_4(2)'$. In the first case, we see using [32] that $i_2(C_G(t_0)) = q-1$, while $i_2(G) = (q-1)(q^2+1)$, so (3.1) holds. In the other cases we can use [9] to obtain the precise values of $i_2(G)$, $i_2(C_G(t_0))$ and $\sum i_2(M)$, and again check that (3.1) holds.

This completes the proof of Theorem 1.1 when G is an exceptional group of Lie type.

4. Proof of Theorem 1.1 for G classical

In this section we prove Theorem 1.1 in the case where G is classical.

Let G be a finite simple classical group over $\mathbb{F}_q$, and exclude $L_2(q)$, $L_3(q)$ and $U_3(q)$. So G is one of the groups

$$L_n^\epsilon(q)\,(n \geq 4),\ PSp_{2m}(q)\,(m \geq 2),\ P\Omega_{2m+1}(q)\,(q \text{ odd}),\ P\Omega_{2m}^\epsilon(q)\,(m \geq 4).$$

Let V be the natural module for G. As in the previous section, write $G = (\bar{G}^F)'$, where F is a Frobenius endomorphism of the corresponding simple adjoint algebraic group $\bar{G}$ over $\bar{\mathbb{F}}_q$. Define

$$d = \begin{cases} (n, q-\epsilon), & \text{if } G = L_n^\epsilon(q) \\ (2, q-1), & \text{if } G = PSp_{2m}(q), P\Omega_{2m+1}(q) \text{ or } P\Omega_{2m}^\epsilon(q). \end{cases}$$

For convenience we handle first the following groups G:

$$\begin{array}{l} L_4^\epsilon(2),\, L_4^\epsilon(3),\, L_4^\epsilon(4),\, ,L_4^\epsilon(5),\, L_5^\epsilon(2),\, L_6^\epsilon(2), \\ Sp_4(2),\, PSp_4(3),\, Sp_4(4),\, Sp_6(2),\, PSp_6(3),\, Sp_8(2), \\ \Omega_7(3),\, \Omega_8^\epsilon(2),\, P\Omega_8^+(3),\, \Omega_{10}^\epsilon(2). \end{array} \tag{4.1}$$

For these groups, generators x, t as in Theorem 1.1 can be found by a search using Magma [4]. So suppose from now on that G is not one of the groups in (4.1).

The proof follows along the same lines as the previous section. There is an element $x \in G$ of order given in Table 4; in all cases we take x to act either irreducibly on the natural module V, or irreducibly on both summands of an orthogonal decomposition $V = V_i + V_i^\perp$, where $\dim V_i = i \leq 2$. Then $T = C_{\bar{G}^F}(x)$ is a maximal torus of order $d|\langle x \rangle|$ and also $C_{Aut(G)}(T) = T$. Again, the involutions in $\mathrm{Aut}(G)$ that invert x are conjugates of t_0, a preimage in $N(T)$ of $-1 = w_0$ or $w_0\tau$ (where τ is an involutory graph automorphism). The possibilities for $C_G(t_0)$ are also given in Table 4. (In the symplectic and orthogonal cases, $C_G(t_0)$ is either the given group, or the given group quotiented by the scalars $\langle -I \rangle$.)

As in the previous section, define $\mathcal{M}(x)$ to be the set of maximal subgroups of G containing x such that $M \neq N_G(\langle x \rangle)$. We claim that $\mathcal{M}(x)$ is as in Table 5. When $\langle x \rangle$ is a Singer subgroup (i.e. the intersection with G of a cyclic subgroup generated by a Singer cycle of $PSL(V)$), this follows from the main theorem of [3]. This covers the cases where $G = L_n(q)$, $U_n(q)$ with n odd, $PSp_{2m}(q)$ and $P\Omega_{2m}^-(q)$. For the other cases we use [2, Thm. 3.1], which classifies subgroups of classical groups of orders divisible by numbers of the form $\Phi_d(q)$, where $d > \frac{1}{2}\dim V$. Working through the possible subgroups, we find that the only ones containing x are those in Table 5. We calculate the number of groups in $\mathcal{M}(x)$ using Lemma 2.2.

