HOMOGENIZATION THEORY FOR ADVECTION–DIFFUSION EQUATIONS WITH MEAN FLOW

By

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ABSTRACT

The problem of periodic homogenization for advection–diffusion equations is considered in this thesis. We study the problem for velocity field which consist of two parts, a spatiotemporally dependent mean flow and a periodically fluctuating part. Under the assumption of scale separation between the characteristic length and time scales of the mean flow and the fluctuations we derive an effective equation which governs the evolution of the passive scalar field at the length and time scales of the mean flow. We are mostly interested in understanding the effect of the mean flow upon the homogenized transport of the passive scalar field.

We show rigorously that for mean flows which are either weak or equal in strength with the fluctuations the effective equation is an advection–diffusion equation with an effective diffusion tensor which is computed through the solution of an auxiliary partial differential equation with periodic boundary conditions, the *cell problem*. We show that the structure of the cell problem depends on the temporal period of oscillations of the fluctuations in the velocity field. A very efficient algorithm of the solution of the cell problem is also developed.

For weak mean flows and in the absence of slow modulations in the fluctuations the effective diffusion tensor is constant, independent of the mean flow. When fluctuations and mean flow are of equal strength the effective diffusion tensor is a function of space and time, with values depending upon the mean flow as well as the slow modulations in the fluctuations. When the mean flow is stronger than the fluctuations one cannot in general obtain an effective equation which is independent of the fast variables. In this regime greater variability of the effective diffusivity can occur, depending upon the specific properties of the mean flow: from no enhacement in the diffusivity to the appearence of resonant enhanced diffusion phenomena that boost the diffusivity far above its bare molecular value. The problem is studied through a combination of formal asymptotic analysis of the cell problem, numerical experiments and rigorous analysis using the method of two–scale convergence.

The symmetry properties of the effective diffusion tensor are also studied. Necessary and sufficient conditions for the symmetry of the effective diffusivity are derived for steady velocity fields and the dependence of the antisymmetric part of the diffusivity on the Peclet number is analyzed. Numerical examples for both steady and and time dependent velocity

field are also presented.

Finally, we propose a systematic way of studying higher order homogenization using the method of two–scale convergence. Our technique enables us to rigorously obtain higher order effective equations in cases where the multiple scales technique breaks down.

CHAPTER 1
STATEMENT OF THE PROBLEM AND REVIEW OF THE LITERATURE

1.1 Introduction

A problem of great practical and theoretical interest is that of the transport of physical entities in fluids. Examples include temperature, potential vorticity, salinity in the ocean and pollutants in the environment. In many instances, such as the transport of ozone in the stratosphere, the transport of nutrients in the ocean or the concentration of dyes used in visualizing flows, it is reasonable to assume that the physical entity that we consider does not affect its fluid environment. In this case, and under the additional assumption that inertial effects can be neglected, it is called a *passive tracer* and its concentration a *passive scalar field*. The problem of understanding the evolution of passive tracers is of great interest in various fields of science and engineering, such as astrophysics (in particular in connection with the effectiveness of mixing produced by internal waves in the sun) [23, 60, 93], transport of contaminants in saturated porous media [15, 18, 43, 51, 61, 64], plasma physics [26, 82, 107], transport in arrays of steady or time dependent convection cells [102, 104, 105], fully developed turbulence [68] and ocean/atmosphere science [31, 50].

Passive tracers are being transported in two ways: ordinary molecular diffusion as well as passive advection by their fluid environment. Consequently, the evolution of the passive scalar field is governed by the following initial value problem:

$$\frac{\partial T(\mathbf{x},t)}{\partial t} + \mathbf{u}\cdot\nabla T(\mathbf{x},t) = \kappa\Delta T(\mathbf{x},t) + f(\mathbf{x},t) \;\text{ in }\; \mathbb{R}^d \times (0,\infty) \tag{1.1a}$$

$$T(\mathbf{x},t=0) = T_0(\mathbf{x}) \;\text{ on }\; \mathbb{R}^d, \tag{1.1b}$$

where κ is the molecular diffusivity, $f(\mathbf{x},t)$ is an external pumping field and $\mathbf{u}(\mathbf{x},t)$ is the velocity field of the fluid in which the passive tracer is immersed and which will always taken to be incompressible, $\nabla \cdot \mathbf{u} = 0$. In principle, the velocity field should be obtained from the solution of a more complex system of equations. For example, when considering the transport of pollutants in the atmosphere, the evolution of the velocity field is governed by the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} + \mathbf{u}(\mathbf{x},t) \cdot \nabla \mathbf{u}(\mathbf{x},t) = -\nabla p(\mathbf{x},t) + \nu \Delta T(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t) \tag{1.2a}$$

$$\nabla \cdot \mathbf{u}(\mathbf{x},t) = 0, \tag{1.2b}$$

where ν is the viscosity, $p(\mathbf{x},t)$ is the pressure field and $\mathbf{F}(\mathbf{x},t)$ is an external force. On the other hand, the velocity field in heterogeneous porous media is determined by Darcy's law [101]:

$$\mathbf{u}(\mathbf{x}) = -\mathcal{K}(\mathbf{x}) \nabla p(\mathbf{x}) \tag{1.3a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.3b}$$

where $\mathcal{K}$ is the permeability tensor and $p(\mathbf{x})$ is the pressure field.

Ideally, one would hope to obtain an explicit representation of the velocity field through the solution of either (1.2) or (1.3), together with the appropriate initial and boundary conditions, and then use this velocity field in (1.1) to solve for the passive scalar field. However, this program is usually impossible to carry out in practice since it is very difficult to obtain exact solutions for the equations governing the velocity field.

Moreover, fluid flows of geophysical and astrophysical interest are usually very complex, being active on a continuum of length and time scales [82]. Consequently, even if a detailed description of the velocity field could be had - based on either experimental measurements or phenomenological theories- then the determination of the evolution of the passive scalar would require the solution of a variable coefficients advection—diffusion equation. The fact that the velocity field should contain a wide range of excited length and time scales would impose severe restrictions on the grid size and consequently the time step for an efficient numerical treatment of (1.1).

Moreover, this detailed information on the distribution of the passive scalar field even if available would not be of much practical use. What is important is the description of the evolution of the passive scalar field on large length and time scales, that is the scales of *observational interest*. Consequently, a more practical reformulation of the original problem, expressed through (1.1) and either (1.2) or (1.3), is as follows: first, we replace the equation for the velocity field with

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0 \tag{1.4a}$$

$$\mathbf{u}(\mathbf{x}, t) \text{ has prescribed statistical structure} \tag{1.4b}$$

and the goal of solving the advection—diffusion equation (1.1) with

$$\begin{gathered}\text{Describe the evolution of } T(\mathbf{x}, t) \text{ at long times and large scales,} \\ \text{so that it "averages" over the small scale statistical structure of } \mathbf{u}(\mathbf{x}, t)\end{gathered} \tag{1.5}$$

The term statistics in (1.4b) is used in a broad sense. By this we mean that, instead of considering velocity fields that are exact solutions of the fluid equations, we would like to consider simplifying models for the small scale fluctuations in the velocity field that are realistic enough so that they incorporate some of the real physical properties of the flow in the problem under investigation, yet they are simple enough so that they render equation (1.1) amenable to analysis and computations.

In the remaining of this introductory chapter we present some background material relevant to the problem that we shall study in this work. We first discuss in some detail the problem of transport of passive tracers in the ocean and atmosphere, which is the major motivation for the undertaking of this project. This physical problem will provide us with the motivation for the types of velocity fields that we shall consider as well for the type of issues that we would like to address. We shall also give a brief review of the literature on the problem of periodic homogenization for advection—diffusion equations, in particular in connection to the effect of the mean flow and the symmetry properties of the effective diffusivity tensor. We also discuss about approaches to this problem other than the method of homogenization. We close this chapter with a brief presentation of new results contained in this thesis and an overview of the contents of the chapters of this manuscript.

1.2 Transport of Passive Tracers in the Ocean and Atmosphere

We are particularly interested in problems related to the transport of passive tracers in the atmosphere and ocean. As examples we mention the transport of ozone in the atmosphere [55, 90, 91, 92] and the transport of pollutants or nutrients in the ocean [21, 46, 49, 50, 66, 81, 116]. Apart from the great practical importance of understanding the transport of passive tracers in the ocean, this problem is also very interesting for modelling purposes: since many tracers can be measured in the real ocean, they provide a useful means to evaluate the fidelity of ocean model simulations [50]. In connection to this, we hope that the study of the transport of passive tracers may also be useful for the understanding of the transport of active tracers, such as salinity in the ocean and moisture in the atmosphere.

The evolution of a passive tracer in the ocean and the atmosphere is influenced by both the mean flow, active at the scales of observational interest, as well as the mesoscale eddies, active at smaller scales. Decomposing the velocity field into the mean flow and the eddy field, $\mathbf{u} = \mathbf{V} + \mathbf{v}$, (1.1) becomes:

$$\frac{\partial T(\mathbf{x},t)}{\partial t} + (\mathbf{V}+\mathbf{v})\cdot\nabla T(\mathbf{x},t) = \kappa\Delta T(\mathbf{x},t) \text{ in } \mathbb{R}^d \times (0,\infty) \tag{1.6a}$$

$$T(\mathbf{x}, t=0) = T_0(\mathbf{x}) \text{ on } \mathbb{R}^d, \tag{1.6b}$$

where for simplicity we have assumed that there are not any external sources or sinks. Moreover, we assume that the domain in the ocean/atmosphere in which we consider the problem is large enough so that we can neglect the boundary conditions and pose the problem in the whole space.

Even with today's computer resources, most ocean components of climate models use a coarse resolution that does not resolve ocean eddies [46]. This is due to computational speed as well as limited observational data. Consequently, the effect of the eddy field on the large scale behavior of the passive scalar has to be modelled in a satisfactory way. This is the problem of *parametrization*.

To be more precise, by taking an appropriately defined average of (1.6) we obtain:

$$\frac{\partial \langle T\rangle(\mathbf{x},t)}{\partial t} + \mathbf{V}\cdot\nabla\langle T\rangle(\mathbf{x},t) = \kappa\Delta\langle T\rangle(\mathbf{x},t) - \nabla\cdot F \;\text{ in }\; \mathbb{R}^d\times(0,\infty) \tag{1.7a}$$

$$\langle T\rangle(\mathbf{x},t=0) = T_0(\mathbf{x}) \;\text{ on }\; \mathbb{R}^d, \tag{1.7b}$$

where the *eddy flux* $\mathbf{F}$ is defined as:

$$\mathbf{F} = \langle \mathbf{v}\, T\rangle \tag{1.8}$$

Thus, the problem reduces to expressing the eddy flux as a functional of the average field $\langle T\rangle$:

$$\mathbf{F} = \mathcal{F}(\langle T\rangle) \tag{1.9}$$

The traditional approach to this type of problems is to model the effect of mesoscale eddies, or more generally of turbulent fluid motion, as standard Fickian diffusion [69]. Thus, we assume that (1.9) can be written as:

$$\mathbf{F} = -\mathcal{K}^*\cdot\nabla\langle T\rangle \tag{1.10}$$

where $\mathcal{K}^*$ ia a constant symmetric second order tensor, usually taken to be a multiple of the identity tensor, the *effective diffusivity* tensor. Substituting (1.10) into (1.7) we obtain an equation for the average passive scalar field $\langle T\rangle$:

$$\frac{\partial \langle T\rangle(\mathbf{x},t)}{\partial t} + \mathbf{V}\cdot\nabla\langle T\rangle(\mathbf{x},t) = \kappa\Delta\langle T\rangle(\mathbf{x},t) + \nabla\cdot(\mathcal{K}^*\cdot\nabla\langle T\rangle(\mathbf{x},t)) \;\text{ in }\; \mathbb{R}^d\times(0,\infty) \tag{1.11a}$$

$$T(\mathbf{x},t=0) = T_0(\mathbf{x}) \;\text{ on }\; \mathbb{R}^d, \tag{1.11b}$$

This parametrization essentially suggests that the only effect of the eddy field is to enhance the diffusion of the passive scalar field, as measured at the length and time scales of the mean flow. Thus, the problem of obtaining a coarse grained description for the evolution of the passive tracers reduces to the determination of the coefficients in $\mathcal{K}^*$. This is usually done through either ad hoc phenomenological assumptions such as mixing length arguments ([112] and references therein), or through fitting with observations, i.e. matching the distribution of known tracers for given mean flows [55, 90, 92].

However, it has been well documented in the atmosphere and ocean science literature that this simple parametrization is not adequate in most instances, especially in connection with tracer transports in the stratosphere as well as the ocean [46, 55, 81, 90, 91, 92]. In particular, both observational and numerical model studies have suggested that tracers are advected over large scales by a velocity different from the mean velocity. Moreover, the effective diffusivity tensor $\mathcal{K}^*$ should be spatially inhomogeneous.

These issues can be resolved in principle by introducing a space-time dependent effective diffusivity tensor $\mathcal{K}^*(\mathbf{x}, t)$. The common practice is to decompose $\mathcal{K}^*(\mathbf{x}, t)$ in the following way:

$$\mathcal{K}^*(\mathbf{x}, t) = \mathcal{S}(\mathbf{x}, t) + \mathcal{A}(\mathbf{x}, t) \tag{1.12}$$

where $\mathcal{S}$ and $\mathcal{A}$ are the symmetric and antisymmetric parts, respectively:

$$\mathcal{S}_{ij} = \frac{1}{2}(\mathcal{K}^*_{ij} + \mathcal{K}^*_{ji}) \tag{1.13a}$$

$$\mathcal{A}_{ij} = \frac{1}{2}(\mathcal{K}^*_{ij} - \mathcal{K}^*_{ji}) \tag{1.13b}$$

This decomposition is natural since the symmetric and antisymmetric parts of the effective diffusion tensor represent, as we shall discuss, physically different phenomena. Using now equation (1.12) in (1.6) we obtain:

$$\begin{aligned} \frac{\partial \langle T \rangle}{\partial t} \quad &+ \quad (\mathbf{V} + \mathbf{U}^s + \mathbf{U}^a) \cdot \nabla \langle T \rangle \\ &= \quad \kappa \Delta \langle T \rangle + \mathcal{S}_{ij} \frac{\partial^2 \langle T \rangle}{\partial x_i \partial x_j} \ \text{ in } \ \mathbb{R}^d \times (0, \infty) \end{aligned} \tag{1.14a}$$

$$T(\mathbf{x}, t = 0) = T_0(\mathbf{x}) \ \text{ on } \ \mathbb{R}^d, \tag{1.14b}$$

where

$$\mathbf{U}^s = -\nabla \cdot \mathcal{S}, \quad \mathbf{U}^a = -\nabla \cdot \mathcal{A} \tag{1.15}$$

According to this parametrization the average field $\langle T \rangle$ is being advected by an effective velocity which consists of the three parts, the mean flow, the velocity $\mathbf{U}^s$ due to the symmetric part of $\mathcal{K}^*$ and the velocity $\mathbf{U}^a$ due to the antisymmetric part of $\mathcal{K}^*$. Various terms have been used for the latter; for example, *bolus velocity* [23] or *eddy-induced transport velocity* [46]. The relationship of $\mathbf{U}^s$ and $\mathbf{U}^a$ to the Stokes drift in the case where the eddy flux is due to wave like structures has also been discussed [81].

