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Class on Thursday:

340: Question done on board.

341: Project 1

[Classification of groups of small order — 8.]

~~3~~342: Quiet working with GTA assistance.

Important to be able to tell when two direct products of the form

$$C_{n_1} \times \dots \times C_{n_r}$$

are isomorphic.

To help, use following:

Prop. 3.3 If m, n are coprime positive integers (i.e. $\gcd(m,n) = 1$), then

$$C_m \times C_n \cong C_{mn}.$$

Pf. Let

$$C_m = \langle x \rangle, \quad C_n = \langle y \rangle.$$

In $C_m \times C_n$ let

$$z = (x, y).$$

By 3.2(4),

$$\begin{aligned} o(z) &= \operatorname{lcm}(o(x), o(y)) \\ &= \operatorname{lcm}(m, n) \\ &= mn \end{aligned}$$

(as m, n are coprime).

Hence

$$\begin{aligned} C_m \times C_n &= \langle z \rangle \\ &\cong C_{mn}. \quad // \end{aligned}$$

Eg. $C_2 \times C_3 \times C_7$

$$\begin{aligned} &\cong C_6 \times C_7 \\ &\cong C_{42} \end{aligned}$$

__All__ finite abelians are classified by the next result:

__Theorem 3.4__ Every finite abelian group is isomorphic to a direct product of cyclic groups

$$C_{n_1} \times \cdots \times C_{n_r}.$$

__Proof__ See the the recommended book by __R. Allenby, p. 254.__

__Eg.__ 1) Up to isomorphism, the abelian groups of order 8 are

$$C_8,\ C_4 \times C_2,\ C_2 \times C_2 \times C_2.$$

2) Find (up to iso) all abelian groups of order 10.

<u>Ans</u> By Thm 3.4, any abelian group of order 10 is isom. to

$$C_{10}, \text{ or } C_5 \times C_2.$$

But by 3.3, $C_5 \times C_2 \cong C_{10}$.

So the only one is

$$C_{10}.$$

4. Groups of small order

In this chapter we'll prove

<u>Theorem 4.1</u> Up to isomorphism, the groups of order ≤ 7 are:

Order	Groups
1	C_1
2	C_2
3	C_3
4	$C_4, C_2 \times C_2$
5	C_5
6	C_6, D_6
7	C_7

<u>Remark</u> For higher orders there can be lots more groups:

Order	Groups
8	abelian (3) D_8, Q_8
9	Only abelian
10	C_{10}, D_{10}
11	C_{11}
12	abelian, D_{12}, A_4, "G_{12}"

To prove Thm 4.1, need some lemmas about "abstract group theory".

<u>Lemma 4.2</u> If p is a prime and $|G| = p$, then $G \cong C_p$.

<u>Pf.</u> Let $1 \neq x \in G$. By Lagrange, $o(x)$ divides $|G| = p$, hence $o(x) = p$.

Hence

$$G = \langle x \rangle \cong C_p. \quad //$$

<u>Lemma 4.3</u> Suppose G is a finite group and $|G|$ is an even number. Then $\exists\, x \in G$ such that $o(x) = 2$.

<u>Pf.</u> Suppose G has <u>no</u> elts of order 2.

Then if $x \in G$ and $x \neq e$, we have

$$x \neq x^{-1}.$$

So we can list all the elements of G as

$$e, x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_k, x_k^{-1}.$$

Hence $|G| = 2k+1$, an odd number $\times\!\!\times$.

Hence G has an element of order 2. //

<u>Lemma 4.4</u> Let G be a group of order $2n$ $(n \geq 3)$. Suppose G has elts x, y such that

1) $o(x) = n$, $o(y) = 2$

2) $yx = x^{-1}y$

3) $G = \{e, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}$

Then $G \cong D_{2n}$.

Pf. Recall D_{2n} has a rotation ρ of order n and reflection σ of order 2 s.t.

a) $D_{2n} = \{e, \rho, \dots, \rho^{n-1}, \sigma, \rho\sigma, \dots, \rho^{n-1}\sigma\}$

b) $\sigma\rho = \rho^{-1}\sigma$

(see Sheet 1, Q2).