Again let $\mathrm{Inv}(x)$ be the set of involutions in $\mathrm{Aut}(G)$ that invert x, and define

$$S = \bigcup_{\alpha \in Inv(x)} I_2(C_G(\alpha)) \cup \bigcup_{M \in \mathcal{M}(x)} I_2(M).$$

TABLE 4. Torus $\langle x\rangle$ and centralizer $C_G(t_0)$, G classical

G	$\lvert\langle x\rangle\rvert$	$C_G(t_0)$
$L_n(q)$, $n \geq 4$	$\frac{1}{d}\frac{q^n-1}{q-1}$	$PSO^{\epsilon}_n(q)$, q or n odd $[q^{n-1}]Sp_{n-2}(q)$, q and n even
$U_n(q)$, $n \geq 4$	$\frac{1}{d}\frac{q^n+1}{q+1}, n$ odd $\frac{1}{d}(q^{n-1}+1), n$ even	$PSO_n(q)$ $PSO^{\epsilon}_n(q)$, q odd $[q^{n-1}]Sp_{n-2}(q)$, q even
$PSp_{2m}(q)$, $m \geq 2$	$\frac{1}{d}(q^m+1)$	$GL^{\epsilon}_m(q).2$, $q \equiv \epsilon \bmod 4$ $[q^{\frac{1}{2}(m^2+3m-2)}].Sp_{m-2}(q)$, q and m even $[q^{\frac{1}{2}(m^2+m)}].Sp_{m-1}(q)$, q even, m odd
$P\Omega_{2m+1}(q)$, $m \geq 3$, q odd	$\frac{1}{d}(q^m+1)$	$(O^{\epsilon}_m(q) \times O^{\epsilon'}_{m+1}(q)) \cap G$
$P\Omega^+_{2m}(q)$, $m \geq 4$	$\frac{1}{d}(q^{m-1}+1)(q+1)$, m odd $\frac{1}{d}(q^{m-1}+1)$, m even	$(O^{\epsilon}_m(q))^2.2 \cap G$, q odd $[q^{\frac{1}{2}(m^2+m-2)}].Sp_{m-2}(q)$, q and m even $[q^{\frac{1}{2}(m^2-m)}].Sp_{m-1}(q)$, q even, m odd
$P\Omega^-_{2m}(q)$, $m \geq 4$	$\frac{1}{d}(q^m+1)$	$(O^{\epsilon}_m(q) \times O^{-\epsilon}_m(q)) \cap G$, q odd $[q^{\frac{1}{2}(m^2+m-2)}].Sp_{m-2}(q)$, q and m even $[q^{\frac{1}{2}(m^2-m)}].Sp_{m-1}(q)$, q even, m odd

We aim to show that $i_2(G) > |S|$, which will complete the proof of Theorem 1.1 when G is classical.

As in (3.2) we have $|S| \leq |T|\, i_2(C_G(t_0)) + \sum_{M\in\mathcal{M}(x)} i_2(M)$, and so it suffices to prove that

$$i_2(G) > |T|\, i_2(C_G(t_0)) + \sum_{M\in\mathcal{M}(x)} i_2(M). \tag{4.2}$$

Lemma 2.3 gives the upper bounds for $i_2(C_G(t_0))$ and for $\sum_{M\in\mathcal{M}(x)} i_2(M)$ in Table 6, where $d(n)$ denotes the number of prime divisors of n. We can use these bounds together with Lemma 2.3 to get an upper bound for the right hand side of (4.2); and Lemma 2.5 gives a lower bound for $i_2(G)$. In this way we check easily that (4.2) holds with the following possible exceptions (recalling that we have already excluded the groups in (4.1)):

(1) $G = L^{\epsilon}_4(q)$, $PSp_4(q)$ or $P\Omega^{\epsilon}_8(q)$;
(2) $G = L^{\epsilon}_5(3)$, $L^{\epsilon}_6(3)$, $L^{\epsilon}_8(2)$, $Sp_6(4)$, $PSp_8(3)$, $Sp_8(4)$ or $\Omega^{\epsilon}_{12}(2)$.

