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Riemannian curvature and stability of monoparametric families of trajectories

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Abstract. For a given holonomic system of two degrees of freedom and for a given monoparametric family of trajectories (not necessarily isoenergetic), generated by the known potential of the system, we find a formula offering the Riemannian curvature associated with the Maupertuis metric. In the light of the inverse problem of dynamics, we introduce also the notion of the family zero-curvature curves (FZCC).

As an application, we derive the pertinent formulae for all members of a family of concentric circular orbits produced by the appropriate potentials, we examine the connection of their stability to the sign of the curvature and we compare with stability deduced by other considerations, including numerical integration.

1. Introduction

For any natural conservative dynamical system of n degrees of freedom, in the region of the configuration space M_n where motion is allowed to take place with a definite value of the total energy, one can define the Maupertuis manifold M_n^*, according to the Maupertuis–Lagrange–Jacobi least-action principle so that real trajectories in M_n (all traced with energy E) become geodesics of M_n^* (Whittaker 1961, p 254, Arnold 1978, p 247). In other words the totality of isoenergetic motions taking place in M_n can be considered as a geodesic flow in M_n^*.

It is known that the Riemannian tensor of the above manifold M_n^*, calculated at any point $P^* \in M_n^*$ can provide information related to the convergence or divergence of two neighbouring geodesics passing in the close vicinity of P^* and corresponding to two trajectories of M_n *of the same total energy*. This fact, in turn, is interpreted to be related to the *stability or instability* of either of the two trajectories in M_n. As reported by Whittaker (1961, p 417) this idea goes back to Synge in the twenties and has been used by various authors since (Aizawa 1972, van Velsen 1980, Barbosu 1996).

On the other hand, several authors have used the Maupertuis principle, in order to study nonintegrability of Hamiltonian systems. Thus, e.g. the Riemann tensor has been evaluated by van Velsen (1980) in his statistical study of certain few-body systems. Considering $n > 2$, van Velsen starts by reminding us that, at any point Q of M_n^*, a pair of tangent vectors $\bar{x}, \bar{y}$ spans a plane section S_Q. The sectional (Gaussian) curvature K along S_Q is defined with the aid of the Riemann tensor and is independent of the choice of $\bar{x}, \bar{y}$ but varies as the section S_Q varies. He then proceeds to show that $\frac{1}{2}n(n-1)$ appropriate sectional curvatures c_{ij} are needed and that *the manifold has negative curvature if all the functions* c_{ij} *are negative*.

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A well-established result, based on the signs of the curvatures c_{ij}, is the following. For compact manifolds M_n^*, negative curvature implies that the system is mixing and ergodic (the flow is then termed a K-system and exhibits strong instability) (Sinai 1960, Anosov 1967, Arnold and Avez 1968). Yet, in a conservative system, not all c_{ij} can become negative everywhere in M_n^* (see also Szydlowski 1994). There appear then weaker instabilities and, for $n > 2$, this seems to be a dominant feature as, e.g., in the case of the general three-body problem (van Velsen 1980, Barbosu 1995, Barbosu and Elmabsout 1998) .

For $n = 2$, just one sectional (the Riemannian) curvature is defined. The role of the Riemannian curvature for $n = 2$ has been explored by Aizawa (1972) in connection with the existence of smooth invariant curves (Hénon and Heiles 1964), i.e. in connection with the question of integrability of the pertinent potential. Aizawa concludes that, for two-dimensional systems, the Riemannian curvature, although not sufficient, 'is useful for discussing global properties of the system'.

Not only the problem of integrability but also the question of how stability is connected to the sign of the curvature of M_2^* is not quite settled in the literature. It appears that the sign of the Riemannian curvature is, in general, *an indicator* of the stability or instability of an orbit. The following statements seem to be generally accepted. (i) If the orbit is bound to travel in a closed region, negative curvature everywhere implies strong instability. (ii) Positive curvature does not suffice to ensure stability. Yet, provided that this curvature is positive enough at places reached by the orbit, its positiveness is a strong indicator of stability (Barbosu 1996).

In what follows we shall take for granted what seems to be commonplace in the literature (see e.g. van Velsen 1980), i.e. that the knowledge of the Riemannian tensor of the Maupertuis manifold is of importance in detecting global properties of the motion. But we shall focus attention on the following fact. Due to the reasoning in their derivation, all formulae giving sectional curvatures refer to a certain potential and to a certain value E of the total energy of the orbit, in other words they refer to *a family of isoenergetic orbits* (Whittaker 1961, p 419).

In this paper, we find a formula giving the Riemannian curvature *for all members of a preassigned, not necessarily isoenergetic, family of trajectories*. To this end we use a generalized version of Szebehely's (1974) equation for the two-dimensional inverse problem of dynamics for holonomic systems (Mertens 1981). We also work out the version of this formula accounting for a material point moving in the Cartesian xy plane. The above formulae are then applied to the case of a family of concentric circles.