Define $\phi : D_{2n} \longrightarrow G$ by

$$\phi(\rho^i \sigma^j) = x^i y^j \quad \forall i, j.$$

Then ϕ is a bijection, and

A) $\phi\left((\rho^{i_1}\sigma^{j_1})(\rho^{i_2}\sigma^{j_2})\right)$

$= \phi\left(\rho^{i_1 \pm i_2}\sigma^{j_1+j_2}\right)$

$(+ \text{ if } j_1 = 0, - \text{ if } j_1 = 1)$

$= x^{i_1 \pm i_2} y^{j_1+j_2}$

B) $\phi(\rho^{i_1}\sigma^{j_1})\,\phi(\rho^{i_2}\sigma^{j_2})$

$= (x^{i_1} y^{j_1})(x^{i_2} y^{j_2})$

$= x^{i_1 \pm i_2} y^{j_1+j_2}$

$(+ \text{ if } j_1 = 0, - \text{ if } j_1 = 1)$

Hence ϕ is an isomorphism. //

<u>Proof of Thm 4.1</u>

Groups of orders 1, 2, 3, 5, 7

– done by Lemma 4.2.

Groups of order 4: (Sheet 2 qn).

Leaves order 6:

<u>Prop 4.5</u> If $|G| = 6$

then $G \cong C_6$ or D_6.

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Test on Tuesday!

Questions will be ......

on Chapters 1–3.

Solutions to Sheet 1: tomorrow.

Pf of Prop 4.5

Let $|G| = 6$.

If $\exists\, x \in G$ of order 6

then

$$G = \langle x \rangle \cong C_6.$$

So assume now that

G has not elts. of

order 6. So by

Lagrange, every elt. of G has order 1 (the identity) or 2 or 3.

If all non-identity elts have order 2, then G is abelian and of order divisible by 4 by Sheet 2, Q7 $\times$

<u>Hence G has an elt. x of order 3</u>

Also by Lemma 4.3, <u>G has an element y of order 2.</u>

(This establishes condition (1) of 4.4.)

Next let

$$H = \langle x \rangle = \{e, x, x^2\}.$$

Then $y \notin H$ (as $o(y) = 2$), so $Hy \neq H$, hence

$G = H \cup Hy$

$= \{e, x, x^2, y, xy, x^2y\}.$

(This establishes (3) of 4.4)

Finally, need to show $yx = x^{-1}y$.

Now $yx \in G$, so is equal to one the elts in the list

: we consider each possibility:

$yx = e \Rightarrow x = y^{-1} = y$ $\#$

$yx = x \Rightarrow y = e$ $\#$

$yx = x^2 \Rightarrow y = x$ $\#$

$yx = y \Rightarrow x = e$ $\#$

So $\underline{yx = xy \text{ or } x^2y}$

<u>If $yx = xy$</u> then

$$(xy)^2 = xyxy = x^2y^2 = x^2 \neq e$$

and

$$(xy)^3 = xyxyxy = x^3y^3 = y \neq e.$$

Hence if ~~xy~~ $yx = xy$ then $o(xy) = 6$ $\#$

Therefore

$$yx = x^2 y = x^{-1} y.$$

(This establishes (2) of 4.4.)

Hence $G \cong D_6$ by Lemma 4.4. //

5. Homomorphisms

These are functions between groups that "preserve multiplication".

<u>Defn</u> Let G, H be groups. A function $\phi : G \to H$ is a <u>homomorphism</u> if

$$\phi(\underbrace{xy}_{\text{mult. in } G}) = \underbrace{\phi(x)\phi(y)}_{\text{mult. in } H} \quad \forall x, y \in G$$

Note: an <u>isomorphism</u> is a homomorphism that is also a bijection.

Eg. 1) G, H any groups. Define $\phi : G \to H$ by

$$\phi(x) = e_H \quad \forall x \in G.$$

This is a homom., called the <u>trivial</u> homom. $G \to H$.