From the above discussion we see that the antisymmetric part of the effective diffusivity is purely advective in nature and does not induce additional dissipation. Moreover, the eddy induced transport velocity is incompressible, $\nabla \cdot \mathbf{U}^a = 0$ and the antisymmetric tensor $\mathcal{A}$ plays the role of the *stream matrix* [42] for this incompressible velocity field. On the other hand, the symmetric part $\mathcal{S}$ is partly diffusive and partly convective in nature. Moreover, the velocity due to the symmetric part $\mathbf{U}^s$ is not solenoidal, neither is it potential in general [81, 91].

The relative strength of advection and diffusion due to the eddies depends on the physical situation under investigation. For example, in [55] and the references therein it has been argued that the eddy fluxes due to planetary scale waves are primarily advective rather than diffusive in nature. However, in general neither advection nor diffusion may be neglected a priori [92].

Since the parametrization based on the introduction of a space-time dependent non-symmetric effective diffusion tensor seems to be compatible with some observations and numerical simulations, it is important to be able to derive such a coarse-grained description for the evolution of the passive scalar field in a systematic and, as far as possible, rigorous way. Various attempts have been made towards this direction. The effect of mesoscale eddies on the large scale evolution of passive tracers in the ocean has been effectively taken into account through the Gent-McWilliams parametrization in [46], see also [49]. The validity of this parametrization was tested in numerical simulations where ocean data were also used. It has also been justified for small amplitude eddy motions in [81, 92]. Methods based on the Lagrangian formulation of the equation that governs the evolution of the passive scalar

	Mean Flow	Eddy Field
Length Scale	L $= 10^6$ m	$\ell = 10^5$ m
Time Scale	T $= 10^7$ s	$\tau = 10^6$ s
Velocity	$\overline{u} \sim \overline{v} \sim 10^{-2}$ ms^{-1}	$u' \sim v' \sim 10^{-1}$ ms^{-1}
Stream Function	$\overline{\psi} \sim \overline{u}L \sim 10^4$ m^{2}s^{-1}	$\psi' \sim u'\ell \sim 10^4$ m^{2}s^{-1}

Table 1.1: Orders of magnitude for mean flow and eddy field

have also been used, [4, 81, 90].

In this work we shall try to derive this parametrization for simple classes of model flows using a different approach, namely the method of periodic homogenization. The simplified model for the velocity field that we shall consider will enable us to study the problem in a systematic way. The goals of this program are:

1. Derive rigorously the advection—diffusion equation that governs the evolution of a passive tracer at length and time scales comparable to those of the mean flow.

2. Compute the effective diffusivity tensor in an efficient and easily implementable way.

3. Study the properties of the effective diffusivity and in particular of its antisymmetric part. More specifically, understand the effect of both the mean flow and of the eddy field on the structure of $\mathcal{K}^*$.

As a motivation of the approach that we shall take, let us consider a simplified model for the transport of a passive tracer in the ocean. Neglecting the effects of stratification as well as of turbulence at small scales, we can decompose the velocity field into the mean flow and the mesoscale eddies. In Table 1.1, which is taken from [32], we present typical orders of magnitudes for the length scales, time scales and strengths of the mean flow and the eddy field. From this table two interesting conclusions can be drawn:

1. There is a clear separation of length and time scales between the mean flow and the eddy field, $\delta \sim \frac{\ell}{L} \sim \frac{\tau}{T} \sim 10^{-1}$

2. The magnitude of the fluctuating part of the velocity field is greater than or comparable to the magnitude of the mean flow.

The fact of scale separation suggests that multiscale techniques could be applied in order to derive an equation for the evolution of the passive scalar field at length and time scales comparable to those of the mean flow. Moreover, it suggests that this multiscale method should include the effects both of the mean flow as well as of the eddy field, taking into account that the fluctuations are comparable to or stronger than the mean flow. Assuming that the mesoscale field can be adequately modelled as having a periodic structure, for example steady periodic solutions of the Navier Stokes equations that were used in [84] or oscillatory shear flows that were considered in [21, 66, 116], it seems that one could approach this problem using the tools of periodic homogenization. Needless to say, the assumption of periodicity is an oversimplification and should be thought of as the first step towards the study of the problem for more realistic velocity fields. Before embarking on this project, let us review the method, in particular as it applies to advection—diffusion equations with oscillating velocity fields.

1.3 Homogenization Theory for Advection–Diffusion Equations with Periodic Incompressible Velocity Fields

1.3.1 Introduction

In this section we introduce the method of homogenization, in particular as it applies to the problem under investigation. After a general discussion we focus on homogenization for advection—diffusion equations. We wish to review the known results on several issues: the derivation of the homogenized equations in the absence of mean flow, the asymptotic behavior of the effective diffusivity tensor with respect to the nondimensional parameters of the problem, the effect of the mean flow on the homogenized equation and the symmetry properties of the effective diffusivity.

The method of homogenization is a powerful method for obtaining effective equations and computing effective parameters in problems where phenomena occur at various well-seperated length and time scales. Thus, under the assumption of scale separation it enables us in principle to study the effect of fluctuations upon the macroscopic behavior of various physical systems. It has been applied with great success to a variety of problems, such as the theory of composite materials [9, 37, 100], the flow in porous media [29, 56],[54, ch. 5], turbulent diffusion [69] and turbulent combustion [71, 72].

From a mathematical point of view, homogenization theory is concerned with the effects of high frequency oscillations in the coefficients upon solutions of partial differential equations

[40, p. 218]. In order to understand this issue, methods of varying degree of sophistication have been developed: formal multiple scale expansions [14], the method of oscillating test functions [14], two-scale convergence [1, 2, 83], and the perturbed test functions approach [39]. As general references for the theory of periodic homogenization we mention the books [14, 28, 58, 99].

1.3.2 Periodic Homogenization for Advection–Diffusion Equations

Let us now focus on the problem of periodic homogenization for the initial value problem (1.1). We consider incompressible velocity fields which consist of a superposition of a mean flow $\mathbf{V}$ and a periodic fluctuating part $\mathbf{v}$ with mean zero. The mean flow varies on length and time scales L, T whereas the fluctuating part varies on scales ℓ, τ:

$$\mathbf{u} = \mathbf{V}(\frac{\mathbf{x}}{L}, \frac{t}{T}) + \mathbf{v}(\frac{\mathbf{x}}{\ell}, \frac{t}{\tau}) \tag{1.16}$$

We assume that there is a clear scale separation between the length and time scales of the mean flow and the fluctuations:

$$\frac{\ell}{L} << 1, \quad \frac{\tau}{T} << 1 \tag{1.17}$$

We also assume that the initial data in (1.1) are slowly varying, at a length scale comparable to that of the mean flow. Under these assumptions equation (1.1) becomes:

$$\frac{\partial T(\mathbf{x},t)}{\partial t} + \left(\mathbf{V}(\frac{\mathbf{x}}{L}, \frac{t}{T}) + \mathbf{v}(\frac{\mathbf{x}}{\ell}, \frac{t}{\tau})\right) \cdot \nabla T(\mathbf{x},t) = \kappa \Delta T(\mathbf{x},t) \ \text{ in } \ \mathbb{R}^d \times (0,\infty) \tag{1.18a}$$

$$T(\mathbf{x}, t=0) = T_0(\frac{\mathbf{x}}{L}) \ \text{ on } \ \mathbb{R}^d, \tag{1.18b}$$

The goal of homogenization theory is to obtain an effective equation that describes the evolution of the passive scalar field at length and time scales comparable to those of the mean flow. We briefly review earlier work on this problem.

A simpler version of this problem that has attracted much attention over the last twenty years is the one in which the mean flow is absent. In this setting the velocity field consists only of an incompressible zero mean periodic component $\mathbf{v}$. Building upon earlier work [14, 87] it was shown in [77], see also [16, 69, 76, 98] that at length and time scales large compared to those of the velocity field the passive scalar field satisfies a diffusion equation with the diffusivity being always enhanced beyond its molecular value. The enhancement in the diffusivity is computed through the solution of an auxiliary partial differential equation with periodic boundary conditions, *the cell problem.* Thus, for mean zero periodic incompressible fields the parametrization (1.10) is justified.

The derivation of the homogenized equation and of the cell problem are valid for all mean zero incompressible velocity fields under mild regularity conditions without any further assumptions. In particular, the homogenized equation is uniformly valid for all values of the molecular diffusion κ and the temporal period of oscillations τ of the velocity field.[1] However, the effective diffusivity tensor does depend on these parameters: $\mathcal{K}^* = \mathcal{K}^*(\kappa, \tau)$. Expressing this in terms of the appropriate nondimensional parameters of the problem we have:

$$\mathcal{K}^* = \mathcal{K}^*(S, Pe) \tag{1.19}$$

where Pe is the *Peclet number* that measures the relative strength of fluid advection as compared with molecular diffusion and S is the *Strouhal number* which measures the ratio of the characteristic velocity sweeping time to the period of its oscillation [17]. Precise definitions and discussion will be given in the next chapter.

The study of the dependence of the effective diffusivity on the nondimensional parameters of the problem has been the subject of many studies. This is a particularly interesting question since for most realistic flows the Peclet number is very large. To this end, a complete theory based on Stieltjes integral representation formulas for the effective diffusivity as well as variational principles was developed by Avellanada and Majda for steady velocity fields [6, 7], see also [41] for variational principles. The integral representation formulas were also extended to time dependent flows in [8]. Based on these representation formulas it was shown that the effective diffusivity in each direction is an increasing function of the Peclet number, for fixed Strouhal number. Moreover, upper and lower bounds on the enhancement in the diffusivity were obtained:

$$C_1(S) \leq \hat{e} \cdot \mathcal{K}^* \cdot \hat{e} \leq C_2(S)\, Pe^2, \tag{1.20}$$

[1] To be more precise, the homogenization theorem holds in the parameter range $\delta << S,\ Pe << \frac{1}{\delta}$ where S, Pe are the Strouhal and Peclet numbers, respectively, and δ the parameter measuring the scale separation

where $\hat{e}$ denotes a unit vector in $\mathbb{R}^d$ and C is a constant independent of the Peclet number. The concepts of *maximally enhanced diffusion*, when the effective diffusivity scales like Pe^2 and *minimally enhanced diffusion*, when $\mathcal{K}^*$ is bounded by a constant independent of the Peclet number where introduced. It was shown that the asymptotic behavior of the effective diffusivity with respect to the Peclet number depends upon the topological properties of the velocity field. For example, it was shown that the effective diffusivity for steady cellular flows in two dimensions scales like $\sqrt{Pe}$ for large Peclet numbers, whereas for two dimensional shear flows it scales like Pe^2 in the direction of the shear and like C in the orthogonal direction, [52]. Necessary and sufficient conditions for maximally and minimally enhanced diffusion for time dependent velocity fields were derived by Mezic et al. in [80] using techniques from ergodic theory and dynamical systems.

1.3.3 The Effect of Mean Flow

The difficulty in extending the homogenization theorem to the case where a spatiotemporally dependent mean flow is present lies on the fact that the time scales associated with transport and diffusion processes are different. Thus, at the diffusion time scale (precise definitions will be given at the next chapter) at which the homogenization theorem that we desribed in the previous paragraphs is valid, the transport due to the mean flow is an $O(\frac{1}{\delta})$ quantity, δ being the parameter measuring the scale separation. Thus, an asymptotic treatment of this problem based on multiple scale expansions requires an extension of the homogenization method to the case where effective equations should contain terms of different orders of magnitude.

This difficulty was circumvented in [97] for steady velocity fields and constant mean flows by keeping the troublesome $\frac{1}{\delta}\mathbf{V}\cdot\nabla$ term together with the $O(1)$ terms in the expansion. The effective equation was later justified using the maximum principle, see also [86]. A different approach was taken in [70, 74], see also [58, ch. 2], [59]. In these works, the aforementioned difficulty was removed by studying the problem in a frame comoving with the mean flow, i.e. by introducing *mean Lagrangian coordinates*. It was shown that in the homogenized equation expressed in mean Lagrangian coordinates is a diffusion equation with the diffusivity being always enhanced.

The presence of a constant mean flow, apart from inducing an $O(\frac{1}{\delta})$ advection term in the effective equation, when expressed in Eulerian coordinates, has a profound effect on the properties of the effective diffusivity, since now, apart from the nondimensional parameters, it also depends on the mean flow:

$$\mathcal{K}^* = \mathcal{K}^*(S, Pe, \mathbf{V}) \tag{1.21}$$

The quantitive study of the dependence of the effective diffusivity on the mean flow was first given by Koch et al. in [64] and later rigorously justified, in a more general framework, by Majda and Mclaughlin for steady two dimensional incompressible velocity fields in [70]. Relevant work was also done by Bhattacharya et al. in [15] and by Mauri in [74]. The surprising result of these studies is that the presence of a constant mean flow has a dramatic effect on the asymptotic behavior of $\overline{\mathcal{K}}^*$ as $Pe \to \infty$. Roughly speaking, mean flows with rationally related components lead to maximally enhanced diffusion in all directions other than the one perpendicular to themselves. In this direction the diffusion will be minimally enhanced. This result holds for most velocity fields, with the exception of shear flows aligned perpendicularly to the mean flow and with flows with only low number nonzero Fourier modes such as the Childress-Soward flow. On the contrary, mean flows with irrational ratios lead, provided some technical conditions are satisfied, to minimally enhanced diffusion in all directions for flows with no stagnation points. The aforementioned rigorous results, valid at the limit $Pe \to \infty$, where shown to be present at finite Peclet numbers through numerical simulations of the cell problem [70]. Without discussing the technical details of the proofs of the above results, we mention that they crucially rely on the ergodic properties of the flow generated by the velocity field on the two-dimensional unit torus $\mathbb{T}^2$. Naturally, the ergodic properties of the velocity field play a very important role in the study of the homogenization of transport equations, i.e. the case of infinite Peclet number, [36, 57, 108]. Let us also remark that a similar phenomenon was reported by Golden et al. in [47] for potential flows when the potential has two characteristic wavelengths: the effective diffusivity depends on whether the ratio of the wavelengths is rational or irrational.

The above analysis concerning the effect of a constant mean flow to the effective diffusivity for steady two dimensional velocity fields was extended to the case of spatiotemporal flows in various ways. Resonant enhanced diffusion due to synchronization between the constant mean flow and the temporal oscillations of the flow for two dimensional nonsteady shear flows was shown to occur by Majda and Kramer in [69]. This phenomenon was also demonstrated by Castiglione et al. in [24] through numerical simulations for random shear flows.

The effect of rapidly fluctuating temporal mean winds on the effective diffusivity was studied by Bonn and McLaughlin in [17]. They considered velocity fields of the form $\mathbf{u} = \mathbf{V} + \mathbf{A}(t/\tau) + \mathbf{v}(\mathbf{x}/\ell)$ with the aim of understanding the extend to which the presence of rapidly fluctuating mean winds alters the sensitive dependence of $\mathcal{K}^*$ on $\mathbf{V}$ as it is predicted from the steady theory [2] .Through asymptotic analysis of the cell problem they showed that at the limit $S \to \infty$, for fixed Pe, the steady problem is retrieved, whereas for at the limit $S \to 0$ the effective diffusion coefficients are represented by an average over the steady geometry. Numerical experiments were also performed and a non-monotonic dependence of the diffusion coefficients on the Peclet number was reported at certain regimes of the parameter space.