We also introduce the notion of the *family zero-curvature curves* (FZCC) and, in specific examples, we see how these curves are positioned together with the *family boundary curves* (FBC) (Bozis and Borghero 1995) of the given family produced by the given potential. The relative position of the FZCC and the FBC helps to relate the stability or instability of any particular orbit (belonging to the family at hand) to the sign of the Riemannian curvature at points of the configuration space visited by this orbit.

Making use of the fact that all potentials creating circles are integrable, we study the stability of this family on this ground. In so doing, we manage to express the curvature of this family in terms (i) of the potential, (ii) of a function B related to the allowed region of motion and (iii) of a function S indicating the stability (or instability) of the circles. We conclude with comments on the above relation. Judging from this particular case (the family of the circles) we are led to understand why the connection of stability and the sign of the Riemannian curvature of M_2^* is rather intricate.

2. The Maupertuis action line element and Szebehely's formula

Let us consider a conservative holonomic system of two degrees of freedom whose configuration space M_2 is parametrized by the curvilinear coordinates u, v. The kinetic energy T of the system may be written as

$$T = \tfrac{1}{2}(g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2), \tag{1}$$

where dots denote differentiation with respect to the time t. The line element of the Riemannian space M_2 is given by $\mathrm{d}s^2 = (2T)\,\mathrm{d}t^2 = g_{ij}\,\mathrm{d}u^i\,\mathrm{d}u^j$, with $u^1 = u, u^2 = v$.

Let $V(u, v)$ be the potential of the system and consider a trajectory (c_0) in M_2 traced with total energy E. For the above triplet of g_{ij}, E and V we can construct the Maupertuis metric g^*_{ij} leading to the line element

$$\mathrm{d}s^{*2} = (E - V)\,\mathrm{d}s^2. \tag{2}$$

We shall keep *denoting by an asterisk* all components of the metric tensor as well as the Christoffel symbols when they refer to the manifold M^*_2.

It is known that to each trajectory (c_0) in M_2 there corresponds a geodesic (c^*_0) in the Riemannian space M^*_2 which is associated with energy equal to E and whose line element is given by (2) (e.g. Arnold 1978, Goldstein 1980).

Also, according to the equation of deviation of geodesics (Arnold 1978, p 310), the convergence or divergence of two geodesics originating in the neighbourhood of a certain point $P^* \in M^*_2$ depends on the sign of the Riemannian sectional curvature $K(u, v)$, which can be calculated at the point P^* with the aid of the Riemann–Christoffel tensor R^*_{ijkl}. For more than two degrees of freedom there will be more than one section at the point P^* but for the two-dimensional space M^*_2 (as is the case in this study) there exists just one such section. If $(1, 0)$ and $(0, 1)$ are taken as the two independent vectors to determine this section at a point of M^*_2, the Riemannian curvature at this point is given (Spain 1965, pp 56 and 52) by

$$K = \frac{1}{g^*} R^*_{1212}, \tag{3}$$

where

$$g^* = \det(g^*_{ij}) = g^*_{11} g^*_{22} - g^{*2}_{12} \tag{4}$$

and

$$R^*_{1212} = \frac{1}{2}\left(2\frac{\partial^2 g^*_{12}}{\partial u \partial v} - \frac{\partial^2 g^*_{11}}{\partial^2 v} - \frac{\partial^2 g^*_{22}}{\partial u^2}\right) + g^{*ts}([12, s]^*[12, t]^* - [22, s]^*[11, t]^*), \tag{5}$$

where the summation convention has been adopted.

In the second term of the r.h.s. of equation (5) there appear the contravariant components $g^{*ts} = \mathrm{cof}(g^*_{ts})/g^*$ of the metric tensor and also the Christoffel symbols of the first kind

$$[ij, k]^* = \frac{1}{2}\left(\frac{\partial g^*_{ik}}{\partial u^j} + \frac{\partial g^*_{jk}}{\partial u^i} - \frac{\partial g^*_{ij}}{\partial u^k}\right). \tag{6}$$

Let us now consider, in an inertial frame, a holonomic system moving in the potential field $V = V(u, v)$ and suppose that

$$f(u, v) = c = \text{constant} \tag{7}$$

is a monoparametric family of trajectories traced in the configuration space uv in the presence of the potential $V(u, v)$. Also denote by

$$E = E(f(u, v)) \tag{8}$$

the total energy $E = T + V$, which is kept constant, of course, along each trajectory.