2) The signature function

$$s : S_n \longrightarrow C_2 \ (= \{\pm 1\})$$

is a homom. since

$$s(xy) = s(x)s(y) \quad (2.1)$$

3) Define

$$\phi : (\mathbb{R}, +) \longrightarrow (\mathbb{C}^*, \times)$$

by

$$\phi(x) \longrightarrow e^{2\pi i x} \quad \forall x \in \mathbb{R}$$

This is a homom as

$$\phi(\underbrace{x+y}_{\substack{\text{"mult"} \\ \text{in } (\mathbb{R},+)}}) = e^{2\pi i(x+y)} = e^{2\pi i x} \cdot e^{2\pi i y} = \underbrace{\phi(x)\,\phi(y)}_{\text{mult. in } (\mathbb{C}^*,\times)}$$

4) Recall

$$D_{2n} = \{e, \rho, \ldots, \rho^{n-1}, \sigma, \rho\sigma, \ldots \rho^{n-1}\sigma\}$$
$$= \{\rho^i\sigma^j \mid 0 \leq i \leq n-1,\ j = 0 \text{ or } 1\}.$$

Define $\phi : D_{2n} \to C_2$ by

$$\phi(\rho^i\sigma^j) = (-1)^j$$

(so ϕ sends

rotations $\to +1$

reflections $\to -1$.).

Then ϕ is homom. ~~[illegible]~~

(Ex.).

We'll study homoms.

via their kernels & images.

First need:

Prop 5.1 Let $\phi : G \to H$ be a homom. Then

1) $\phi(e_G) = e_H$.
2) $\phi(x^{-1}) = \phi(x)^{-1} \quad \forall x \in G$
3) For $x \in G$, $o(\phi(x))$ divides $o(x)$.

Pf. 1) Proved as in 1.2.

2) Let $x \in G$. Then by (1)

$$e_H = \phi(e_G) = \phi(x x^{-1})$$
$$= \phi(x)\phi(x^{-1}).$$

Hence $\phi(x)^{-1} = \phi(x^{-1})$.

3) Let $o(x) = n$.

Then $x^n = e_G$, so

$$e_H = \phi(e_G)$$
$$= \phi(x^n)$$
$$= \phi(x)^n \quad (\phi \text{ homom})$$

Therefore $o(\phi(x))$ divides n.

//.

<u>Image.</u>

For a homom. $\phi : G \to H$, the *image* is

$$\mathrm{Im}(\phi) = \{\phi(x) : x \in G\} \subseteq H.$$

<u>Prop 5.2</u> $\mathrm{Im}(\phi)$ is a *subgroup* of H.

Pf. 1) $e_H = \phi(e_G) \in \mathrm{Im}(\phi)$.

2) If $g, h \in \mathrm{Im}(\phi)$, then $g = \phi(x)$, $h = \phi(y)$ with $x, y \in G$, so

$$gh = \phi(x)\phi(y) = \phi(xy) \in \mathrm{Im}(\phi).$$

3) If $g \in \mathrm{Im}(\phi)$, then $g = \phi(x)$ and

$$g^{-1} = \phi(x)^{-1} = \phi(x^{-1}) \in \mathrm{Im}(\phi).$$

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Eg. 1) $G = S_3$, $H = C_3$

Does $\exists$ a homom. $\phi : G \to H$?

Ans Yes, trivial homom

$$\phi(x) = 1 \quad \forall x \in S_3.$$

2) Does $\exists$ non-trivial homom $\phi : S_3 \to C_3$?

Ans No!

Pf. Suppose

$$\phi : S_3 \to C_3$$

is a homom.

Let $(1\,2) = t \in S_3$ and consider

$$\phi(t) \in C_3.$$

Now $o(t) = 2$, so $\phi(t)$ has order dividing 2 by 5.1(3).

As C_3 has no elts of order 2, this implies $\phi(t)$ has order 1, so

$$\phi(t) = 1.$$

Similarly, for 2-cycle $(ij) \in S_3$,

$$\phi(ij) = 1.$$

Also $(123) = (13)(12)$, a product of 2-cycles, so

$$\phi(123) = \phi((13)(12))$$

$$= \phi(13)\,\phi(12)$$

$$= 1$$

Similarly $\phi(132) = 1$.

So ϕ is the trivial homom.

//

Kernels

Defn If $\phi : G \to H$ is a homom., the kernel

$$\operatorname{Ker}(\phi) = \{x \in G : \phi(x) = e_H\}.$$

E.g. 1) Trivial homom. $G \to H$ has kernel G.

2) Signature homom

$$s : S_n \to C_2.$$

$$\operatorname{Ker}(s) = \{x \in S_n : s(x) = 1\}$$

$$= A_n.$$

3) Homom

$$\phi : (\mathbb{R}, +) \to (\mathbb{C}^*, \times)$$

defined by

$$\phi(x) = e^{2\pi i x} \quad (x \in \mathbb{R}).$$

This has kernel

$$\operatorname{Ker}(\phi) = \{x \in \mathbb{R} : \phi(x) = 1\}$$

$$= \mathbb{Z}$$

$= \{x \in \mathbb{R} : e^{2\pi i x} = 1\}$

$= \mathbb{Z}$.