The results reported so far exhibit the very complicated, nonlinear, dependence of the effective diffusivity on the mean flow. However, they do not provide us with a complete description to the problem since they are restricted to constant mean flows. It is important to extend these results to nontrivial, space-time dependent mean flows. A first step towards this direction was taken by Majda and Kramer in [69], see also [72]. In these works a nontrivial mean flow which is $O(\delta)$ weak compared to the fluctuations was introduced. The fact that the mean flow is weak compared to the fluctuations ensures that at the diffusion time scale the advection due to the mean flow and the enhanced diffusion due to the rapid oscillations of the velocity field are of the same order of magnitude, so the homogenization theorem can be easily extended to this problem. It was shown that the effective equation is an advection—diffusion equation with the advection being governed by the mean flow alone and the effective diffusivity being constant, independent of the mean flow.

The presence of a spatiotemporally dependent mean flow which is of the same order of magnitude as the fluctuations was studied by Mazzino in [75] and by Castiglione et al. in [24]. Using formal multiple scale expansions, an effective equation was obtained, together with the cell problem. Now the mean flow enters into the cell problem, as in the case of constant mean flow, and consequently the effective diffusivity is a function of both space and time and depends on the local properties of the mean flow. These equations were then used

[2] We emphasize the fact that in the framework that we shall develop in this work $\mathbf{A}(t/\tau)$ will not be considered a rapidly fluctuating mean wind but, rather, a part of the fluctuating component of the velocity field

to study the effect of the mean flow on the effective correlation times of turbulent transport in [75]. Unfortunately, the homogenized equation that was derived in these papers is not correct, since it consists of an advection—diffusion equation in which the advection term due to the mean flow is of the same order of magnitude as the enhanced diffusion. We have already remarked that, since the time scales of advection and diffusion are different, it is not possible to find a rescaling in time where the effects of diffusion and advection due to the mean flow are balanced. We shall discuss this issue in the next chapter.

A more careful study of this problem was conducted by Majda and Souganidis in [72]. They showed rigorously that at the time scale defined from advection due to the mean flow, to leading order the fluctuations in the velocity field have no effect on the homogenized transport: the passive scalar field is transported due to the mean flow alone. In particular, the small scale inhomogeneities of the velocity field do not influence the effective transport of the passive scalar.

A related problem for solute transport in porous media was investigated by Bourgeat, Jurak and Piatnitski in [18]. The authors argued that, since the steady velocity field is due, according to Darcy's law (1.3), to the presence of a pressure gradient, it is not realistic to neglect the presence of a non trivial mean flow. They studied the initial-boundary value problem for an advection—diffusion equation with the velocity field satisfies Darcy's law on a strip in $\mathbb{R}^d$. By performing a careful study of the boundary layers as well the initial layer and using interior Holder estimates and the maximum principle the authors were able to obtain the complete asymptotic expansion for the passive scalar field as well as the velocity field. They then used this expansion to obtain an effective advection—diffusion equation, with a spatially dependent effective diffusivity which is $O(\delta)$ weak compared to the advection due to the mean flow.

Bhattacharya in [12, 13] considered steady periodic mean flows. Thus, he studied the homogenization problem for velocity fields of the form (1.16), with both $\mathbf{V}(\delta\,\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ being periodic with the same period. He then studied the asymptotic behavior of the solutions to the advection—diffusion equation (1.14) at long time scales. He showed that at the diffusion time scale of the mean flow the passive scalar field satisfies an effective diffusion equation and the diffusivity is always enhanced. He also studied the behavior of the passive scalar field at long time scales which are small compared to the mean flow diffusion time and showed that the passive scalar field goes through a non diffusive behavior, prior to the final phase which is described by the effective diffusion equation. We emphasize that this is problem is different than the one we shall consider in this work, since it is restricted to periodic mean flows. It is closely related to the method of reiterated homogenization [3].

1.3.4 Symmetry Properties of the Effective Diffusivity Tensor

The investigation that we presented so far in this section was concerned with the properties of the symmetric part of the effective diffusivity. The reason why the symmetry properties of the effective diffusivity have not been studied in depth is that, for constant effective diffusivities, the antisymmetric part has no influence on the large scale, long time evolution of the passive scalar field. Indeed, the right hand side of the effective equation (1.11) can be rewritten as:

$$\begin{aligned} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\mathcal{K}^*_{ij}\frac{\partial \langle T\rangle}{\partial x_j}\right) &= \sum_{i,j=1}^{d} \mathcal{K}^*_{ij}\frac{\partial^2 \langle T\rangle}{\partial x_i \partial x_j} \\ &= \sum_{i,j=1}^{d} \mathrm{symm}(\mathcal{K}^*_{ij})\frac{\partial^2 \langle T\rangle}{\partial x_i \partial x_j} \end{aligned} \tag{1.22}$$

However, as discussed above in section 1.2, when the velocity field consists of a spatiotemporally dependent mean flow with periodic fluctuations, the resulting effective diffusivity is a function of both space and time, and the antisymmetric part contributes to the effective drift (equations (1.14) and (1.15)). Therefore, it is important to study the symmetry properties of $\mathcal{K}^*$. Here we comment on relevant studies by other investigators.

Koch and Brady in [62] studied the symmetry properties of the effective diffusivity in anistropic porous media. They formally showed that the antisymmetric part of the effective diffusivity results from anisotropic porous media that lack a center of reflectional symmetry. It was also argued that the presence of an antisymmetric part is related to the breaking of the Onsager reciprocal relations of non-equilibrium thermodynamics [34]. The same conclusion was also presented by Moffatt in [82]. Based on physical reasoning Koch and Brady also argued that the antisymmetric part of $\mathcal{K}^*$ scales like Pe^3 for small values of the Peclet number and like Pe for large values of Pe, independently of the properties of the mean flow. We emphasize that this conclusion, which was derived formally in [62], is in contrast with the behavior of the symmetric part, whose scaling for large values of the Peclet number depends very sensitively upon the presence and the properties of the mean flow.

Sufficient conditions that ensure the symmetry of the effective diffusivity for two dimensional steady velocity fields were derived by Fannjiang and Papanicolaou in [41]. They proved that a sufficient condition for the effective diffusivity to be symmetric is that it is independent of the sign of the stream function. Based on this observation they derived a set of sufficient conditions that the stream function should satisfy in order for the effective diffusivity to be symmetric. Namely, if the stream function possesses either a translational antisymmetry with respect to a vector, reflectional antisymmetry with respect to a vector or rotational antisymmetry with respect to a point, then the resulting effective diffusivity is symmetric. However, they mentioned that the general necessary and sufficient conditions that ensure the symmetry of $\mathcal{K}^*$ are not known (their findings will be presented in more detail in chapter 4).

In this section we have focused upon the aspects of the work on periodic homogenization for advection—diffusion equations that form the basis for the results that we shall present in the following chapters. However, the problem of effective diffusivity for rapidly oscillating velocity fields has been approached through a variety of different techniques. In the next section we shall present some of the relevant results that were obtained through different approaches.

1.4 Alternative Approaches, Experimental Results

1.4.1 Introduction

Since the problem of the transport of passive tracers in incompressible fluids is of such fundamental importance in various areas of science and engineering, it is natural that it has been approached through various techniques, apart from the method of homogenization. Therefore, it is necessary to review the pertinent results that were obtained through different methods and compare them to the predictions of homogenization theory. This is particularly important since the method of homogenization is an asymptotic theory which is valid in principle at the limit of infinite scale separation. Moreover, the asymptotic analysis of the effective diffusivity with respect to the Peclet number involves another limiting procedure and one would like get some confidence on the applicability of the results that were predicted in the previous section for small but finite δ (the parameter which measures scale separation) and large but finite values of the Peclet number.

In this section we shall present results that were obtained with different methods and are relevant for the issues that were discussed in the previous section: First the the fact that the behavior of the passive scalar field for long but *finite* times and large, finite scales is diffusive and the fact that the diffusivity is always enhanced. Second, the fact that the effective diffusion tensor depends sensitively on the mean flow, in particular in connection to the the asymptotic behavior of the effective diffusivity with respect to the non dimensional parameters of the problem. Third, the possibility for resonant enhanced diffusion. Fourth, the symmetry properties of the effective diffusivity and finally the validity of the homogenization theorem in the presence of a mean flow which is stronger than the fluctuations.

Of course, there is a vast literature on the subject and we do not make any attempt for a complete overview of all the existing literature. For general considerations related to the problem of turbulent diffusion we refer to the review paper by Majda and Kramer [69]. We shall only try to describe very briefly work relevant to the problem of transport of passive tracers in periodic incompressible velocity fields. First, we shall comment on the Taylor dispersion theory and its extensions by Brenner and coworkers. Then we shall mention some exact solutions of (1.1) for simplified flow geometries which result on the explicit computation of the effective diffusivity. We shall also describe results from boundary layer analysis for steady flows, in particular in connection with the scaling of the effective diffusivity with respect to the Peclet number. We shall also present some experimental results related to the transport of tracers in convection cells which show agreement with the predictions of homogenization theory. We shall also present an alternative, Lagrangian, definition of the effective diffusivity and mention works on the computation of the effective diffusivity through Monte Carlo simulations. We shall close this section by discussing briefly the problem of diffusive approximation of the motion of particles in random velocity fields.

1.4.2 Taylor-Aris Dispersion, Exact Solutions, Boundary Layer Techniques

Based on the pioneering work of G. I. Taylor on dispersion in cylindrical capillaries [109, 110, 111] and using the method of moments introduced by Aris in [5], Brenner developed a theory for dispersion in spatially periodic porous media in [22]. This work was later extended in various ways by Koch, Brady and coworkers [61, 62, 63]. Despite the different methodology the predictions of Brenner's theory are the same as those of homogenization theory for steady flows. In particular, diffusion is always enhanced and the computation of the enhancement in the diffusivity reduces to the solution of the cell problem that was obtained from homogenization theory. Mauri and Rubinstein in [97] showed that homogenization theory reduces to the Taylor-Aris theory for the special case of porous media composed of parallel straight tubes [3].

It is in general impossible to derive explicit solutions for the initial value problem (1.1) for general periodic flows. This can be accomplished only under simplifying assumptions upon the geometry of the flow, in particular for shear flow geometries. For example, for oscillatory two dimensional shear flows the advection—diffusion equation (1.1) is exactly solvable. The exact solution was obtained by Zeldovich in [117]. Similar calculations were performed by Kullenberg in [66] and by Young, Rhines and Garrett in [116]. The enhanced diffusion at long times computed through these exact solutions is exactly the one predicted from homogenization theory. In the case of shear flow geometries the cell problem is also exactly solvable [69] and the comparison between the predictions of the two approaches is immediate.

The effective diffusivity for steady cellular flows has also been studied through the application of elaborate matched asymptotic expansions to the advection—diffusion equation by various authors. The investigations of Childress in [26], Shraiman in [102], Rosenbluth et al. in [96] and Soward in [106] led to the conlusion that the effective diffusivity at the limit of large Peclet number has the form $\mathcal{K}^* = \alpha \, Pe^{1/2} \, \mathcal{I}$. The value of the prefactor α was also computed. This result is in complete accordance with the predictions of homogenization theory that were obtained through rigorous analysis of the cell problem [41, 52].

The boundary layer techniques were later applied to flows with more complex geometries consisting of a combination of open channels and vortices by Childress and Soward in [27, 107]. The results reported in [27] are in agreement with the predictions of Majda and McLaughlin which is based on rigorous analysis and numerical solution of the cell problem [70]. Moreover, through boundary layer analysis Childress and Soward were able to prove in [107] the sensitive dependence of the effective diffusivity on the mean flow under the additional assumption that the mean flow be weak compared to the fluctuations.

The studies of Shraiman [102] and Ronsebluth et al. [96] as well as those of Sagues

[3] Despite the fact that homogenization theory and Brenner's theory produce the same result, homogenization theory has a number of advantages: first its predictions can be rigorously justified; second it can be extended to cover much more general situations; for example, we can consider velocity fields with many scales and introduce nonlinear reaction terms, whereas Brenner's method is more problem—dependent; third, it enables us to obtain a complete asymptotic expansion for the passive scalar field in which the higher order coefficients are obtained through the solution of additional cell problems.

and Horsthemke [98] and McCarty and Horsthemke [76] that were mentioned in the previous section were directly influenced from the problem of diffusive transport in two dimensional Rayleigh-Benard convection cells [25, ch. 2]. This problem was studied experimentally by Solomon and Gollub in [105]. The results of their experiments showed that the passive tracers that they utilized behave diffusively after a transient time. Moreover, the enhancement in the diffusivity that they measured is in qualitative agreement with the predictions of the homogenization theory of a wide range of Peclet numbers.

Solomon and Gollub also performed experiments with passive tracers in time dependent Rayleigh-Benard cells. The very interesting result was that the enhancement of the diffusivity is several orders of magnitude larger than the one they measured for steady cells. They attributed this great enhancement to chaotic advection[4]. The difference in the effective transport between steady and time dependent cellular flows was exhibited through numerical solution of the corresponding cell problem by Biferale et al. in [16]. A more rigorous explanation for this phenomenon can be given through the theory of maximally enhanced diffusion for time dependent flows that was developed by Mezic et al. in [80].

1.4.3 Lagrangian Effective Diffusivity, Monte Carlo Simulations

The theoretical investigations that we have been discussing so far are based on the analysis of the initial value problem for the advection—diffusion equation (1.1) which governs the evolution of the concentration of the passive scalar. The method of homogenization as we presented it so far, as well as the boundary layer techniques that we mentioned in this section are based on the *Eulerian* definition of the effective diffusivity as the average flux due to a given concentration gradient [64]. However, an alternative description of the problem is possible through the study of the motion of a single tracer particle which is transported due to the incompressible velocity field $\mathbf{u}(\mathbf{x}, t)$ and diffused due to the molecular diffusion κ. The equations of motion for a tracer particle located initially at $\mathbf{x}_0$ are:

$$d\,\mathbf{X}(t) = \sqrt{2\kappa}\,d\,\mathbf{W}(t) + \mathbf{u}(\mathbf{X}(t), t)\,dt \tag{1.23a}$$

[4]Due to the incompressibility of the velocity field the system of ODEs $\dot{\mathbf{x}} = \mathbf{v}$ is a Hamiltonian system with the stream function being the Hamiltonian. A steady two-dimensional flow leads to a one dimensional autonomous Hamiltonian system which is completely integrable. On the other hand, time dependent two dimensional flows lead to nonautonomous one dimensional Hamiltonian systems which are not in general integrable and can exhibit chaotic behavior.

$$\mathbf{X}(t=0) = \mathbf{x}_0 \tag{1.23b}$$

This is a system of stochastic differential equations. $\mathbf{W}(t)$ represents the d-dimensional Brownian motion [85, ch. 2]. The advection—diffusion equation, with δ-concentrated initial data at $\mathbf{x}_0$, governs the transition probability density for the tracer particle which was initially located at $\mathbf{x} = \mathbf{x}_0$ [45, ch. 3].