Then, according to Szebehely (1974), there exist infinitely many potentials $V(u, v)$ giving rise to the family (7) with energy distribution (8) along the members of the family. In order to write down Szebehely's original equation for the case at hand, we need to introduce (Mertens 1981) the following notation:

$$c_1 = g_{22} f_u - g_{12} f_v, \qquad c_2 = -g_{12} f_u + g_{11} f_v, \tag{9a}$$

$$A = g_{11} f_v^2 - 2g_{12} f_u f_v + g_{22} f_u^2, \tag{9b}$$

$$B_1 = \tfrac{1}{2} g_{11,u} f_v^2 + [g_{12,v} - \tfrac{1}{2} g_{22,u}] f_u^2 - g_{11,v} f_u f_v, \tag{9c}$$

$$B_2 = \tfrac{1}{2} g_{22,v} f_u^2 + [g_{12,u} - \tfrac{1}{2} g_{11,v}] f_v^2 - g_{22,u} f_u f_v, \tag{9d}$$

$$w_0 = g(f_v^2 f_{uu} - 2 f_u f_v f_{uv} + f_u^2 f_{vv}) - B_1 c_1 - B_2 c_2 \tag{9e}$$

and

$$\alpha = \frac{A c_1}{2 w_0}, \qquad \beta = \frac{A c_2}{2 w_0}, \tag{9f}$$

where g is given by the non-asterisked formula (4) and where lettered subscripts denote partial differentiation.

Then, Szebehely's linear partial differential equation for the potential $V(u, v)$, as first offered by Mertens (1981), reads

$$\alpha V_u + \beta V_v + V = E. \tag{10}$$

Equation (10) is one of the basic tools in considerations having to do with the inverse problem of dynamics (Bozis 1995). In particular it answers the following question: given the family (7), traced with the preassigned energy-dependence function (8), what are the potentials $V(u, v)$ which can generate (for adequate initial conditions, of course) this family?

In general, for any specific solution $V(u, v)$ of equation (10), not wholly and not all the geometrical entities given by (7) will be adopted as real trajectories. Apparently only those trajectories (or parts of trajectories) satisfying the inequality

$$E(f(u, v)) \geqslant V(u, v) \tag{11a}$$

will be substantially traced.

Thus, when we refer to a family (7), we have in mind a set of trajectories with equation (7). Sets created by different potentials $V(u, v)$ generally include different members, all with the same equation (7), of course.

In general, inequality (11a) defines, in the uv-configuration space, the allowed and nonallowed regions where members of the family (7) can exist as real orbits produced by the potential $V(u, v)$ (Bozis and Borghero 1995). The curves

$$\alpha V_u + \beta V_v = 0, \tag{11b}$$

if they exist, are called *family boundary curves* (FBC). In general, they are different from the well known zero-velocity curves which refer to one orbit with a certain value of the constant energy E or to a family of isoenergetic orbits. In this latter case the FBC coincide with the zero-velocity curves.

3. The Riemannian curvature for a family of trajectories

For a two-dimensional conservative system, the components g_{ij} and g_{ij}^* of the metrics of M_2 and M_2^*, respectively, are related by

$$g_{ij}^* = (E - V) g_{ij}. \tag{12}$$

For any specific dynamical system, the metric g_{ij} and the potential $V(u, v)$ are known. Therefore, for any definite value of the total energy E, we can calculate from (12) the metric g^*_{ij} of M^*_2 and from (3) the curvature K. In doing the algebra, we treat E as a constant, as if we had in mind one trajectory or one family of trajectories all having the same energy E.

Second-order derivatives both of V and g_{ij} with respect to u and v will enter into the calculations but, due to the form of the r.h.s. of equation (12), the energy E (in fact $E - V$, as a whole) will survive. So, for a given energy $E =$ constant, we obtain

$$K = \frac{1}{4g^2(E-V)^3}[K_0 + K_1(E-V) + K_2(E-V)^2], \tag{13}$$

where

$$K_0 = K_{0,11}V_u^2 + K_{0,12}V_uV_v + K_{0,22}V_v^2, \tag{14a}$$

$$K_1 = 2g(g_{22}V_{uu} - 2g_{12}V_{uv} + g_{11}V_{vv}) + K_{1,1}V_u + K_{1,2}V_v \tag{14b}$$

and

$$\begin{aligned} K_2 = -2g(g_{22,uu} - 2g_{12,uv} + g_{11,vv}) &+ g_{11}(g_{11,v}g_{22,v} - 2g_{12,u}g_{22,v} + g^2_{22,u}) \\ &+ g_{12}(g_{11,u}g_{22,v} - g_{22,u}g_{11,v} - 2g_{12,u}g_{22,u} - 2g_{11,v}g_{12,v} + 4g_{12,u}g_{12,v}) \\ &+ g_{22}(g_{11,u}g_{22,u} - 2g_{11,u}g_{12,v} + g^2_{11,v}). \end{aligned} \tag{14c}$$

We observe that the whole expression (14*c*) for K_2 depends merely on the metric g_{ij} and that so do the coefficients $K_{0,11}$, $K_{0,12}$, $K_{0,22}$ of K_0 and $K_{1,1}$, $K_{1,2}$ of K_1 according to the following formulae:

$$K_{0,11} = 2gg_{22}, \qquad K_{0,12} = -4gg_{12}, \qquad K_{0,22} = 2gg_{11} \tag{15a}$$

and

$$K_{1,1} = -2g_{11}g_{22}g_{12,v} + g_{12}g_{22}g_{11,v} + 2g_{12}g_{22}g_{12,u} - g^2_{22}g_{11,u} + (g - g^2_{12})g_{22,u} + g_{11}g_{12}g_{22,v}, \tag{15b}$$

$$K_{1,2} = (g - g^2_{12})g_{11,v} - 2g_{11}g_{22}g_{12,u} + 2g_{11}g_{12}g_{12,v} + g_{12}g_{22}g_{11,u} + g_{11}g_{12}g_{22,u} - g^2_{11}g_{22,v}. \tag{15c}$$

Equation (13) is valid for any trajectory produced by $V(u, v)$ and having total energy E.