For the groups under (1), we show that (4.2) still holds, by improving the lower bound on $i_2(G)$ from Lemma 2.5 and the upper bound on $i_2(C_G(t_0))$ in Table 6; the improved bounds are in Table 7. For $i_2(G)$, these improvements follow using the bounds in Table 1 together with Lemma 2.4; and for $i_2(C_G(t_0))$ they follow by direct calculation in $C_G(t_0)$ (this is straightforward for q odd, and done as in [21, 5.12] for q even).

Table 5. Maximal overgroups of $\langle x\rangle$, G classical

G	Groups in $\mathcal{M}(x)$	Number
$L_n(q)$	$(GL_{\frac{n}{r}}(q^r).r)\cap G$, r prime, $r\|n$	1 for each r
$U_n(q)$	$(GU_{\frac{n}{r}}(q^r).r)\cap G$, n odd, r prime, $r\|n$ $GU_{n-1}(q)$, n even	1 for each r 1
$PSp_{2m}(q)$	$Sp_{\frac{2m}{r}}(q^r).r$, r prime, $r\|m$ $O^-_{2m}(q)$, q even	1 for each r 1
$P\Omega_{2m+1}(q)$	$O^-_{2m}(q)\cap G$	1
$P\Omega^+_{2m}(q)$	$(O^-_{2m-2}(q)\times O^-_2(q))\cap G$ $(GU_m(q).2)\cap G$, m even $\Omega_7(q)$ (irred.), $m=4$	1 2 d
$P\Omega^-_{2m}(q)$	$(O^-_{\frac{2m}{r}}(q^r).r)\cap G$, r prime, $r\|m$ $(GU_m(q).2)\cap G$, m odd	1 for each r 1

Table 6. Upper bounds for $i_2(C_G(t_0))$ and $\sum i_2(M)$, G classical

G	$i_2(C_G(t_0))<$	$\sum i_2(M)\leq$
$L^\epsilon_n(q)$, n even	$q^{n-1}i_2(Sp_{n-2}(q))$	$2(q+1)^2q^{\frac{1}{2}(n^2-n-4)}$
$L^\epsilon_n(q)$, n odd	$i_2(PSO_n(q))$	$2d(n)\,(q^3+1)q^{\frac{1}{6}(n^2+3n)}$
$PSp_{2m}(q)$, m even	$q^{\frac{1}{2}(m^2+3m-2)}i_2(Sp_{m-2}(q))$	$i_2(O^-_{2m}(q)+d(m)\,i_2(Sp_m(q^2).2)$
$PSp_{2m}(q)$, $m\geq 3$ odd	$q^{\frac{1}{2}(m^2+m)}i_2(Sp_{m-1}(q))$	$i_2(O^-_{2m}(q))+d(m)\,i_2(Sp_{2m/r}(q^r))$, r largest prime divisor of m
$P\Omega^\epsilon_{2m}(q)$, $m\geq 6$ even	$q^{\frac{1}{2}(m^2+m-2)}i_2(Sp_{m-2}(q))$	$2(q+1)^2q^{(m-1)^2-1}+$ $2(q+1)^2q^{\frac{1}{2}(m^2-m+4)}$
$P\Omega^\epsilon_{2m}(q)$, $m\geq 5$ odd	$q^{\frac{1}{2}(m^2-m)}i_2(Sp_{m-1}(q))$	as above
$P\Omega_{2m+1}(q)$, $m\geq 3, q$ odd	$i_2(O^\epsilon_m(q)\times O^{\epsilon'}_{m+1}(q))$	$i_2(O^-_{2m}(q))$

Using the improved bounds in Table 7, it is straightforward to check that (4.2) again holds for the groups in (1).

Finally, for the groups under (2) above, we again improve the bounds on $i_2(G)$ and $i_2(C_G(t_0))$ by direct calculation to show that (4.2) holds.

This completes the proof of Theorem 1.1 when G is a classical group.