Using now (1.23) we can introduce an alternative definition of the effective diffusivity based on the statistics of tracer particles for long times. The *Lagrangian* effective diffusivity is defined as the time rate of change of the mean squared displacement of the tracer particle at long times:

$$\mathcal{K}_L^* = \lim_{t\to\infty} \frac{d}{dt} \langle (\mathbf{X}(t) - \mathbf{x}_0) \otimes (\mathbf{X}(t) - \mathbf{x}_0) \rangle_W \tag{1.24}$$

where $\otimes$ denotes the tensor product and $\langle \cdot \rangle_W$ denotes averaging over the statistics of the Brownian motion. In terms of the transition probability density the Lagrangian effective diffusivity can be expressed as:

$$\mathcal{K}_L^* = \lim_{t\to\infty} \frac{1}{2\,t} \int_{R^d} (\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0)\, T(\mathbf{x}, t) dx \tag{1.25}$$

The Lagrangian effective diffusivity is obviously symmetric, in contrast with the Eulerian effective diffusivity which is in general nonsymmetric. Indeed, it was argued in [62] that $\mathcal{K}_L^*$ is equal to the symmetric part of the Eulerian effective diffusivity $\mathcal{K}^*$. Consequently, an analysis based on formula (1.24) cannot be helpful when studying the antisymmetric part of the effective diffusivity. However, the Lagrangian definition (1.24) can be used to test the predictions of homogenization theory at finite times and to estimate the transient time before the diffusive behavior predicted from homogenization theory takes hold .

The stochastic differential equations can be solved exactly for simplified flow geometries, in particular for two dimensional shear flows. Explicit computation of the statistics of the tracer particle by McLaughlin in [78], see also [69], showed complete agreement between the Lagrangian effective diffusivity and the Eulerian one predicted from homogenization theory. Moreover, it showed that the transient time before the asymptotic diffusive behavior is of the order of magnitude of the *cell diffusion time*, the time it takes for a tracer particle to diffuse through a period cell due to the molecular diffusion κ.

More complicated flow geometries require the numerical solution of the equations of motion and computation of the relevant statistics. This can be accomplished through Monte-Carlo simulations of (1.23). These simulations were performed, for example, by Rosenbluth et al. in [96] for cellular flows, Crisanti et al. in [30] for the three dimensional ABC flow. Explicit comparisons between the effective diffusivity computed through the solution of cell problem and through Monte Carlo simulations were performed by Biferale et al. in [16] for a variety of steady and time dependent mean zero periodic incompressible flows. It was found that there is an excellent agreement between the results predicted form these two methods. McLaughlin in [78] showed that the sensitive dependence of the effective diffusivity upon the mean flow is not an artifact of the method of homogenized averaging. Monte Carlo simulations for two dimensional steady velocity fields in the presence of a constant mean flow clearly showed the sensitive dependence of the effective diffusivity on the ratio of the two components of the mean flow at finite times and for finite Peclet numbers.

We also mention that probabilistic arguments based on the analysis of (1.23) can be used in order to prove the homogenization theorem. This approach was taken by a variety of authors, for example Lions et al. [14, ch. 3], Bhattacharya [11] and Pardoux [88].

1.4.4 Motion of Particles in Random Velocity Fields

From the above discussion it is clear that the method of periodic homogenization is a very efficient and accurate way of describing the evolution of passive scalar fields at finite long times and large scales. Moreover, comparison with experimental results in convection cells [104, 105] as well as in flow in porous media [43] justify, at least qualitatively, the theoretical predictions concerning the dependence of the effective diffusivity upon the nondimensional parameters of the problem. The assumption of periodicity simplifies the mathematical analysis of the problem, however some of its consequences are really unrealistic and should be critically examined. In particular, the sensitive dependence of the effective diffusivity upon the mean flow is strictly a consequence of the periodicity assumption. As we shall see in this work this sensitive dependence becomes even more dramatic when the mean flow is stronger than the fluctuations in which case the very structure of the homogenized equation depends upon the ratio between the two components of the mean flow (restricting ourselves to the two dimensional case). We juxtapose this surprising conclusion against a similar problem where the fluctuations in the velocity field are random, as opposed to periodic.

All of the works on periodic homogenization that we have referred to so far have been concerned with velocity fields where the mean flow is either absent or, when present, at most comparable in magnitude with the fluctuations. The case of mean flows that are stronger than the fluctuations has not been, to our knowledge, considered within the framework of periodic homogenization. We close this literature survey by reporting a relevant result in the problem of diffusive approximation to the behavior of a particle trajectory in a random velocity field that was studied in [59]. The authors considered the equations of motion (1.23) for a tracer particle moving in an incompressible velocity field in the absence of molecular diffusion. The velocity field consists of a constant mean flow perturbed by weak random fluctuations:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{V} + \delta\mathbf{v}(\mathbf{x}), \tag{1.26a}$$

$$\mathbf{x}(t=0) = \mathbf{x}_0 \tag{1.26b}$$

The fluctuating part $\mathbf{v}(\mathbf{x})$ is a zero-mean stationary random field with rapidly decaying correlation tensor.

They authors proved that for δ small and t large (i.e. $t \sim \delta^{-2}$) $\mathbf{x}(t) - \mathbf{V}t$ behaves like a diffusion process. To be more precise, they proved that in relative coordinates comoving with the mean flow and after a long time of order $O(1/\delta^2)$, the mean density

$$\langle \phi(\frac{t}{\delta^2}, \zeta - \frac{\mathbf{V}t}{\delta^2}) \rangle \tag{1.27}$$

tends $\delta \to 0$ to the limit density $\overline{\phi}(t,\zeta)$ as $\delta \to 0$. The limit density satisfies the constant coefficients diffusion equation

$$\frac{\partial \overline{\phi}}{\partial t} = \sum_{i,j=1}^{d} a_{ij}(\mathbf{V}) \frac{\partial^2 \overline{\phi}}{\partial \zeta_i \partial \zeta_j} \quad in\ \mathbb{R}^d \times (0,\infty) \tag{1.28a}$$

$$\overline{\phi}(t=0,\zeta) = \phi_0(\zeta) \quad on\ \mathbb{R}^d \tag{1.28b}$$

The diffusion tensor $a_{ij}(\mathbf{V})$, which is spatially homogeneous, is given by the classical Kubo formula:

$$a_{ij}(\mathbf{V}) = \int_0^\infty R_{ij}(\mathbf{V}s)\,ds \tag{1.29a}$$

$$R_{ij}(\mathbf{x}) = \langle v_i(\mathbf{x}+\mathbf{y})v_j(\mathbf{y})\rangle \tag{1.29b}$$

where $R_{ij}(x)$ denotes the correlation tensor of the velocity field and $\langle\cdot\rangle$ denotes averaging over the statistics of $\mathbf{v}$. The assumption that the components of the correlation tensor $R_{ij}(x)$ are rapidly decaying ensures that the integral in (1.29) is well defined. In the original Eulerian coordinates this result can be stated as follows: For small, finite δ the mean density $\langle\phi(t,x)\rangle$ behaves like $\psi(t,x)$ that satisfies the equation

$$\frac{\partial\psi}{\partial t} + \mathbf{V}\cdot\nabla\psi = \delta^2\nabla(a(\mathbf{V})\nabla\psi) \;\; in \; \mathbb{R}^d \times (0,\infty) \tag{1.30a}$$

$$\psi(t=0,x) = \phi_0(x) \;\; on \; \mathbb{R}^d \tag{1.30b}$$

We emphasize that the above result requires that the fluctuations in the velocity field be random. In order for the long time behavior of particles moving in a periodic velocity field to be diffusive, the presence of nonzero molecular diffusion is necessary. Two important conclusions can be drawn from the the above result: First, the effective diffusivity a_{ij} depends on the mean flow. On the other hand, the structure of the effective equation is independent of the detailed properties of the mean flow, provided that it is nonzero. The analysis of the similar problem for weak periodic fluctuations will exhibit a dramatic dependence of the homogenized equation on the properties of the mean flow.

1.5 Overview of the Thesis

In this section we wish to present a brief overview of the new results that are included in this thesis. The major contribution of this work is the rigorous derivation and systematic study of the effective equation the governs the transport of a passive scalar field in a class of incompressible model flows consisting of a superposition of a large–scale mean flow with a small–scale periodic structure. It is shown that this effective equation is an advection–

dominated advection–diffusion equation with an effective diffusion tensor which is computed through the solution of the cell problem. We show that the structure of the effective equation and of the cell problem are determined by the strength of the mean flow relative to the fluctuations and the magnitude of the characteristic velocity sweeping time relative to the period of oscillations.

The most interesting case is the one where the mean flow is of equal strength to the fluctuations. We show that in this case the effective diffusion tensor is a function of space and time whose values depend upon both the mean flow and the fluctuations.

In view of the importance of the eddy induced transport velocity on the transport of passive tracers in the atmosphere and ocean we undertake a systematic study of the symmetry properties of the effective diffusion tensor. We derive necessary and sufficient conditions for the symmetry of $\mathcal{K}^*$ for steady velocity fields and study the asymptotic behavior of the antisymmetric part of $\mathcal{K}^*$ with respect to the Peclet number.

An efficient method for the numerical solution of the cell problem is also developed based on a Fourier spectral method and the solution of the corresponding complex nonhermitian linear system of equations using an iterative method and preconditioning.

The homogenization theorem is also studied using the method of two–scale convergence. This method has not been, to our knowledge, previously applied to problems of higher order homogenization and we believe that our approach can be useful to a variety of homogenization problems.

1.6 Organization of Chapters

The remainder of this work is distributed over four chapters. We briefly present their contents. In chapter 2 we introduce the model for the velocity field that we shall study and we nondimensionalize the advection—diffusion equation, identifying the relevant nondimensional parameters of the problem. We also state and prove the homogenization theorem for the case where the mean flow is weak compared to the fluctuations.

In chapter 3 we study the case where the mean flow is equal in strength to the fluctuations. We prove the homogenization theorem for this case. Various numerical examples are presented and their physical meaning is discussed. The case where the mean flow is stronger than the fluctuations is also analyzed through numerical examples and formal asymptotics of the cell problem. Finally, the numerical method that we use in order to solve the cell problem is discussed.

The study of the antisymmetric part of the effective diffusivity is presented in chapter 4. Necessary and sufficient conditions for $\mathcal{K}^*$ to be symmetric are derived for steady velocity fields, together with the asymptotic behavior of the antisymmetric part for large and small Peclet numbers. Numerical examples for both steady and time dependent velocity fields are also presented.

The method of two—scale convergence is introduced in chapter 5. We then apply the method to the problem under investigation with particular emphasis to the case when the mean flow is stronger than the fluctuations. A rigorous and systematic way for studying higher order homogenization is presented. A discussion about the results of this work, together with a presentation of various interesting problems that, we believe, can be studied in the framework developed in this thesis are presented in chapter 6.

Finally, an alternative proof of the homogenization theorem using techniques from semigroup theory, together with some properties of periodic incompressible velocity fields that are being used in the main text are presented in the appendix.

CHAPTER 2
HOMOGENIZATION FOR WEAK MEAN FLOWS

2.1 Introduction

In this chapter we start our study of the problem of periodic homogenization for advection–diffusion equations with mean flow. In section 2.2 we introduce the model velocity field that we shall consider, nondimensionalize the advection–diffusion equations, and identify the relevant nondimensional parameters. In section 2.3 we present the homogenization theorem. We show that, while the effective equation has always the same structure in the parameter range that we consider in this chapter, the structure of the cell problem depends upon the period of temporal fluctuations. The derivation of the effective equations is presented in section 2.4.

2.2 Scaling and Nondimensional Mixing Parameters

We start with the dimensional form of the advection–diffusion equation that governs the evolution of the passive scalar field:

$$\frac{\partial T(\mathbf{x},t)}{\partial t} + \mathbf{u}\cdot\nabla T(\mathbf{x},t) = \kappa\Delta T(\mathbf{x},t) \ \text{ in } \mathbb{R}^d \times (0,\infty) \tag{2.1a}$$

$$T(\mathbf{x},t=0) = T_{in}(\mathbf{x}) \ \text{ on } \mathbb{R}^d, \tag{2.1b}$$

where κ is the molecular diffusivity and $\mathbf{u}$ is a smooth given velocity field. The velocity field is incompressible ($\nabla\cdot\mathbf{u} = 0$) and consists of a superposition of a mean flow $\mathbf{V}$, varying at large length and time scales, with periodic fluctuations $\mathbf{v}$. With the mean flow we associate characteristic length and time scales L_0, T_0. We also assume that L_0 is characteristic length scale of the initial conditions as well. The characteristic length and time scales of the periodic fluctuations are identified to be the spatial and temporal periods L_p and T_p, respectively. For simplicity we shall take the spatial period to be the same in all directions:

$$\mathbf{v}(\mathbf{x},t+T_p) = \mathbf{v}(\mathbf{x},t)$$

$$\mathbf{v}(\mathbf{x} + L_p \hat{e}_j, t) = \mathbf{v}(\mathbf{x}, t), \;\; j = 1, \ldots d,$$

where $\{\hat{e}\}_{j=1}^{d}$ denotes the unit vector in the jth coordinate direction. We also define the magnitudes of the mean flow and the fluctuations using the maximum norm over the entire space and time and the period cell, respectively:

$$V_0 = \max_{x \in \mathbb{R}^d,\, t \in [0,\infty)} |\mathbf{V}|$$

$$v_0 = \max_{x \in [0,L_p]^d,\, t \in [0,T_p]} |\mathbf{v}|$$

furthermore, we assume that the periodic fluctuations have mean zero over the period cell:

$$\langle \mathbf{v} \rangle := \frac{1}{T_p\, L_p^d} \int_{[0,L_p]^d} \int_{[0,T_d]} \mathbf{v}(\mathbf{x}, t)\, d\,\mathbf{x}\, d\,t = \mathbf{0} \tag{2.2}$$

Putting everything together, we can write the velocity field that we shall consider in the following form:

$$\mathbf{u} = V_0\, \mathbf{V}'(\frac{\mathbf{x}}{L_0}, \frac{t}{T_0}) + v_0\, \mathbf{v}'(\frac{\mathbf{x}}{L_p}, \frac{t}{T_p}) \tag{2.3}$$

where $\mathbf{V}'$, $\mathbf{v}'$ are nondimensional vector functions. The periodic function $\mathbf{v}'(\mathbf{y}, \tau)$ has period 1 in time and in each spatial coordinate direction. We assume that there is a clear separation of length and time scales between the fluctuations and the mean flow: $L_p << L_0$, $T_p << T_0$. We express this scale separation by introducing two nondimensional parameters δ, η:

$$\delta := \frac{L_p}{L_0}, \quad \eta := \frac{T_p}{T_0}, \quad \delta << 1, \;\; \eta << 1 \tag{2.4}$$

We also introduce a nondimensional parameter a which measures the strength of the fluctuations relative to that of the mean flow:

$$a := \frac{v_0}{V_0} \tag{2.5}$$

We emphasize that no assumption regarding the magnitude of a is made at this point.

The velocity field (2.3) induces the following set of time scales, in addition to the characteristic times T_0, T_p:

$$T_{ls} := \frac{L_0}{V_0}, \quad T_K := \frac{L_0^2}{\kappa} \tag{2.6a}$$

$$\tau_{sw} := \frac{L_p}{v_0}, \quad \tau_{\kappa} := \frac{L_p^2}{\kappa} \tag{2.6b}$$

The *large-scale sweeping time* T_{ls} measures the time needed for a particle moving with the characteristic velocity of the mean flow V_0 to travel the characteristic length L_0. The *large-scale diffusion time* is defined as the time needed for a finely concentrated spot of the passive scalar field to spread over the length scale L_0. Similar definitions hold for the *small-scale sweeping time* τ_{sw} and the *cell diffusion time*: τ_k.