As a next step, let us then classify this trajectory to a family of trajectories (7), not necessarily isoenergetic. In fact, as we switch from one member of (7) to another, the energy on each specific member (for which equation (13) is satisfied) changes as it is given by (8). Then, for any specific orbit of the family (7), the expression $E - V$ appearing in (13) may be obtained from Szebehely's equation (10). Thus, we obtain the curvature (13) for the entire family. After some straightforward algebra there results:

$$\begin{aligned} K_f = \frac{1}{4g^2(\alpha V_u + \beta V_v)^3}[&2g(\alpha V_u + \beta V_v)(g_{22}V_{uu} - 2g_{12}V_{uv} + g_{11}V_{vv}) \\ &+ (L_{11}V_u^2 + L_{12}V_uV_v + L_{22}V_v^2)], \end{aligned} \tag{16}$$

with the coefficients

$$L_{11} = K_{0,11} + \alpha K_{1,1} + \alpha^2 K_2, \tag{17a}$$

$$L_{12} = K_{0,12} + (\alpha K_{1,2} + \beta K_{1,1}) + 2\alpha\beta K_2, \tag{17b}$$

$$L_{22} = K_{0,22} + \beta K_{1,2} + \beta^2 K_2 \tag{17c}$$

now depending on the metric g_{ij} and on the family (7).

Equation (16) is valid for all members of (7) produced by the potential $V = V(u, v)$ and exclusively for those. It offers the Riemannian curvature, for the (unique) orbit having equation (7) and passing through the point (u, v) of the configuration space uv.

For a unit-mass point moving in the Cartesian xy-*plane* under the influence of the potential $V(x, y)$ it is $u = x$, $v = y$, $g_{11} = g_{22} = 1$, $g_{12} = 0$, $g = 1$, and, from (9), we find

$$\alpha = -\frac{1+\gamma^2}{2\Gamma}, \qquad \beta = -\frac{\gamma(1+\gamma^2)}{2\Gamma}, \tag{18}$$

where

$$\gamma = \frac{f_y}{f_x}, \qquad \Gamma = \gamma\gamma_x - \gamma_y. \tag{19}$$

Thus,

(i) formula (13) reduces to

$$K = \frac{1}{2(E-V)^2}\left[V_{xx} + V_{yy} + \frac{1}{E-V}(V_x^2 + V_y^2)\right], \tag{20}$$

a result also found by Aizawa (1972) and by van Velsen (1980).

(ii) Formula (16) reads

$$K_f = \frac{2\Gamma^2}{(1+\gamma^2)(V_x+\gamma V_y)^2}\left[V_{xx} + V_{yy} - \frac{2\Gamma(V_x^2+V_y^2)}{(1+\gamma^2)(V_x+\gamma V_y)}\right]. \tag{21}$$

Thus, e.g. for all orbits (conic sections) of total energy E, traced in the (central) Newtonian potential $V = -1/r$, $(r = (x^2 + y^2)^{\frac{1}{2}})$, the curvature (20) is (see also Barbosu 1996)

$$K = -\frac{E}{2(1+Er)^3}, \tag{22}$$

whereas, for any circle of radius r the curvature (21) is given by

$$K_f = \frac{2}{r}. \tag{23}$$

Selecting, out of all conics passing through a point at distance r, the circle (whose energy is $E = -1/2r$), from (22) we find (23).

In fact we are interested in detecting the sign of the function K_f, as given by equation (16). To avoid unnecessary complication of the calculations (mainly due to the presence of the fractions α and β, given by $(9f)$), we can equivalently look for the sign of the expression $s_f = 2g^2A^2(c_1V_u + c_2V_v)^2K_f$, given by

$$s_f = \frac{w_0}{A(c_1V_u + c_2V_v)}[4gw_0A(c_1V_u + c_2V_v)(g_{22}V_{uu} - 2g_{12}V_{uv} + g_{11}V_{vv}) \\ +(M_{11}V_u^2 + M_{12}V_uV_v + M_{22}V_v^2)], \tag{24}$$

where

$$M_{11} = 4w_0^2K_{0,11} + 2Ac_1w_0K_{1,1} + A^2c_1^2K_2, \tag{25a}$$
$$M_{12} = 4w_0^2K_{0,12} + 2Aw_0(c_1K_{1,2} + c_2K_{1,1}) + 2A^2c_1c_2K_2, \tag{25b}$$
$$M_{22} = 4w_0^2K_{0,22} + 2Aw_0c_2K_{1,2} + A^2c_2^2K_2. \tag{25c}$$

4. Family zero-curvature curves

For a two-degrees of freedom holonomic system with given metric for which a family of trajectories (7) is known, the Riemann curvature K_f, as given by equation (16), is a function of the position coordinates u, v only. The energy E has been replaced by Szebehely's formula (10). Regarding the domain D of definition of this function $K_f(u, v)$, one sees immediately that the points at which the denominator $\alpha V_u + \beta V_v$ vanishes must be excluded from D. Those are exactly, if they exist, the points of the FBC, defined by the equation (11*b*).