Table 7. Some improved bounds

G	$i_2(G) >$	$i_2(C_G(t_0)) <$
$L_4^\epsilon(q)$	$\frac{1}{2}q^7(q-1)$	$2q^4$
$PSp_4(q)$	$\frac{1}{2}q^5(q-1)$	$2q^3$
$P\Omega_8^\epsilon(q)$, q even	$\frac{1}{2}q^{16}$	q^{11}
$P\Omega_8^\epsilon(q)$, q odd	$\frac{1}{8}q^{16}$	$4q^8$

5. Proof of Theorem 1.1 for G an alternating group

The existence of a pair x, t in A_n satisfying (i)–(iii) of Theorem 1.1 for every $n \geq 8$ can be extracted from papers of Conder [5, 6], a more recent paper by Conder et al. [7], and some easy Magma computations. Gareth Jones recently gave the following pairs x, t in his plenary lecture at the conference "Symmetries and Covers of Discrete Objects" (Queenstown, New Zealand, February 2016).

For even $n \geq 8$, take

$$x = (2, 3, \dots, n) \text{ and } t = (1, 2)(3, 4)$$

And for odd $n \geq 9$, take

$$x = (1, 2, \dots, n) \text{ and } t = (1, 2)(3, 6)$$

Elementary arguments show that $\langle x, t\rangle = A_n$, and it is an easy exercise to show that x, t also satisfy (iii) of Theorem 1.1.

6. Proof of Theorem 1.1 for G sporadic

In this section, we show that each of the 26 sporadic simple groups has at least one pair x, t satisfying (i)–(iii) of Theorem 1.1. We shall use the fact that such pairs give abstract chiral polyhedra, as explained in the preamble to the theorem. When we give a pair x, y of generators of G, the corresponding pair x, $t := xy$ is the one satisfying conditions (i)–(iii) of Theorem 1.1. We define the *type* of a chiral polyhedron to be $\{p, q\}$, where x, y have orders p, q.

6.1. **Mathieu groups.** The groups M_i with $i = 11, 12, 22$ were fully investigated in [12]. They respectively have 66, 118 and 242 non-isomorphic chiral polyhedra, hence they have that many pairs x, t satisfying (i)–(iii) of Theorem 1.1.

The following generators of M_{23} give a chiral polyhedron of type $\{11, 15\}$:

$$x := (1, 14, 17, 21, 10, 5, 2, 16, 18, 12, 8)(3, 6, 19, 22, 15, 9, 20, 23, 4, 7, 11),$$

$$y := (1, 8, 6, 10, 21, 22, 19, 12, 11, 7, 4, 5, 3, 18, 9)(2, 23, 20, 16, 13)(14, 15, 17),$$

and $t := xy$. The pair x, t satisfies (i)–(iii) of Theorem 1.1.

The following generators of M_{24} give a chiral polyhedron of type $\{23,15\}$:

$$x := (1,17,23,21,2,7,3,15,4,20,10,6,16,13,19,22,11,18,5,14,9,8,12),$$

$$y := (1,20,2)(3,7,4,17,21,5,18,24,11,22,19,9,14,23,15)(8,13,16,10,12),$$

and $t := xy$. The pair x,t satisfies (i)–(iii) of Theorem 1.1.

6.2. **Janko groups.** The groups J_1 and J_2 were investigated in [12]. They have respectively 1056 and 888 non-isomorphic chiral polyhedra.

A Magma search gave a chiral polyhedron of type $\{19,8\}$ for J_3. We do not give its generators here as these are permutations on 6516 points.

The group J_4 has $i_2(J_4) = 51,747,149,311$. It also has a unique class of maximal subgroups containing elements of order 29. Take $\sigma \in J_4$ of order 29. The normalizer $N_{J_4}(\langle\sigma\rangle) = C_{29} : C_{28}$ is maximal in J_4. We have $i_2(C_{29} : C_{28}) = 29$, and all the 29 involutions are conjugate in J_4. There are two conjugacy classes $2A$ and $2B$ of involutions in J_4. Using the character table in Atlas [9], we compute that the structure constant for the classes $2B, 2B, 29A$ is 1, and hence involutions in $C_{29} : C_{28}$ are of type $2B$.

Now, the centralizer of an involution of type $2B$ has structure $2^{11} : (M_{22} : 2)$. This subgroup has exactly 280831 involutions. Since $29 \cdot 280831 << i_2(J_4)$, there must exist at least one chiral polyhedron with automorphism group J_4.