Using now the time scales of the problem we can form various nondimensional quantities. Following earlier work of Rubinstein and Mauri [74] concerning steady velocity fields with a constant mean flow we shall define both local and global nondimensional quantities: the *local Peclet* and the *local Strouhal* numbers:

$$Pe_l = \frac{\tau_{\kappa}}{\tau_{sw}}, \quad S_l = \frac{\tau_{sw}}{T_p}, \tag{2.7}$$

as well as their global counterparts:

$$Pe_g = \frac{T_k}{T_{ls}}, \quad S_g = \frac{T_{ls}}{T_0}, \tag{2.8}$$

The global Strouhal number measures the ratio between the large scale sweeping time over the characteristic time scale of the mean flow. The local Strouhal number is the ratio between the characteristic velocity sweeping time to period time. The local Peclet number measures the ratio between the cell diffusion time over the small-scale sweeping time; the global Peclet number measures the ratio between the large-scale diffusion time over the large-scale sweeping time.

A comment concerning the terminology is in order. The local Peclet number describes the relative strength of advection due to the fluctuations $\mathbf{v}$ to molecular diffusion within the period cell. Similarly, the local Strouhal describes how fast the fluctuating part of the velocity field oscillates in time, relative to the local time scales associated with the period cell. Hence the term local. On the other hand, the global dimensionless numbers are defined through the length and time scales of the mean flow, which define the scales of observational interest. The term global seems appropriate.

Using the definitions of the time scales (2.6) we can express (2.7) and (2.8) in the form:

$$Pe_l = \frac{v_0\, L_p}{\kappa}, \quad S_l = \frac{L_p}{T_p\, v_0}, \tag{2.9}$$

and:

$$Pe_g = \frac{V_0\, L_0}{\kappa}, \quad S_g = \frac{L_0}{T_0\, V_0}, \tag{2.10}$$

Since we have already assumed a relationship between the two sets of length and time scales in (2.4), the global and local dimensionless numbers are not independent. In fact, using the definitions (2.9) and (2.10) as well as the definitions of a, δ and η we get:

$$S_g = \frac{\eta\, a}{\delta} S_l, \quad Pe_g = \frac{1}{a\,\delta} Pe_l \tag{2.11}$$

No assumption regarding the orders of magnitudes of $\{Pe_l,\, S_l\}$, or equivalently $\{Pe_g,\, S_g\}$, is made at this point. These two dimensionless numbers are enough to describe the physical system under investigation. In other words, two systems whose evolution is described through (2.1) and have the same Peclet and Strouhal numbers are *dynamically equivalent*, [10, pp. 211-216].

Now we are interested in obtaining an effective equation which governs the dynamics of the passive scalar field at long times and large scales, namely the length and time scales of the mean flow. For this purpose, it is convenient to nondimensionalize (2.1) with respect to the characteristic length and time scales of the mean flow.

To this end, we introduce the primed nondimensional independent variables $\mathbf{x}'$, t' and the nondimensional velocity fields $\mathbf{V}'$, $\mathbf{v}'$ introduced previously:

$$\mathbf{x}' \equiv \frac{\mathbf{x}}{L_g}, \quad t' \equiv \frac{t}{T_g}, \quad \mathbf{V}' \equiv \frac{\mathbf{V}}{V_0}, \quad \mathbf{v}' \equiv \frac{\mathbf{v}}{v_0} \tag{2.12}$$

The new passive scalar field T° and the new initial conditions T_0° are defined through

$$T(\mathbf{x},t) = T^\circ(\mathbf{x}',t'), \quad T_{in}(\mathbf{x}) = T^\circ_{in}(\mathbf{x}') \tag{2.13}$$

Inserting (2.12) into (2.1) and using (2.13) as well as the definition of Pe_g and S_g we obtain:

$$S_g \frac{\partial T(\mathbf{x},t)}{\partial t} + \left(\mathbf{V}(\mathbf{x},t) + a\,\mathbf{v}(\frac{\mathbf{x}}{\delta},\frac{t}{\eta})\right)\cdot \nabla T(\mathbf{x},t) = \frac{1}{Pe_g}\Delta T(\mathbf{x},t) \ \text{ in } \mathbb{R}^d \times (0,\infty) \tag{2.14a}$$

$$T(\mathbf{x},t=0) = T_{in}(\mathbf{x}) \ \text{ on } \mathbb{R}^d, \tag{2.14b}$$

where, for notational simplicity, the primes and circles have been dropped.

As has already been mentioned, we want to obtain effective equations at length scales at least as large as those of the mean flow and time scales at least comparable to the large scale eddy turnover time. Moreover, we want to ensure that the large scale properties of the system are independent of the location of the small scale. Thus, we assume:

$$S_g \sim O(1) \tag{2.15}$$

We also need to make an assumption regarding the relative strength of advection and diffusion at the local scale. A physically interesting case is to suppose that advection and diffusion are of comparable strength at the local scale[5]:

$$Pe_l \sim O(1) \tag{2.16}$$

Using now (2.11) we can rewrite (2.13) using the $O(1)$ nondimensional numbers Pe_l, S_g:

$$S_g \frac{\partial T(\mathbf{x},t)}{\partial t} + \left(\mathbf{V}(\mathbf{x},t) + a\,\mathbf{v}(\frac{\mathbf{x}}{\delta},\frac{t}{\eta})\right)\cdot \nabla T(\mathbf{x},t) = \frac{a\,\delta}{Pe_l}\Delta T(\mathbf{x},t) \ \text{ in } \mathbb{R}^d \times (0,\infty) \tag{2.17a}$$

$$T(\mathbf{x},t=0) = T_0(\mathbf{x}) \ \text{ on } \mathbb{R}^d, \tag{2.17b}$$

[5] More precisely, we assume that the local Peclet number is independent of the parameter δ.

The fluctuating part of the velocity field $\mathbf{v}(\mathbf{y},\tau)$ has period 1 in time and in each spatial coordinate direction, with $\mathbf{y} = \frac{\mathbf{x}}{\delta}$, $\tau = \frac{t}{\eta}$. The average over the rescaled periodicity cell $[0,1]^d \times [0,1]$ becomes:

$$\langle \cdot \rangle := \int_{[0,1]^d} \int_{[0,1]} \cdot \; d\mathbf{y}\, d\tau \tag{2.18}$$

We shall also have the occasion to use the spatial average, denoted by $\langle \cdot \rangle_y$, as well as the temporal average $\langle \cdot \rangle_\tau$:

$$\langle \cdot \rangle_y := \int_{[0,1]^d} \cdot \; d\mathbf{y} \tag{2.19a}$$

$$\langle \cdot \rangle_\tau := \int_{[0,1]} \cdot \; d\tau \tag{2.19b}$$

The magnitude of the dimensionless parameters S_l, Pe_g can be determined from (2.11) and depends on the relationship between δ, η, a. In the rest of the paper we shall study the effective equations that result from (2.17) under various assumptions on the relative strengths of the parameters δ, η, a.

2.3 Distinguished Limits

Apart from the Peclet and the Strouhal numbers, there are three, independent, dimensionless quantities that appear in (2.17): a, measuring the strength of the fluctuations relative to the mean flow, δ measuring the separation of length scales and η measuring the separation of time scales. Consequently, there are various different distinguished limits that we can consider. In this work we shall make the assumption that both η and a depend upon δ. To this end, we assume that a, η have the form:

$$a = \delta^\alpha, \quad \eta = \delta^\gamma \tag{2.20}$$

Under this assumption (2.17) becomes:

$$S_g \frac{\partial T(\mathbf{x},t)}{\partial t} + \left(\mathbf{V}(\mathbf{x},t) + \delta^\alpha \mathbf{v}(\frac{\mathbf{x}}{\delta}, \frac{t}{\delta^\gamma}) \right) \cdot \nabla T(\mathbf{x},t) = \frac{\delta^{\alpha+1}}{Pe_l} \Delta T(\mathbf{x},t) \;\text{ in}\, \mathbb{R}^d \times (0,\infty) \tag{2.21a}$$

$$T(\mathbf{x}, t = 0) = T_0(\mathbf{x}) \text{ on } \mathbb{R}^d, \tag{2.21b}$$

As we discussed in the first chapter, in the case of transport of passive tracers in the ocean the mean flow is weaker or equal in strength to the mesoscale eddies. Consequently, the physically relevant parameter range to consider is $\alpha \in [-1, 0]$. On the other hand, we shall not make any a priori assumptions on the temporal oscillations. We simply take $\gamma > 0$. Weaker mean flows, $alpha < -1$, will not affect the homogenized equation. On the other hand, for mean flows much stronger than the fluctuations, we expect that the turbulent diffusivity will be small compared to the molecular diffusivity, except when resonant enhanced diffusion phenomena occur. This problem will be studied numerically in the next chapter and rigorously in chapter 5.

As examples of specific choices of the parameters α, γ that have been studied in the literature we mention the choice $\alpha = -1,\ \gamma = 2$: Now the magnitude of the mean flow is $O(\delta)$ compared to that of the fluctuations and this implies that the global Peclet number is an $O(1)$ quantity. Moreover, the choice $\gamma = 2$, for $\alpha = -1$ implies that both the local and global Strouhal numbers are $O(1)$ quantities and thus no distinction between them needs to be made. This problem was studied in [69], see also [72]. An alternative proof of the homogenization theorem for this scaling based on semigroup theoretic arguments will be presented in the appendix. The choice $\alpha = 0,\ \gamma = 1$ was partially studied in [24, 75]. However, the effective equations obtained in the paper are not correct, since they imply that advection and diffusion are of the same order of magnitude at the macroscale. This choice of parameters leads to an $O(\frac{1}{\delta})$ global Peclet number and advection dominates diffusion at the macroscale.

In this chapter we shall study the regime $\alpha \in (-1, 0)$. We shall discuss the case where the mean flow is of equal strength to the fluctuations in the next chapter. Alternative proofs of the results using the method of two—scale convergence will be presented in the chapter 5.

Clearly, the strength of the diffusion relative to the advection by the mean flow at the macroscale will depend on the global Peclet number. From (2.11) and the assumption that the local Peclet number is an $O(1)$ quantity, we see that Pe_g will depend only upon the exponent α and not on the separation of time scales which is controlled by γ. On the other hand, γ will determine the structure of the cell problem. In particular, there are three possible cell problems: one for $\gamma \in (0, 1-\alpha)$, a second for $\gamma = 1-\alpha$ and a third one for $\gamma > 1-\alpha$. Let us remark that the critical value $\gamma = 1-\alpha$ corresponds to the case $S_l \sim O(1)$.

In this chapter we shall prove the following theorem:

THEOREM 2.1 *Consider the initial value problem (2.21) where the mean flow and the fluctuations are smooth and incompressible and the initial conditions are also smooth. Assume further that $\langle \mathbf{v} \rangle = 0$. Then for δ sufficiently small there exists a constant C independent of δ and t_0 such for any $t_0 > 0$ the following estimates hold:*

$$||T - \overline{T}||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))} \leq \delta \left(1 + \frac{t_0}{S_g}\right) C(Pe_l, S_g) \tag{2.22a}$$

$$||\nabla T - \nabla \overline{T}||_{L^2((0,t_0)\times\mathbb{R}^d))} \leq \delta^{\frac{1-\alpha}{2}} \sqrt{Pe_l \left(\frac{S_g}{2} + 1 + \frac{t_0}{S_g}\right)}\, C(Pe_l, S_g) \tag{2.22b}$$

where $\overline{T}$ satisfies the following equation:

$$S_g \frac{\partial \overline{T}(\mathbf{x}, t)}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla \overline{T}(\mathbf{x}, t) = \frac{\delta^{\alpha+1}}{Pe_l} \nabla \cdot (\mathcal{K}^* \cdot \nabla \overline{T}(\mathbf{x}, t)) \;\; in\, \mathbb{R}^d \times (0, +\infty) \tag{2.23a}$$

$$\overline{T}(\mathbf{x}, t = 0) = T_{in}(\mathbf{x}) \;\; on\, \mathbb{R}^d, \tag{2.23b}$$

The effective diffusion tensor $\mathcal{K}^*$ is:

$$\begin{aligned} \mathcal{K}^*_{ij} &= \delta_{ij} - Pe_l \langle v_i \chi_j \rangle \\ &:= \delta_{ij} + \overline{\mathcal{K}}_{ij}, \end{aligned} \tag{2.24}$$

where $\boldsymbol{\chi}$ is the solution of the following cell problems:

1. *For $\gamma < 1-\alpha$:*

$$\mathbf{v} \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} - \frac{1}{Pe_l} \Delta_{\mathbf{y}} \boldsymbol{\chi} = -(\mathbf{v} - \langle \mathbf{v} \rangle_y) \tag{2.25}$$

2. $\gamma = 1 - \alpha$*:*

$$S_g \frac{\partial \boldsymbol{\chi}}{\partial \tau} + \mathbf{v} \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} - \frac{1}{Pe_l} \Delta_{\mathbf{y}} \boldsymbol{\chi} = -\mathbf{v} \tag{2.26}$$

3. $\gamma > 1 - \alpha$*:*

$$\langle \mathbf{v} \rangle_\tau \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} - \frac{1}{Pe_l} \Delta_{\mathbf{y}} \boldsymbol{\chi} = -\langle \mathbf{v} \rangle_\tau \tag{2.27}$$

Before presenting the proof of this theorem let us make a few comments. First, we observe that the effective diffusion tensor is constant, independent of the mean flow (as will become clear from the derivation this is correct only to leading order: higher order corrections to the effective diffusion tensor are space-time dependent with values depending on the mean flow). Thus, the effect of the fluctuations to leading order is to enhance dissipation at the macroscale, without affecting the velocity field with which the passive tracers are advected at the length and time scales of the mean flow. There are two reasons why the effective diffusion tensor is, to leading order, constant. First, the mean flow is weak compared to the fluctuations. Second, we have assumed that the fluctuations are independent of the large scale variables. If we were to study slowly modulated fluctuations (as an example one can consider two dimensional shear flows with slowly varying amplitude, $\mathbf{v}(\mathbf{x}, t, \mathbf{y}, \tau) = (f(x_2, t)\, v_1(y_2, \tau), 0))$ then $\mathcal{K}^*$ would be a function of space and time with values depending on the modulations of the fluctuations. We shall address this issue in the next chapter, together with the strong mean flow case.

Moreover, the effective diffusion tensor will not be symmetric in general. Since it is constant only its symmetric part is relevant for the effective equation (2.23), see equation (1.22) in chapter 1. Now, it is easy to show that the symmetric part of the enhancement in the diffusivity can be expressed as [69, p. 252]:

$$\mathrm{symm}(\overline{\mathcal{K}}_{ij}) = \langle \nabla_y \chi_i \cdot \nabla_y \chi_j \rangle \tag{2.28}$$

which is positive semidefinite. Consequently, the dissipation is always enhanced.

Using now the definition of the global Peclet number we can rewrite equation (2.23) in the form:

$$S_g \frac{\partial \overline{T}(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla\overline{T}(\mathbf{x},t) = \frac{1}{Pe_g}\nabla\cdot(\mathcal{K}^*\nabla\cdot\overline{T}(\mathbf{x},t)) \;\; in\ \mathbb{R}^d\times(0,\infty) \tag{2.29a}$$

$$T(\mathbf{x},t=0) = T_{in}(\mathbf{x}) \;\; on\ \mathbb{R}^d, \tag{2.29b}$$

with Pe_g being an $O(\delta^{\alpha+1})$ quantity. Thus, the effective equation is an advection dominated advection–diffusion equation. Naturally, at the limit $Pe_g \to \infty$, that is, for $\delta \to 0$, the effective equation is simply a transport equation.