For the reasons to be mentioned at the end of this section we are interested in the sign of the function (24) *at all regions* of the uv-plane where the orbit *is or is not* travelling. We observe that this sign may change as we cross: either (i) a FBC (consisting of points of discontinuity of $K_f(u, v)$, in the neighbourhood of which the curvature tends to infinity) or (ii) the curve

$$4gw_0A(c_1V_u + c_2V_v)(g_{22}V_{uu} - 2g_{12}V_{uv} + g_{11}V_{vv}) + (M_{11}V_u^2 + M_{12}V_uV_v + M_{22}V_v^2) = 0 \quad (26)$$

(consisting of points where the curvature (16) vanishes).

We propose to call the above curves (26) *family zero-curvature curves* (FZCC). For a given metric, associated with the kinetic energy of the natural system, by definition, the FZCC, if they exist, correspond to a preassigned family (7) and a certain potential (out of the infinitely many) generating this family.

If there exists just one FZCC (open or closed) and the whole uv-plane is allowed for the family (7), i.e. if $\alpha V_u + \beta V_v > 0$ everywhere, then this FZCC will separate the configuration space uv into two regions: the positive and the negative curvature region for the members of (7). These two regions would also exist if $\alpha V_u + \beta V_v < 0$ everywhere (i.e. if motion on curves with equation (7) was forbidden everywhere) but it would be meaningless to draw them.

If, apart from one FZCC, there exist also one or more FBC, the picture regarding the sign of $K(u, v)$, as this sign is given by (24), changes and has to be studied separately for each specific case.

We are, of course, essentially interested in detecting and observing, for the family (7), the sign of K in the *allowed* region of the configuration space, where, according to inequality (11*a*), $\alpha V_u + \beta V_v \geqslant 0$. However, for a preassigned metric g_{ij}, we observe that:

(i) if the equation (10) is satisfied by the triplet (f, V, E), it is also satisfied by the triplet $(f, -V, -E)$,

(ii) if to equation (16) we apply the transformation

$$f \to f, \qquad V \to -V, \qquad \text{then} \quad K \to -K. \quad (27)$$

The meaning of the above remarks is the following: nonallowed regions for a family (7) generated by $V(u, v)$ become allowed regions by selecting the potential $-V(u, v)$ to create members of the family. The positive (negative) curvature regions are then turned to negative (positive) curvature regions and real orbits travel there with the opposite energy.

For the above reasons, in the examples of section 6 we shall draw FZCC both in the allowed and nonallowed regions of the configuration space uv and we shall comment about stability of orbits both for triplets (f, V, E) and $(f, -V, -E)$. In section 7, when forming conclusions regarding stability, we must keep in mind that the perturbed orbits used to check the stability are all lying on the same three-dimensional energy submanifold of the four-dimensional phase space.

5. Riemannian curvature and stability for a family of circular orbits

Consider the monoparametric family of concentric circular orbits

$$f(r, \theta) = r = c = \text{constant}, \quad (28)$$

each one created (for adequate initial conditions) by a material point of unit mass. All potentials which can generate the above family are given, in polar coordinates r, θ, by

$$V(r,\theta) = G(r) + \frac{1}{r^2}H(\theta), \tag{29}$$

where G and H are arbitrary functions of their respective arguments (Broucke and Lass 1977). Each orbit (28) is traced with total energy

$$E = G(r) + \tfrac{1}{2}rG'(r), \tag{30}$$

where the prime denotes differentiation with respect to r. Not all circles (28) or not all parts of a circle are actual orbits produced by a certain potential (29). Inequality (11a), for the present case, reads

$$B = r^3G'(r) - 2H \geqslant 0 \tag{31}$$

and defines the FBC for the pair (28), (29).

Apart from any other interest that the example (28) may present, it was selected because it lends itself to analytic treatment of the curvature formula (16) along all members of (28) produced by any of the potentials (29). Actually this is among the few pairs (f, V) for which the totality (a multitudiness introduced by two arbitrary functions) of potentials $V(u, v)$ producing $f(u, v) = c$ is known (Bozis 1995).