<u>Prop 5.3</u> If $\phi : G \to H$ is a homom, then $\ker(\phi)$ is a subgroup of G.

<u>Pf.</u> 1) $e_G \in \ker(\phi)$ since $\phi(e_G) = e_H$ by 5.1(1).

2) $x, y \in \ker(\phi)$ 4

$\Longrightarrow \phi(x) = \phi(y) = e_H$

$\Longrightarrow \phi(xy) = \phi(x)\phi(y) = e_H$

$\Longrightarrow xy \in \ker(\phi)$.

3) $x \in \ker(\phi)$

$\Longrightarrow \phi(x^{-1}) = \phi(x)^{-1}$ (5.1) $= e_H^{-1} = e_H$

$\Longrightarrow x^{-1} \in \ker(\phi)$. //

Kernels of homoms are "special" kinds of subgroups called <u>normal subgroups</u>!

6. <u>Normal subgroups</u>

<u>Defn</u> Let G be a group and $N \subseteq G$. We say N is a <u>normal subgroup</u> of G if

1) N is a subgrp of G
2) $g^{-1}Ng = N, \forall g \in G$

where

$$g^{-1}Ng = \{g^{-1}ng : n \in N\}.$$

If N is a normal subgroup of G, write

$$\underline{N \lhd G.}$$

<u>Eg</u>- 1) G any group.

Then $\{e\}$ and G are normal subgroups of G, since

$$g^{-1}\{e\}g = \{e\} \quad \forall g \in G$$

and

$$g^{-1}Gg = G \quad \forall g \in G$$

(Ex).

2) If G is <u>abelian</u> then <u>every</u> subgroup N of G is a normal subgroup, since

$$g^{-1}Ng = \{g^{-1}ng : n \in N\}$$

$$= \{n : n \in N\} = N$$

(as $g^{-1}ng = n$ in abelian gr

3) Let $G = S_3$.

Subgroups of G are

1) $\{e\}$, G
2) $\langle (ij) \rangle$ $\quad (\cong C_2)$
3) $\langle (123) \rangle$ $\quad (\cong C_3)$.

Which of these are normal subgps?

<u>Ans</u> 1) Normal

2) These are not normal,

eg. $\langle (12) \rangle = \{e, (12)\}$ is <u>not normal</u>, since taking $g = (13)$,

$$g^{-1} \langle (12) \rangle g$$

Containing $g^{-1}(12)g = (13)(12)(13$

$= (123) \notin \langle (12) \rangle$.

Hence $g^{-1} \langle (12) \rangle g \neq \langle (12) \rangle$.

3) In fact

$$\langle (123) \rangle \lhd S_3$$

(see next example).

To check subgroups are normal, here's a useful lemma.

<u>Lemma 6.1</u> Let N be a subgroup of G. Then

$$N \lhd G \iff g^{-1}Ng \subseteq N \quad \forall g \in G.$$

<u>Pf</u> ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose

$\underline{g^{-1}Ng \subseteq N \quad \forall g \in G.}$

Let $x \in G$. By

$$\underline{x^{-1}Nx \subseteq N}$$

Also, applying (☞) with g replaced by x^{-1},

$$xNx^{-1} \subseteq N.$$

Hence

$$\underline{N \subseteq x^{-1}Nx}$$

Therefore

$$x^{-1}Nx = N.$$

Hence $N \lhd G$. //

Eg (4)

Claim $A_n \lhd S_n$.

[So $A_3 = \langle (123) \rangle \lhd S_3$, completing previous example].

Pf. Let $g \in S_n$.

Then for $x \in A_n$,

$$\begin{aligned} s(g^{-1}xg) &= s(g^{-1})s(x)s(g) \\ &= s(g^{-1}) \cdot 1 \cdot s(g) \end{aligned}$$

$= 1$

hence $g^{-1}xg \in A_n$.

This shows

$$g^{-1}A_n g \subseteq A_n \quad \forall g \in S_n.$$

So by 6.1,

$$A_n \triangleleft S_n.$$