The second interesting observation is that the structure of the cell problem depends upon the magnitude of the local Strouhal number. For $\gamma < 1-\alpha$ the S_l is an $O(\delta^\sigma)$, $\sigma > 0$ quantity which means that the characteristic sweeping time is much smaller than the temporal period. In this case, the temporal fluctuations are too slow and they don't play any role in the cell problem, which is an elliptic equation with the fast—time like variable τ entering merely as a parameter.

When $\gamma = 1-\alpha$ the local Strouhal number is an $O(1)$ quantity and the characteristic velocity sweeping time is comparable to the period time. On the other hand, by assumption, the local Peclet number is $O(1)$ and advection and diffusion are of the same order of magnitude at the microscale. Thus, in this case all three terms, time derivative with respect to the fast time, local advection and local diffusion contribute to the cell problem. On the other hand, when $\gamma > 1-\alpha$ the local Strouhal number is an $O(\delta^{-\sigma})$, $\sigma > 0$ quantity and the characteristic sweeping time is larger than the time period. In this case the oscillations at the microscale are too fast and the system cannot adjust to them instantaneously. This is why in this regime the cell problem is an elliptic equation with the velocity fluctuations replaced by their time averages.

In the next section we shall derive the effective equation (2.23) together with the cell problems (2.25), (2.26) and (2.27). We shall also obtain the necessary estimates that justify the validity of the effective equation.

2.4 Derivation of the Effective Equations

In this section we shall derive the homogenized equations. In order to prove the homogenization theorem we shall need the following lemma, which is a direct consequence of Fredholm's theorem, [40, ch. 6]:

LEMMA 2.1 *Let $\mathcal{L}_0$ denote the operator*

$$\mathcal{L}_0 := S\frac{\partial}{\partial \tau} + \mathbf{u}(\mathbf{y},\tau)\cdot\nabla_y - \frac{1}{Pe}\Delta_{\mathbf{y}}$$

where $\mathbf{u}(\mathbf{y},\tau)$ is a smooth, incompressible velocity field which is periodic in both $\mathbf{y}$ and τ. Further, let $g(\mathbf{y},\tau)$ be a smooth function which is periodic in both $\mathbf{y},\tau$. Then the equation

$$\mathcal{L}_0\, f(\mathbf{y},\tau) = g(\mathbf{y},\tau) \tag{2.30}$$

has a unique, up to a constant, solution if and only if $g(\mathbf{y},\tau)$ has zero mean. Uniqueness is ensured by requiring $f(\mathbf{y},\tau)$ to have zero mean. In particular, the only solutions of the homogeneous equation

$$\mathcal{L}_0\, f(\mathbf{y},\tau) = 0 \tag{2.31}$$

are constants.

We emphasize that the incompressibility of the velocity field is necessary for the above lemma to hold.

In the subsequent analysis we shall also need to estimate the solution of parabolic equations in terms of the inhomogeneous term and the initial conditions. We present the result that we shall need in the following lemma. For background material on parabolic partial differential equations we refer to the books [40, ch. 7], [44]

LEMMA 2.2 *Consider the following parabolic initial value problem:*

$$S_g\,\frac{\partial T(\mathbf{x},t)}{\partial t} + \left(\mathbf{V}(\mathbf{x},t) + \delta^{\alpha}\,\mathbf{v}(\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\gamma}})\right)\cdot\nabla T(\mathbf{x},t) - \frac{\delta^{\alpha+1}}{Pe_l}\Delta T(\mathbf{x},t) = \delta\, F(\mathbf{x},t) \;\; in\, \mathbb{R}^d\times(0,\infty) \tag{2.32a}$$

$$T(\mathbf{x}, t = 0) = \delta\, f(\mathbf{x}) \;\; on\, \mathbb{R}^d, \tag{2.32b}$$

with $F(\mathbf{x}, t) \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$, $f(\mathbf{x}) \in L^2(\mathbb{R}^d)$ *and the velocity fields* $\mathbf{V}$, $\mathbf{v}$ *are smooth and incompressible. Then for every* $t_0 > 0$ *we have the following estimates:*

$$||T||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))} \leq \delta \left(1 + \frac{t_0}{S_g}\right) C \tag{2.33a}$$

$$||\nabla T||_{L^2((0,t_0)\times\mathbb{R}^d))} \leq \delta^{\frac{1-\alpha}{2}} \sqrt{Pe_l \left(\frac{S_g}{2} + 1 + \frac{t_0}{S}\right)}\, C \tag{2.33b}$$

where $C := \max\left(||F||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))},\, ||f||_{L^2(\mathbb{R}^d)}\right)$.

Proof: We multiply (2.32) by T, integrate over $\mathbb{R}^d$ and integrate by parts and use the fact that the solutions of (2.32) decay sufficiently rapidly at infinity and the incompressibility of the velocity field to obtain the basic energy identity:

$$S_g \frac{1}{2}\frac{d}{dt}||T||^2_{L^2(\mathbb{R}^d)} + \frac{\delta^{1+\alpha}}{Pe_l}||\nabla T||^2_{L^2(\mathbb{R}^d)} = \delta\,(F, T)_{L^2(\mathbb{R}^d)} \tag{2.34}$$

Thus:

$$\begin{aligned} ||T||_{L^2(\mathbb{R}^d)}\frac{d}{dt}||T||_{L^2(\mathbb{R}^d)} = \frac{1}{2}\frac{d}{dt}||T||^2_{L^2(\mathbb{R}^d)} &= \frac{\delta}{S_g}\,(F, T)_{L^2(\mathbb{R}^d)} \\ &\leq \frac{\delta}{S_g}\,||F||_{L^2(\mathbb{R}^d)}||T||_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{2.35}$$

on account of the Cauchy-Schwarz inequality. Integrating now in time we obtain:

$$\begin{aligned} ||T||_{L^2(\mathbb{R}^d))}(t) &\leq ||T||_{L^2(\mathbb{R}^d)}(t = 0) + \frac{\delta}{S_g}\int_0^t ||F||_{L^2(\mathbb{R}^d)}(s)\, ds \\ &\leq \delta\,||f||_{L^2(\mathbb{R}^d)} + \frac{\delta}{S_g}t_0\,||F||_{L^\infty((0,T_0);L^2(\mathbb{R}^d))} \end{aligned} \tag{2.36}$$

for $t \in (0, t_0]$. Consequently:

$$\begin{aligned}
\sup_{t\in(0,t_0]} ||T||_{L^2(\mathbb{R}^d))}(t) &\leq \delta\, ||f||_{L^2(\mathbb{R}^d)} + \frac{\delta}{S_g} t_0\, ||F||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))} \\
&\leq \delta \left(1 + \frac{t_0}{S_g}\right) \max\left(||F||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))},\ ||f||_{L^2(\mathbb{R}^d)}\right) \qquad (2.37)
\end{aligned}$$

and estimate (2.33a) follows.

To get estimate (2.33b) we integrate the energy identity (2.34) in time to obtain:

$$\begin{aligned}
\frac{\delta^{1+\alpha}}{Pe_l} \int_0^{t_0} ||\nabla T||^2_{L^2(\mathbb{R}^d)}\, dt &= \delta \int_0^{t_0} (F, T)_{L^2(\mathbb{R}^d)} + \frac{S_g}{2} ||T||^2_{L^2(\mathbb{R}^d)}(0) - \frac{S_g}{2} ||T||^2_{L^2(\mathbb{R}^d)}(t_0) \\
&\leq \delta\, t_0\, ||F||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))} ||T||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))} + \frac{S_g}{2} \delta^2 ||f||^2_{L^2(\mathbb{R}^d)} \\
&\leq \delta^2 \left(1 + \frac{t_0}{S_g}\right) t_0\, C\, ||F||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))} + \frac{S_g}{2} \delta^2 ||f||^2_{L^2(\mathbb{R}^d)} \\
&\leq \delta^2 \left(\frac{S_g}{2} + 1 + \frac{t_0}{S_g}\right) C^2 \qquad (2.38)
\end{aligned}$$

Multiplying now (2.38) by $\frac{Pe_l}{\delta^{1+\alpha}}$ and taking the square root of both sides we arrive at (2.33b). The proof of the lemma is complete.

Now we are ready to prove the homogenization theorem. The plan will be to look for a solution in the form of a multiple scales expansion. Using then lemma 1 we shall obtain equations for the first two terms in the expansion. Then we shall obtain bounds for the error caused by neglecting higher order terms in the expansion. Using these bounds we shall estimate the difference between the solution $T(\mathbf{x}, t)$ to the original equation (2.21) and the sum of the first two terms $\overline{T}$ and we shall justify that it is small, in the appropriate norm. As we have already mentioned, in this chapter we are concerned with the parameter regime $\alpha \in (-1, 0)$, $\gamma >$.

We shall use the notation:

$$\mathcal{L}^\delta := S_g \frac{\partial}{\partial t} + (\mathbf{V}(\mathbf{x}, t) + \delta^\alpha \mathbf{v}(\frac{\mathbf{x}}{\delta}, \frac{t}{\delta^\gamma})) \cdot \nabla - \frac{\delta^{1+\alpha}}{Pe_l} \Delta \qquad (2.39)$$

The initial value problem (2.21) can be written as [6] :

$$\mathcal{L}^\delta\, T^\delta(\mathbf{x},t) = 0 \;\; in\ \mathbb{R}^d \times (0,\infty) \tag{2.40a}$$

$$T^\delta(\mathbf{x}, t=0) = T_{in}(\mathbf{x}) \;\; on\ \mathbb{R}^d, \tag{2.40b}$$

According to the recipe of multiple scale expansions we shall treat the "slow" variables $\mathbf{x}$, t and the "fast" variables $\mathbf{y} := \frac{\mathbf{x}}{\delta}$, $\tau := \frac{t}{\delta^\gamma}$ as independent. The differential operators transform as:

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \frac{1}{\delta^\gamma}\frac{\partial}{\partial \tau} \tag{2.41a}$$

$$\nabla \rightarrow \nabla_{\mathbf{x}} + \frac{1}{\delta}\nabla_{\mathbf{y}} \tag{2.41b}$$

Using (2.41) we rewrite $\mathcal{L}^\delta$ in the following form:

$$\begin{aligned}
\mathcal{L}^\delta &= S_g\,\frac{\partial}{\partial t} + \left(\mathbf{V}(\mathbf{x},t) + \delta^\alpha \mathbf{v}(\frac{\mathbf{x}}{\delta}, \frac{t}{\delta^\gamma})\right)\cdot\nabla - \frac{\delta^{1+\alpha}}{Pe_l}\,\Delta \\
&= S_g\left(\frac{\partial}{\partial t} + \frac{1}{\delta^\gamma}\frac{\partial}{\partial \tau}\right) + \left(\mathbf{V}(\mathbf{x},t) + \delta^\alpha \mathbf{v}(\mathbf{y},\tau)\right)\cdot\left(\nabla_{\mathbf{x}} + \frac{1}{\delta}\nabla_{\mathbf{y}}\right) \\
&- \frac{\delta^{1+\alpha}}{Pe_l}\left(\Delta_{\mathbf{x}} + \frac{2}{\delta}\nabla_{\mathbf{x}}\nabla_{\mathbf{y}} + \frac{1}{\delta^2}\Delta_{\mathbf{y}}\right) \\
&= S_g\frac{1}{\delta^\gamma}\frac{\partial}{\partial \tau} + \frac{1}{\delta^{1-\alpha}}\left(\mathbf{v}\cdot\nabla_{\mathbf{y}} - \frac{1}{Pe_l}\Delta_{\mathbf{y}}\right) + \frac{1}{\delta}\mathbf{V}\cdot\nabla_{\mathbf{y}} \\
&+ \delta^\alpha\left(\mathbf{v}\cdot\nabla_{\mathbf{x}} - \frac{2}{Pe_l}\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}\right) + \left(S_g\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla_{\mathbf{x}}\right) - \frac{\delta^{\alpha+1}}{Pe_l}\Delta_{\mathbf{x}} \\
&:= \frac{1}{\delta^\gamma}\mathcal{R}_0 + \frac{1}{\delta^{1-\alpha}}\mathcal{R}_1 + \frac{1}{\delta}\mathcal{R}_2 + \delta^\alpha\mathcal{R}_3 + \mathcal{R}_4 + \delta^{\alpha+1}\mathcal{R}_5,
\end{aligned} \tag{2.42}$$

where:

$$\mathcal{R}_0 := S_g\,\frac{\partial}{\partial\,\tau} \tag{2.43a}$$

[6] In this section, in order to emphasize the dependence of the solution of equation (2.21) on δ, we shall use the notation $T^\delta(\mathbf{x},t)$ as opposed to $T(\mathbf{x},t)$ that we have been using in the previous section.

$$\mathcal{R}_1 := \mathbf{v}(\mathbf{y},\tau)\cdot\nabla_{\mathbf{y}} - \frac{1}{Pe_l}\Delta_{\mathbf{y}} \tag{2.43b}$$

$$\mathcal{R}_2 := \mathbf{V}(\mathbf{x},t)\cdot\nabla_{\mathbf{y}} \tag{2.43c}$$

$$\mathcal{R}_3 := \mathbf{v}(\mathbf{y},\tau)\cdot\nabla_x - \frac{2}{Pe_l}\nabla_x\nabla_{\mathbf{y}} \tag{2.43d}$$

$$\mathcal{R}_4 := S_g\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla_x \tag{2.43e}$$

$$\mathcal{R}_5 := -\frac{1}{Pe_l}\Delta_{\mathbf{x}} \tag{2.43f}$$

The structure of the multiple scales expansion depends upon the values of γ and α. We shall present the detailed derivation for the simplest case, namely for $\alpha = -\frac{1}{2}$ and $\gamma = 1-\alpha = \frac{3}{2}$ and then explain how to proceed with the other cases. Throughout the calculations that follow we shall use both vector notation as well as the summation convention.