We firstly write formula (16), in view of equations (28) and (29). Having selected polar coordinates, we have $g_{11} = 1$, $g_{12} = 0$, $g_{22} = r^2$, $g = r^2$. From formulae (9) we obtain, for the case at hand,

$$c_1 = r^2, \qquad c_2 = 0, \qquad A = r^2 \qquad B_1 = -r,$$
$$B_2 = 0, \qquad w_0 = r^3, \qquad a = \tfrac{1}{2}r, \qquad \beta = 0.$$

From (15) and (14c) we find, in view of the above,

$$\begin{aligned} &K_{0,11} = 2r^4, \qquad K_{0,12} = 0, \qquad K_{0,22} = 2r^2 \\ &K_{1,1} = 2r^3, \qquad K_{1,2} = 0, \qquad K_2 = 0 \end{aligned} \tag{32}$$

and then from (17) we find

$$L_{11} = 3r^4, \qquad L_{12} = 0, \qquad L_{22} = 2r^2. \tag{33}$$

Substituting (33) into formula (16), we obtain for the potential (29) and the family (28),

$$K_{f,\text{circle}} = \frac{2}{(r^3G' - 2H)^3}[2H'^2 + (r^3G' - 2H)[H'' + r^3(rG'' + 3G')]], \tag{34}$$

where the primes in G and H stand for derivatives with respect to r and θ respectively.

Remark. *In particular if $H(\theta) = 0$, the potential (29) becomes central and the above formula (34) reduces to*

$$K_{f,\text{central}} = \frac{rG'' + 3G'}{r^3G'^2}. \tag{35}$$

It is seen that the sign of the curvature depends on the sign of the expression $rG'' + 3G'$. This brings in agreement the following two facts:

(i) the well known fact that circular orbits produced by an *attractive* central force $F(r) < 0$ (notice that $F(r) = -V_r(r) = -G'(r) < 0$ means $G'(r) > 0$) are stable if and only if (Goldstein 1980, p 91)

$$\frac{F'(r)}{F(r)} + \frac{3}{r} > 0. \tag{36}$$

(ii) The fact that these orbits keep positive Riemannian curvature as the point P travels in the configuration space.

On the other hand, we know that all potentials (29) are integrable. Besides the energy integral

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + G + \frac{1}{r^2}H \tag{37}$$

they possess the second integral Φ, given by

$$\Phi = r^4\dot{\theta}^2 + 2H. \tag{38}$$

We shall take advantage of this coincidence to find independently a criterion of stability (inequality (44), below) and then compare to the curvature behaviour.

Solving equations (37) and (38) for $\dot{r}^2$ and $\dot{\theta}^2$ we obtain

$$\dot{r}^2 = 2(E - G) - \frac{\Phi}{r^2} \tag{39}$$

$$\dot{\theta}^2 = \frac{\Phi - 2H}{r^4} \tag{40}$$

and differentiating (39) with respect to the time t, we obtain

$$\ddot{r} = \frac{\Phi}{r^3} - G'. \tag{41}$$

A circular orbit $r = r_0$ having second-integral value equal to Φ is an equilibrium point $r = r_0$ of the one-dimensional (rectilinear) motion described by equation (41) and satisfies the equation

$$r_0^3 G_0' = \Phi, \tag{42}$$

where $G_0' = G'(r_0)$.

Let $r = r_0 + \xi$ be a neighbouring point with the same value of Φ which, of course, no longer corresponds to a circular orbit. Keeping linear terms in ξ, we obtain from (41), in view of (42) also (Boccaletti and Pucacco 1995, p 142),

$$\ddot{\xi} = -\left(\frac{3}{r_0}G_0' + G_0''\right)\xi. \tag{43}$$

This result implies that circular orbits are (linearly) stable if and only if

$$S = rG'' + 3G' > 0. \tag{44}$$

6. Comments—examples

The necessary and sufficient condition (44) for a circle (28) to be stable was derived on the grounds of the integrability of all potentials (29). It is worth noticing that, according to (44), the stability of a circular orbit created by any of the potentials (29) depends merely on the function $G(r)$ and not on the function $H(\theta)$. On the other hand, for $H(\theta) \neq 0$, both the allowed region (31) and the Riemannian curvature (34) depend on G and H.

In other words noncentral potentials (29) having common G and different H create circles of the same character (stable or unstable). These circles (or arcs of circles), of course, according to (31) *exist as real orbits* in different regions of the plane, depending on the function $H(\theta)$ also. In particular, the sign of the curvature K_f, as can be seen from equations (34), (31) and (44), is the same with the sign of the expression

$$k^* = 2H'^2 + B(H'' + r^3 S). \tag{45}$$

For real circular motion, B is permanently positive in (45) and for stable (or unstable) motion, S is definitely positive (or negative). But, due to the presence of $H''(\theta)$ in (45), no general definite statement can be made on the connection of the signs of k^* and S. So, if such a connection of the Riemannian curvature to stability cannot be established for circles, no wonder that it cannot be established for orbits in general.