6.3. **Conway groups.** A non-exhaustive computer search with Magma gives a chiral polyhedron of type $\{23,23\}$ for Co_3, one of type $\{14,23\}$ for Co_2 and one of type $\{3,60\}$ for Co_1.

6.4. **Fischer groups.** The group Fi_{24}' has $Out = 2$. In Fi_{24}', take σ of order 29. We have $N_{Aut(Fi_{24}')}(\langle\sigma\rangle) = C_{29} : C_{28}$. This contains 29 involutions, all in Fi_{24}'. These involutions belong to class $2B$, and their centralizer C in Fi_{24} satisfies $i_2(C) = 5741695$. Therefore $i_2(C) \cdot 29 << i_2(Fi_{24}')$ and so Fi_{24}' is the automorphism group of at least one chiral polyhedron.

The group Fi_{23} has $Out = 1$, and for an element σ of order 23, $N_{Fi_{23}}(\langle\sigma\rangle) = C_{23} : C_{11}$. This latter group does not contain any involutions. Moreover, there is obviously at least one involution that will, with σ, generate the whole of Fi_{23}.

Finally, a non-exhaustive computer search with Magma gives a chiral polyhedron of type $\{11,13\}$ for Fi_{22}.

6.5. **The Monster and the Baby Monster.** The Monster M has $Out = 1$ and a unique class of maximal subgroups of order divisible by 71, namely subgroups $L_2(71)$. Moreover, $N_{L_2(71)}(C_{71}) = C_{71} : C_{35}$, a group of odd order. Therefore, no element of order 71 in M is conjugate to its inverse. Take x of order 71 in M. The x is contained in a unique subgroup $L_2(71)$ of M. Therefore, picking t an involution of M not in the $L_2(71)$ containing $\langle x\rangle$, we have $\langle t,x\rangle = M$. The pair x,t satisfies (i)–(iii) of Theorem 1.1.

The Baby Monster BM has $Out = 1$ and a unique class of maximal subgroups containing elements of order 47. Take x an element of order 47 in BM. We have $N_{BM}(\langle x\rangle) = C_{47} : C_{23}$. Any involution t of BM will give $\langle x, t\rangle = BM$. The pair x, t satisfies (i)–(iii) of Theorem 1.1.

6.6. **The remaining sporadics.** The Thompson group Th has $Out = 1$, and for an element x of order 31, $N_{Th}(\langle x\rangle) = C_{31} : C_{15}$. This latter group does not contain any involutions. Moreover, there is obviously at least one involution that will, with x, generate the whole of Th.

The Lyons group Ly has $Out = 1$ and a unique class of maximal subgroups of order divisible by 37, namely groups $C_{37} : C_{18}$. Moreover, Ly has a unique conjugacy class of involutions, and these have centralizers $2 \cdot A_{11}$. We have $i_2(2 \cdot A_{11}) = 34650$. Therefore $i_2(2 \cdot A_{11}) \cdot 37 << i_2(Ly)$ and Ly is the automorphism group of at least one chiral polyhedron.

For the O'Nan group $O'N$, we refer to [8] where all possible types of chiral polyhedra for $O'N$ have been determined.

Finally, a non-exhaustive computer search with Magma gives a chiral polyhedron of type $\{19, 20\}$ for HN, one of type $\{5, 7\}$ for He, one of type $\{14, 29\}$ for Ru, one of type $\{13, 24\}$ for Suz, one of type $\{11, 15\}$ for McL, and one of type $\{11, 6\}$ for HS.

This completes the proof of Theorem 1.1.

7. Concluding remarks

It would be interesting to prove similar results to Theorem 1.1 and also that of Nuzhin described in the Introduction, for almost simple (rather than just simple) groups. Some results in this vein are known. For instance in [13], it is proved that every almost simple group G with socle $Sz(q)$ is the automorphism group of at least one abstract chiral polyhedron. And in [20], it is shown that the only almost simple groups with socle $L_2(q)$ that are not automorphism groups of abstract chiral polyhedra are $L_2(q)$, $PGL(2, q)$, and a group of the form $L_2(9).2$.

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