2.4.1 The Case $\alpha = -\frac{1}{2}$

2.4.1.1 The Case $\gamma = \frac{3}{2}$

In this case $\mathcal{L}^\delta$ becomes

$$\mathcal{L}^\delta = \frac{1}{\delta^{\frac{3}{2}}}\mathcal{L}_0 + \frac{1}{\delta}\mathcal{L}_1 + \delta^{\frac{1}{2}}\mathcal{L}_2 + \mathcal{L}_3 + \delta^{\frac{1}{2}}\mathcal{L}_4 \tag{2.44}$$

where:

$$\mathcal{L}_0 := \mathcal{R}_0 + \mathcal{R}_1 \tag{2.45a}$$

$$\mathcal{L}_1 := \mathcal{R}_2 \tag{2.45b}$$

$$\mathcal{L}_2 := \mathcal{R}_3 \tag{2.45c}$$

$$\mathcal{L}_3 := \mathcal{R}_4 \tag{2.45d}$$

$$\mathcal{L}_4 := \mathcal{R}_5 \tag{2.45e}$$

Upon using (2.44) we write equation for (2.40) in the form:

$$\left(\frac{1}{\delta^{\frac{3}{2}}}\mathcal{L}_0 + \frac{1}{\delta}\mathcal{L}_1 + \delta^{\frac{1}{2}}\mathcal{L}_2 + \mathcal{L}_3 + \delta^{\frac{1}{2}}\mathcal{L}_4\right) T^\delta(\mathbf{x}, t) = 0 \tag{2.46}$$

For these particular values of α and γ it is enough to consider an expansion of the following form:

$$\begin{aligned} T^\delta(\mathbf{x}, t) \;&\approx\; T_0(\mathbf{x}, t, \mathbf{y}, \tau) + \delta^{\frac{1}{2}}\, T_1(\mathbf{x}, t, \mathbf{y}, \tau) + \delta\, T_2(\mathbf{x}, t, \mathbf{y}, \tau) \\ &+\; \delta^{\frac{3}{2}}\, T_3(\mathbf{x}, t, \mathbf{y}, \tau) + \delta^2\, T_4(\mathbf{x}, t, \mathbf{y}, \tau) + \ldots \end{aligned} \tag{2.47}$$

where the functions T_i, $i = 0, 1, 2, \ldots$ are periodic in $\mathbf{y}$ and τ. We substitute (2.47) into (2.46) and, by equating the coefficients of powers of δ to 0 we obtain the following sequence of equations:

$$O(\delta^{-\frac{3}{2}}):\quad \mathcal{L}_0\, T_0 = 0 \tag{2.48a}$$

$$O(\delta^{-1}):\quad \mathcal{L}_0\, T_1 + \mathcal{L}_1\, T_0 = 0 \tag{2.48b}$$

$$O(\delta^{-\frac{1}{2}}):\quad \mathcal{L}_0\, T_2 + \mathcal{L}_1\, T_1 + \mathcal{L}_2\, T_0 = 0 \tag{2.48c}$$

$$O(1):\quad \mathcal{L}_0\, T_3 + \mathcal{L}_1\, T_2 + \mathcal{L}_2\, T_1 + \mathcal{L}_3\, T_0 = 0 \tag{2.48d}$$

$$O(\delta^{\frac{1}{2}}):\quad \mathcal{L}_0\, T_4 + \mathcal{L}_1\, T_3 + \mathcal{L}_2\, T_2 + \mathcal{L}_3\, T_1 + \mathcal{L}_4\, T_0 = 0 \tag{2.48e}$$

Let us now analyze equations (2.48). We first observe that lemma 1 applied to the $O(\delta^{-\frac{3}{2}})$ equation gives that the first term in the expansion is independent of the fast variables: $T_0 = T_0(\mathbf{x}, t)$. Consequently, the action of $\mathcal{L}_1$ to T_0 gives zero:

$$\mathcal{L}_1\, T_0 = \mathbf{V} \cdot \nabla_{\mathbf{y}} T_0 = 0 \tag{2.49}$$

Thus, the $O(\delta^{-1})$ becomes $\mathcal{L}_0\, T_1 = 0$ and we conclude that the second term in the expansion is also independent of the fast variables, $T_1 = T_1(\mathbf{x}, t)$. Using now the fact that $\mathcal{L}_1\, T_0 = 0$ the $O(\delta^{-\frac{1}{2}})$ simplifies to:

$$\mathcal{L}_0\, T_2 + \mathcal{L}_2\, T_0 = 0 \tag{2.50}$$

The solvability condition for this equation is immediately satisfied since

$$\begin{aligned} \langle \mathcal{L}_2\, T_0 \rangle &= \langle \left(\mathbf{v}(\mathbf{y}, \tau) \cdot \nabla_{\mathbf{x}} - \frac{2}{Pe_l} \nabla_{\mathbf{x}} \nabla_{\mathbf{y}} \right) T_0(\mathbf{x}, t) \rangle \\ &= \langle \mathbf{v}(\mathbf{y}, \tau) \rangle \cdot \nabla_{\mathbf{x}} T_0(\mathbf{x}, t) = 0, \end{aligned} \tag{2.51}$$

on account of the assumption that the velocity field has mean zero over the periodicity cell. We can solve (2.51) through separation of variables. To this end, we look for a solution in the form

$$T_2(\mathbf{x}, t, \mathbf{y}, \tau) = \boldsymbol{\chi}(\mathbf{x}, t, \mathbf{y}, \tau) \cdot \nabla_{\mathbf{x}} T_0(\mathbf{x}, t) \tag{2.52}$$

Substituting now (2.52) into (2.51) we obtain equations for the auxiliary vector function $\boldsymbol{\chi} = \{\chi_l\}_{l=1}^{d}$ [7] . It is the unique mean zero solutions of the cell problem:

$$\mathcal{L}_0\, \boldsymbol{\chi}(\mathbf{y}, \tau) = -\mathbf{v}(\mathbf{y}, \tau) \tag{2.53}$$

[7] In the terminology of homogenization theory $\boldsymbol{\chi}$ is the *corrector field*.

We proceed now with the analysis of the $O(1)$ equation. The solvability condition reads:

$$\langle \mathcal{L}_1 T_2 + \mathcal{L}_2 T_1 + \mathcal{L}_3 T_0 \rangle = 0 \tag{2.54}$$

Since T_1 is independent of the fast variables and $\mathbf{v}$ is incompressible the average of the second term in the above equation vanishes. Thus, we have:

$$\begin{aligned} -\langle \mathcal{L}_1 T_2 \rangle &= -\langle \mathbf{V} \cdot \nabla_{\mathbf{y}} (\boldsymbol{\chi} \cdot \nabla_{\mathbf{x}} T_0) \rangle \\ &= -V_j \langle \frac{\partial \chi_l}{\partial y_j} \rangle \frac{\partial T_0}{\partial x_l} \\ &= 0 \;=\; \langle \mathcal{L}_3 T_0 \rangle \\ &= S_g \frac{\partial T_0}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}} T_0 \end{aligned} \tag{2.55}$$

In the above calculation we have used the fact that, since $\boldsymbol{\chi}(\mathbf{y}, \tau)$ is periodic in $\mathbf{y}$, the average of its gradient is zero. Consequently, the equation for the first term in the expansion is a transport equation:

$$S_g \frac{\partial T_0}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}} T_0 = 0, \tag{2.56}$$

together with the initial condition:

$$T_0(\mathbf{x}, t) = T_{in}(\mathbf{x}) \tag{2.57}$$

This result was to be expected: Since the global Peclet number is an $O(\delta^{-1-\alpha})$ quantity, the $O(1)$ effect of $\mathbf{V} + \delta^{\alpha} \mathbf{v}$ is that of transport due to the mean flow.

The equation for T_3 becomes:

$$\mathcal{L}_0 T_3 = -V_j \frac{\partial \chi_l}{\partial y_j} \frac{\partial T_0}{\partial x_l} - v_l \frac{\partial T_1}{\partial x_l} \tag{2.58}$$

We solve this equation through separation of variables:

$$T_3 = \boldsymbol{\sigma}(\mathbf{x}, t, \mathbf{y}, \tau) \frac{\partial T_0}{\partial x_l} + \boldsymbol{\chi}(\mathbf{y}, \tau) \frac{\partial T_1}{\partial x_l} \tag{2.59}$$

with

$$\mathcal{L}_0\, \boldsymbol{\sigma} = -\mathbf{V}(\mathbf{x},t)\cdot\nabla_{\mathbf{y}}\boldsymbol{\chi}(\mathbf{y},\tau) \tag{2.60}$$

We remark that $\boldsymbol{\sigma}$ is a function of both the fast and the slow variables.

Now we study the $O(\delta^{\frac{1}{2}})$ equation. The solvability condition reads:

$$\langle \mathcal{L}_1\, T_3 + \mathcal{L}_2\, T_2 + \mathcal{L}_3\, T_1 + \mathcal{L}_4\, T_0 \rangle = 0 \tag{2.61}$$

We shall treat each term in (2.61) separately. We have:

$$\langle \mathcal{L}_1\, T_3 \rangle = \langle \mathbf{V}\cdot\nabla_{\mathbf{y}}\left(\boldsymbol{\chi}\cdot\nabla_{\mathbf{x}}T_1 + \boldsymbol{\sigma}\cdot\nabla_{\mathbf{x}}T_0\right)\rangle = 0 \tag{2.62}$$

using the same argument as in (2.55). The second term in (2.61) gives:

$$\begin{aligned}\langle \mathcal{L}_2\, T_2 \rangle &= \langle \left(\mathbf{v}(\mathbf{y},\tau)\cdot\nabla_{\mathbf{x}} - \frac{2}{Pe_l}\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}\right)\boldsymbol{\chi}\cdot\nabla_{\mathbf{x}}T_0\rangle \\ &= \langle v_j\,\chi_l\rangle\frac{\partial^2 T_0}{\partial x_j\partial x_l} - \frac{2}{Pe_l}\langle\frac{\partial \chi_l}{\partial y_j}\rangle\frac{\partial^2 T_0}{\partial x_j\partial x_l} \\ &= \langle v_j\,\chi_l\rangle\frac{\partial^2 T_0}{\partial x_j\partial x_l}\end{aligned} \tag{2.63}$$

The third and fourth term in (2.61) are independent of the fast variables and they immediately give:

$$\langle \mathcal{L}_3\, T_1 \rangle = S_g\,\frac{\partial T_1}{\partial\, t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}T_1 \tag{2.64}$$

and

$$\langle \mathcal{L}_4\, T_0 \rangle = -\frac{1}{Pe_l}\Delta_{\mathbf{x}}T_0 \tag{2.65}$$

Putting everything together we obtain the equation for $T_1(\mathbf{x},t)$:

$$\frac{\partial T_1(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}T_1(\mathbf{x},t) = \frac{1}{Pe_l}\nabla_{\mathbf{x}}\cdot\big(\big(\mathcal{I} + \overline{\mathcal{K}}(\mathbf{x},t)\big)\cdot\nabla_{\mathbf{x}}T_0(\mathbf{x},t)\big)\,, \tag{2.66}$$

together with the initial condition:

$$T_1(\mathbf{x}, t=0) = 0 \tag{2.67}$$

The enhancement in the diffusivity is defined as:

$$\overline{\mathcal{K}}_{ij} = -Pe_l \, \langle v_i \, \chi_j \rangle, \tag{2.68}$$

and the effective diffusivity is:

$$\mathcal{K}^*_{ij} = \mathcal{I} + \overline{\mathcal{K}}_{ij} \tag{2.69}$$

We have already obtained the effective equations for the first two terms in the expansion. However, in order to obtain the necessary estimates that will enable us to prove the homogenization theorem we shall need to compute T_4. We have:

$$\begin{aligned} \mathcal{L}_0 T_4 &= -\left(\mathcal{L}_1 T_3 + \mathcal{L}_2 T_2 + \mathcal{L}_3 T_1 + \mathcal{L}_4 T_0\right) \\ &= -V_j \frac{\partial \chi_l}{\partial y_j}\frac{\partial T_1}{\partial x_l} - V_j \frac{\partial \sigma_l}{\partial y_j}\frac{\partial T_0}{\partial x_l} - \left(v_j \, \chi_l - \langle v_j \, \chi_l \rangle - \frac{2}{Pe_l}\frac{\partial \chi_l}{\partial y_j}\right) \frac{\partial^2 T_0}{\partial x_l \partial x_j} \end{aligned} \tag{2.70}$$

We look for solutions of (2.70) in the form:

$$T_4(\mathbf{x}, t, \mathbf{y}, \tau) = \psi_{jl}(\mathbf{y}, \tau) \frac{\partial^2 T_0}{\partial x_l \partial x_j} + \sigma_l(\mathbf{x}, t, \mathbf{y}, \tau) \frac{\partial T_1}{\partial x_l} + \rho_l(\mathbf{x}, t, \mathbf{y}, \tau) \frac{\partial T_0}{\partial x_l} \tag{2.71}$$

Substitution of (2.71) in (2.70) gives the equations that the auxiliary functions ψ_{jl}, ρ_l satisfy:

$$\mathcal{L}_0 \, \psi_{jl} = -v_j \, \chi_l + \langle v_j \, \chi_l \rangle + \frac{2}{Pe_l}\frac{\partial \chi_l}{\partial y_j}, \quad j, l = 1, .., d \tag{2.72a}$$

$$\mathcal{L}_0 \, \rho_l = -V_j(\mathbf{x}, t) \frac{\partial \sigma_l}{\partial y_j}(\mathbf{x}, t, \mathbf{y}, \tau), \quad l = 1, .., d \tag{2.72b}$$

and $\boldsymbol{\sigma}$ is given by the solution of equation (2.60). We note that the right hand sides of the above equations have zero average and thus equations 2.72a and 2.72b have smooth and unique, up to a constant, solutions. We ensure uniqueness by requiring the solutions to have zero average.

From the multiple scale expansion and the solvability conditions we obtained a solution to original initial value problem (2.21) which has the form:

$$
\begin{aligned}
T(\mathbf{x},t) &= T_0(\mathbf{x},t) + \delta^{\frac{1}{2}}\, T_1(\mathbf{x},t) + \delta\, T_2(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}}) \\
&+ \delta^{\frac{3}{2}}\, T_3(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}}) + \delta^2\, T_4(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}}) + T_{err} \\
&= T_0(\mathbf{x},t) + \delta^{\frac{1}{2}}\, T_1(\mathbf{x},t) + \delta\, \chi_l(\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}})\frac{\partial T_0}{\partial x_l} \\
&+ \delta^{\frac{3}{2}} \left(\sigma_l(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}})\frac{\partial T_0}{\partial x_l} + \chi_l(\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}})\frac{\partial T_1}{\partial x_l}\right) \\
&+ \delta^2 \left(\psi_{jl}(\mathbf{y},\tau)\frac{\partial^2 T_0}{\partial x_l \partial x_j} + \sigma_l(\mathbf{x},t,\mathbf{y},\tau)\frac{\partial T_1}{\partial x_l} + \rho_l(\mathbf{x},t,\mathbf{y},\tau)\frac{\partial T_0}{\partial x_l}\right) \\
&+ T_{err}(\mathbf{x},t), \qquad (2.73)
\end{aligned}
$$

where $T_0(\mathbf{x},\mathbf{t})$ and $T_1(\mathbf{x},\mathbf{t})$ are given by (2.56) and (2.66), respectively, together with the corresponding initial conditions.