Yet (45) *can serve to indicate* either positive curvature when stability is assumed or instability when negative curvature is given. Indeed,

(a) the path of a real $(B > 0)$ stable $(S > 0)$ circular orbit $(r = r_0 > 0)$ consists of points in the plane for which the curvature (34) is positive, i.e. $k^* > 0$ (a situation which is very likely, yet not strictly necessary). This statement is made in view of (45) and also in view of the fact that functions $H(\theta)$ in (29) must be continuous and 2π*-periodic* and, as such, bounded. It may be however that $k^* < 0$ in the vicinity of the minimum (negative) value H''_{min} of the function $H''(\theta)$, although $B > 0$ and $S > 0$.

(b) If the curvature is negative $(k^* < 0)$ for even one point reached by the circular orbit, it is expected that $S < 0$, i.e. the orbit will be unstable (a situation which is very likely, yet not strictly necessary). Yet it may be that $S > 0$ at some places and that it is the term $H''(\theta)$ which contributes to the negativeness of k^*. But, again due to the periodicity of $H(\theta)$, the term $H''(\theta)$ cannot be negative along the entire circle, so, at some points on the circle, k will assume negative values and the orbit will be unstable.

The above comments are now supplemented by two examples, both referring to the concentric circles (28). Two paired figures corresponding to (f, V) and $(f, -V)$ are drawn for each example to show the pertinent FBC (11*b*) and FZCC (26) and to make evident the consequences of the transformation (27).

The figures are drawn under the following conventions: (i) the nonallowed region for the pair (f, V) is shaded, while the allowed region for the family is kept unshaded; (ii) a (+) or (−) sign indicates the sign of the curvature in each region and a letter S or U indicates the presence of stable or unstable orbits in the allowed region; (iii) numerically tested circular orbits and arcs, mentioned below, are not drawn in the figures.

The system of the equations of motion is

$$\begin{aligned} \ddot{x} &= -\frac{x}{r}G' + \frac{1}{r^4}(2xH + yH'), \\ \ddot{y} &= -\frac{y}{r}G' + \frac{1}{r^4}(2yH - xH'), \end{aligned} \tag{46}$$

where dots denote differentiation with respect to time and primes in G, H, as already agreed, denote differentiation with respect to r and θ. To produce the circles (28) numerically, we generally used the following type of initial conditions:

$$x = x_0, \qquad y = 0, \qquad \dot{x} = 0, \qquad \dot{y}_0 = \pm\left(|x_0|G'_0 - \frac{2}{x_0^2}H_0\right)^{\frac{1}{2}} \tag{47}$$

where x_0 is chosen arbitrarily from the allowed region (31), $G'_0 = G'(r = |x_0|)$ and H_0 is the value of $H(\theta)$ at $\theta = 0$ for $x_0 > 0$ or at $\theta = \pi$ for $x_0 < 0$.

So, as a rule, we can identify an orbit solely by the initial conditions $(x_0, \dot{y}_0)$, as given by (47).

Figure 1(*a*) was drawn for $G = -1/r$, $H = -\cos\theta$. The FBC is the unit circle (f): $r + 2\cos\theta = 0$, centred at the point $(-1, 0)$. The FZCC (c): $r = r(\theta)$ is defined by the quadratic $r^2 + 3r\cos\theta + 2 = 0$ and lies entirely inside the (closed) FBC (f) in the nonallowed region having in common with it the point of tangency $(-2, 0)$. The curvature K is positive everywhere outside (f) and also inside (c).

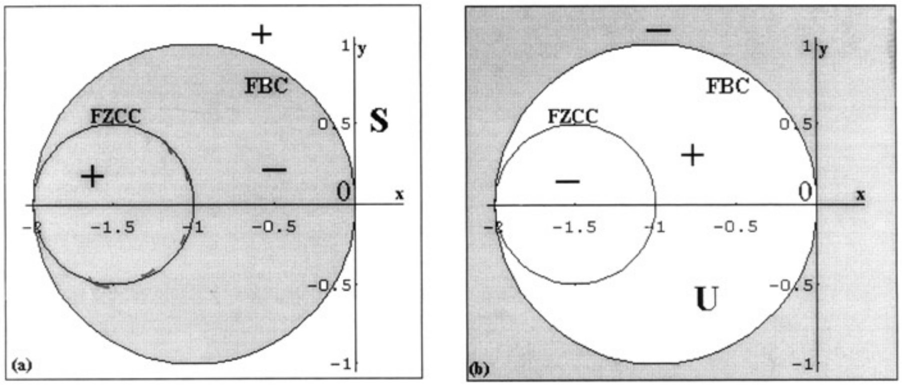

Figure 1. (*a*) All circles ($r > 2$) and circular arcs ($r < 2$) are lying entirely in the positive-curvature unbounded region (unshaded region) and are stable. (*b*) The circular arcs ($r < 2$) in the mixed-sign curvature, bounded region (unshaded region) are unstable.

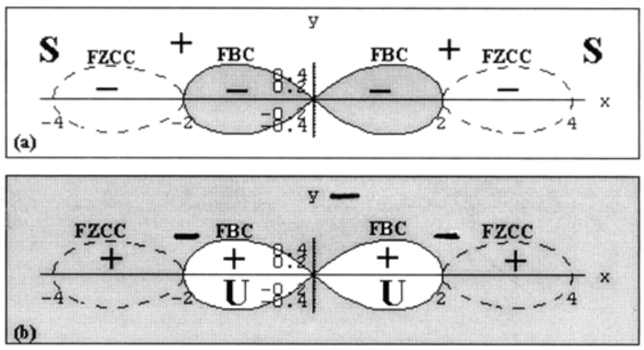

Figure 2. (*a*) All circles ($r > 2$) and circular arcs ($r < 2$) in the unbounded allowed region are stable. Some circles (those with radii $r_0 : 2 < r_0 < 4$) pass through a narrow region of negative curvature, yet they are also stable. (*b*) Only unstable circular arcs ($r < 2$) exist in the bounded allowed region (as in figure 1(*b*)), in spite of the fact the curvature is positive everywhere.

The circular arc ($x_0 = 1$, $\dot{y}_0 = \sqrt{3}$), the asymptotic semicircle ($x_0 = 2$, $\dot{y}_0 = 1$) and the full circle ($x_0 = -\frac{5}{2}$, $\dot{y}_0 = -\frac{\sqrt{2}}{5}$) were tested numerically. As expected from (44), they were all stable.

Figure 1(*b*) is drawn for $G = 1/r$, $H = \cos\theta$. The allowed and nonallowed regions, for the circles (28) to survive, now interchange. Also, according to the transformation (27), regions of negative curvature now become regions of positive curvature. The FZCC now lies entirely in the allowed region. The circular arc with $x_0 = -1$, $\dot{y}_0 = 1$ was tested numerically for stability. Although it travels in the positive curvature arc, it was found, as expected by testing (44), unstable.

Figure 2(*a*) corresponds again to the family (28) and to $G = -1/r$, $H = \cos 2\theta$. The allowed region is outside the figure-eight FBC curve (f): $r = 2\cos 2\theta$. There exist in this case two (symmetric to the axes x and y) FZCC (c_1) and (c_2) given by the quadratic: $r^2 - 6r\cos 2\theta + 8 = 0$ and lying outside (f). The curvature is positive in the allowed region everywhere except for the interior of (c_1) and (c_2).

Although the orbit with $x_0 = 3$, $\dot{y}_0 = \frac{1}{3}$ starts in (and periodically passes through) a small region of negative curvature, it is stable.

The circular orbit with $x_0 = 5$, $\dot{y}_0 = \frac{\sqrt{3}}{5}$ and the small circular arc created with the initial conditions $x_0 = 0$, $y_0 = 1$, $\dot{x}_0 = \sqrt{3}$, $\dot{y}_0 = 0$ were also found to be stable.

Finally we drew figure 2(*b*) for $G = 1/r$, $H = -\cos 2\theta$. We checked the orbits $x_0 = \frac{3}{2}$, $\dot{y}_0 = \frac{\sqrt{2}}{3}$ and $x_0 = 1$, $\dot{y}_0 = -1$ in the allowed region, i.e. in the interior of the FBC of figure 2(*b*). They start and were expected to travel in the positive curvature region, yet they were found to be highly unstable.

7. Concluding remarks

The basic finding of this paper is formula (16). It offers the Riemannian curvature K_f of the Maupertuis' manifold M_2^* corresponding to a two degrees of freedom conservative dynamical system with known metric g_{ij}. In contrast to similar formulae found in the literature, the family needs not include members of the same total energy E. Instead, it is assumed that all members of the family have a common analytical representation $f(u, v) = c$ in the configuration space u, v.

The FZCC were introduced and their relationship with the FBC, known from the inverse problem, was discussed.

Application of polar coordinates (r, θ) and the example of the family of concentric circles (28) produced by any of the potentials (29) to formula (16) has led to the formula (34) for the curvature K of the circles. Independently derived integrability of all potentials (29) has also shown that stability of the circles depends on the sign of the function S, given by (44). The combination of these facts led to formula (45) relating the signs of the Riemannian curvature to the sign of S. Judging from this particular case of circular orbits, we come to understand that we cannot draw a definite and rigorous conclusion regarding the connection of curvature and stability. We can, however, state in a flexible frame that, as a rule, stability is accompanied by positive curvature and negative curvature implies instability. Inversely, positive curvature may be accompanied by stability (figure 1(*a*)) or instability (figure 2(*b*)) and, of course, instability of an orbit does not suffice to provide information about the sign of the curvature at all points visited by the orbit (figure 1(*b*)).

The state of affairs regarding possible connection of the integrability (Aizawa 1972) of the potential to the sign of the Riemannian curvature must be more or less similar. Although *all the potentials* (29) generating the circles (28) are integrable, the circles travel both in positive and negative curvature regions.

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