Our goal now is to show that T_{err} is indeed small in the appropriate norm. We shall accomplish this by obtaining a parabolic equation for T_{err} and then applying lemma 2.2 to this equation. To this end, we apply the operator $\mathcal{L}^\delta$ to (2.73):

$$
\begin{aligned}
0 = \mathcal{L}^\delta\, T^\delta &= \mathcal{L}^\delta\, T_0(\mathbf{x},t) + \delta^{\frac{1}{2}}\, \mathcal{L}^\delta\, T_1(\mathbf{x},t) + \delta\, \mathcal{L}^\delta\, T_2(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}}) \\
&+ \delta^{\frac{3}{2}}\, \mathcal{L}^\delta\, T_3(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}}) + \\
&+ \delta^2\, \mathcal{L}^\delta\, T_4(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}}) + \mathcal{L}^\delta\, T_{err}(\mathbf{x},t) \qquad (2.74)
\end{aligned}
$$

For the initial conditions we have:

$$
\begin{aligned}
T_{in} = T^\delta|_{t=0} &= T_0(\mathbf{x},t)|_{t=0} + \delta^{\frac{1}{2}}\, T_1(\mathbf{x},t)|_{t=0} + \delta\, T_2(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}})|_{t=0} \\
&+ \delta^{\frac{3}{2}}\, T_3(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}})|_{t=0} \\
&+ \delta^2\, T_4(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta^{\frac{3}{2}}})|_{t=0} + T_{err}(\mathbf{x},t)|_{t=0} \\
&= T_{in} + \delta\, T_2|_{t=0} + \delta^{\frac{3}{2}}\, T_3|_{t=0} + \delta^2\, T_4|_{t=0} + T_{err}|_{t=0} \qquad (2.75)
\end{aligned}
$$

From the above equations we obtain an initial value problem for T_{err}:

$$\mathcal{L}^{\delta} T_{err} = -\left(\mathcal{L}^{\delta}\, T_0 + \delta^{\frac{1}{2}}\, \mathcal{L}^{\delta}\, T_1 + \delta\, \mathcal{L}^{\delta}\, T_2 + \delta^{\frac{3}{2}}\, \mathcal{L}^{\delta}\, T_3 + \delta^2\, \mathcal{L}^{\delta}\, T_4\right) \tag{2.76a}$$

$$T_{err}(\mathbf{x}, t=0) = -\left(\delta\, T_2|_{t=0} + \delta^{\frac{3}{2}}\, T_3|_{t=0} + \delta^2\, T_4|_{t=0}\right) \tag{2.76b}$$

Let us study each term in the right hand side of (2.76a) separately. We have:

$$\begin{aligned} \mathcal{L}^{\delta}\, T_0 &= \left(\mathcal{L}_3 + \delta^{-\frac{1}{2}}\mathbf{v}\cdot\nabla - \frac{1}{Pe_l}\delta^{\frac{1}{2}}\Delta\right) T_0 \\ &= \delta^{-\frac{1}{2}}\mathbf{v}\cdot\nabla T_0 - \frac{1}{Pe_l}\delta^{\frac{1}{2}}\Delta T_0 \end{aligned} \tag{2.77}$$

where equation (2.56) was used. The action of $\mathcal{L}^{\delta}$ on T_1 gives:

$$\begin{aligned} \mathcal{L}^{\delta}\, T_1 &= \left(\mathcal{L}_3 + \delta^{-\frac{1}{2}}\mathbf{v}\cdot\nabla - \frac{1}{Pe_l}\delta^{\frac{1}{2}}\right) T_1 \\ &= \frac{1}{Pe_l}\Delta T_0 - \langle v_i\, \chi_j\rangle \frac{\partial^2 T_0}{\partial x_i \partial x_j} + \delta^{-\frac{1}{2}}\mathbf{v}\cdot\nabla T_1 - \frac{1}{Pe_l}\delta^{\frac{1}{2}}\Delta T_1 \end{aligned} \tag{2.78}$$

In deriving (2.78) we used (2.66). Now we proceed with the next term. We have to use the fact that $\boldsymbol{\chi}$ depends on $\mathbf{x}$ and t through $\mathbf{y} = \frac{x}{\delta}$ and $\tau = \frac{t}{\delta^{\frac{3}{2}}}$:

$$\begin{aligned} \mathcal{L}^{\delta}\, T_2 &= \left(\left(\frac{1}{\delta^{\frac{3}{2}}}\mathcal{L}_0 + \frac{1}{\delta}\mathcal{L}_1 + \delta^{-\frac{1}{2}}\mathcal{L}_2 + \mathcal{L}_3 + \delta^{\frac{1}{2}}\mathcal{L}_4\right)\chi_l(\mathbf{y},\tau)\frac{\partial T_0}{\partial x_l}\right)\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} \\ &= \frac{1}{\delta^{\frac{3}{2}}}\frac{\partial T_0}{\partial x_l}(\mathcal{L}_0\chi_l)|_{y=\frac{x}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} + \frac{1}{\delta}V_j\frac{\partial T_0}{\partial x_l}\frac{\partial \chi_l}{\partial y_j}\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} \\ &+ \delta^{-\frac{1}{2}}\frac{\partial^2 T_0}{\partial x_l \partial x_j}\left(v_j\,\chi_l - \frac{2}{Pe}\frac{\partial \chi_l}{\partial y_j}\right)\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} + \left(S_g\chi_l\frac{\partial^2 T_0}{\partial x_j \partial t} + V_j\,\chi_l\frac{\partial^2 T_0}{\partial x_l \partial x_j}\right)\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} \\ &+ \delta^{\frac{1}{2}}\left(-\frac{1}{Pe}\frac{\partial^3 T_0}{\partial x_l \partial x_j^2}\,\chi_l\right)\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} \end{aligned} \tag{2.79}$$

In computing $\mathcal{L}^{\delta}\, T_3$, we have to pay attention to the fact that $\boldsymbol{\sigma}$ depends on both the fast and the slow variables. We have:

$$\begin{aligned}\mathcal{L}^\delta\, T_3 &= \frac{1}{\delta^{\frac{3}{2}}}\left(\frac{\partial T_1}{\partial x_l}(\mathcal{L}_0\chi_l)+\frac{\partial T_0}{\partial x_l}(\mathcal{L}_0\sigma_l)\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}+\frac{1}{\delta}V_j\left(\frac{\partial T_1}{\partial x_l}\frac{\partial \chi_l}{\partial y_j}+\frac{\partial T_0}{\partial x_l}\frac{\partial \sigma_l}{\partial y_j}\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\\
&+ \delta^{-\frac{1}{2}}(\frac{\partial^2 T_1}{\partial x_l\partial x_j}\left(v_j\,\chi_l-\frac{2}{Pe}\frac{\partial \chi_l}{\partial y_j}\right)+\frac{\partial^2 T_0}{\partial x_l\partial x_j}\left(v_j\,\sigma_l-\frac{2}{Pe_l}\frac{\partial \sigma_l}{\partial y_j}\right)\\
&+ \frac{\partial T_0}{\partial x_l}\left(v_j\frac{\partial \sigma_l}{\partial x_j}-\frac{2}{Pe_l}\frac{\partial^2\sigma_l}{\partial x_j\partial y_j}\right))|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\\
&+ (S_g\sigma_l\frac{\partial^2 T_0}{\partial x_j\partial t}+S_g\frac{\partial \sigma_l}{\partial t}\frac{\partial T_0}{\partial x_j}+V_j\,\frac{\partial \sigma_l}{\partial x_j}\frac{\partial T_0}{\partial x_l}+V_j\,\sigma_l\frac{\partial^2 T_0}{\partial x_l\partial x_j}\\
&+ S_g\chi_l\frac{\partial^2 T_1}{\partial x_j\partial t}+V_j\,\chi_l\frac{\partial^2 T_1}{\partial x_l\partial x_j})|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\\
&+ \delta^{\frac{1}{2}}\left(-\frac{1}{Pe}\frac{\partial^3 T_1}{\partial x_l\partial x_j^2}\,\chi_l-\frac{1}{Pe}\frac{\partial^3 T_0}{\partial x_l\partial x_j^2}\,\sigma_l-\frac{1}{Pe}\frac{\partial T_0}{\partial x_l}\,\Delta_{\mathbf{x}}\sigma_l\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\end{aligned} \tag{2.80}$$

We compute now $\mathcal{L}^\delta\, T_4$:

$$\begin{aligned}\mathcal{L}^\delta\, T_4 &= \frac{1}{\delta^{\frac{3}{2}}}\left(\frac{\partial T_1}{\partial x_l}(\mathcal{L}_0\,\sigma_l)+\frac{\partial T_0}{\partial x_l}(\mathcal{L}_0\,\rho_l)+\frac{\partial^2 T_0}{\partial x_l\partial x_j}(\mathcal{L}_0\,\psi_{jl})\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\\
&+ \frac{1}{\delta}V_k\left(\frac{\partial T_1}{\partial x_l}\frac{\partial \sigma_l}{\partial y_k}+\frac{\partial T_0}{\partial x_l}\frac{\partial \rho_l}{\partial y_k}+\frac{\partial^2 T_0}{\partial x_l\partial x_j}\frac{\partial \psi_{jl}}{\partial y_k}\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\\
&+ \frac{1}{\delta^{\frac{1}{2}}}(\frac{\partial^2 T_1}{\partial x_l\partial x_j}\left(v_j\,\sigma_l-\frac{2}{Pe_l}\frac{\partial \sigma_l}{\partial y_j}\right)+\frac{\partial T_1}{\partial x_l}\left(v_j\frac{\partial \sigma_l}{\partial x_j}-\frac{2}{Pe_l}\frac{\partial^2\sigma_l}{\partial x_j\partial y_j}\right)\\
&+ \frac{\partial^2 T_0}{\partial x_l\partial x_j}\left(v_j\,\rho_l-\frac{2}{Pe_l}\frac{\partial \rho_l}{\partial y_j}\right)+\frac{\partial T_0}{\partial x_l}\left(v_j\frac{\partial \rho_l}{\partial x_j}-\frac{2}{Pe_l}\frac{\partial^2\rho_l}{\partial x_j\partial y_j}\right)\\
&+ \frac{\partial^3 T_0}{\partial x_l\partial x_k\partial x_j}\left(v_k\,\psi_{jl}-\frac{2}{Pe}\frac{\partial \psi_{jl}}{\partial y_k}\right))|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\\
&+ (S_g\sigma_l\frac{\partial^2 T_1}{\partial x_j\partial t}+S_g\frac{\partial \sigma_l}{\partial t}\frac{\partial T_1}{\partial x_j}+V_j\,\frac{\partial \sigma_l}{\partial x_j}\frac{\partial T_1}{\partial x_l}+V_j\,\sigma_l\frac{\partial^2 T_1}{\partial x_l\partial x_j}\\
&+ S_g\rho_l\frac{\partial^2 T_0}{\partial x_j\partial t}+S_g\frac{\partial \rho_l}{\partial t}\frac{\partial T_0}{\partial x_j}+V_j\,\frac{\partial \rho_l}{\partial x_j}\frac{\partial T_0}{\partial x_l}+V_j\,\rho_l\frac{\partial^2 T_0}{\partial x_l\partial x_j}\\
&+ \psi_{jl}\left(S_g\frac{\partial^3 T_0}{\partial x_j\partial x_l\partial t}+V_k\,\frac{\partial^3 T_0}{\partial x_l\partial x_j\partial x_k}\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\\
&+ \delta^{\frac{1}{2}}(-\frac{1}{Pe_l}\frac{\partial^3 T_1}{\partial x_l\partial x_j^2}\,\sigma_l-\frac{1}{Pe_l}\frac{\partial T_1}{\partial x_l}\,\Delta_{\mathbf{x}}\sigma_l\\
&- \frac{1}{Pe_l}\frac{\partial^3 T_0}{\partial x_l\partial x_j^2}\,\rho_l-\frac{1}{Pe_l}\frac{\partial T_0}{\partial x_l}\,\Delta_{\mathbf{x}}\rho_l-\frac{1}{Pe_l}\psi_{jl}\frac{\partial^4 T_0}{\partial x_l\partial x_l\partial x_k^2})|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\end{aligned} \tag{2.81}$$

Now we are ready to compute the right hand side of (2.76a). Using the cell problems for the auxiliary functions $\boldsymbol{\chi}$, $\boldsymbol{\sigma}$, $\boldsymbol{\psi}$ we shall show that the right hand side of (2.76a) is an $O(\delta)$ quantity. In order to simplify the notation, we denote the right hand side of (2.76a) by RHS and we write it in the following form:

$$RHS = -\delta^{-\frac{1}{2}}\, F_1 - \, F_2 - \delta^{\frac{1}{2}}\, F_3 - \delta\, F_4 + o(\delta) \tag{2.82}$$

Only $\mathcal{L}^\delta\, T_0$ and $\mathcal{L}^\delta\, T_2$ contribute to the $O(\delta^{-\frac{1}{2}})$ term:

$$\begin{aligned} F_1(\mathbf{x},t) &= \mathbf{v}(\frac{x}{\delta},\frac{t}{\delta^{\frac{3}{2}}})\cdot\nabla T_0 + \frac{\partial T_0}{\partial x_l}(\mathcal{L}_0\chi_l)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} \\ &= \left(\mathbf{v}(\mathbf{y},\tau)+\mathcal{L}_0\boldsymbol{\chi})\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}}\cdot\nabla T_0 \\ &= 0, \end{aligned} \tag{2.83}$$

on account of the cell problem for $\boldsymbol{\chi}$. $\mathcal{L}^\delta\, T_1$, $\mathcal{L}^\delta\, T_2$ and $\mathcal{L}^\delta\, T_3$ contribute to the $O(1)$ term:

$$\begin{aligned} F_2(\mathbf{x},t) &= \left(\mathbf{v}\cdot\nabla T_1 + V_j\frac{\partial T_0}{\partial x_l}\frac{\partial \chi_l}{\partial y_j} + \frac{\partial T_1}{\partial x_l}(\mathcal{L}_0\chi_l) + \frac{\partial T_0}{\partial x_l}(\mathcal{L}_0\sigma_l)\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} \\ &= \left(\mathbf{v}\cdot\nabla T_1 + \frac{\partial T_1}{\partial x_l}(\mathcal{L}_0\chi_l)\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} \\ &+ \left(V_j\frac{\partial T_0}{\partial x_l}\frac{\partial \chi_l}{\partial y_j} + \frac{\partial T_0}{\partial x_l}(\mathcal{L}_0\sigma_l)\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta^{\frac{3}{2}}}} \\ &= 0 \end{aligned} \tag{2.84}$$

Now we consider the $O(\delta^{\frac{1}{2}})$ term. As before, it is understood that we evaluate the expression at $\mathbf{y}=\frac{\mathbf{x}}{\delta}$, $\tau=\frac{t}{\delta^{\frac{3}{2}}}$ and we will not repeat it:

$$\begin{aligned} F_3(\mathbf{x},t) &= -\langle v_i\,\chi_j\rangle\frac{\partial^2 T_0}{\partial x_i\partial x_j} + \frac{\partial^2 T_0}{\partial x_l\partial x_j}\left(v_j\,\chi_l - \frac{2}{Pe}\frac{\partial\chi_l}{\partial y_j}\right) \\ &+ V_j\left(\frac{\partial T_0}{\partial x_l}\frac{\partial\sigma_l}{\partial y_j} + \frac{\partial T_1}{\partial x_l}\frac{\partial\chi_l}{\partial y_j}\right) + \left(\frac{\partial T_1}{\partial x_l}(\mathcal{L}_0\,\sigma_l) + \frac{\partial T_0}{\partial x_l}(\mathcal{L}_0\,\rho_l) + \frac{\partial^2 T_0}{\partial x_l\partial x_j}(\mathcal{L}_0\,\psi_{jl})\right) \\ &= \frac{\partial T_1}{\partial x_l}\left(\mathcal{L}_0\,\sigma_l + V_j\frac{\partial\chi_l}{\partial y_j}\right) + \frac{\partial T_0}{\partial x_l}\left(\mathcal{L}_0\,\rho_l + V_j\frac{\partial\sigma_l}{\partial y_j}\right) \\ &+ \frac{\partial^2 T_0}{\partial x_l\partial x_j}\left(\mathcal{L}_0\,\psi_{jl}(\mathbf{y},\tau) - \langle v_i\,\chi_j\rangle + v_j\,\chi_l - \frac{2}{Pe}\frac{\partial\chi_l}{\partial y_j}\right) \\ &= 0, \end{aligned} \tag{2.85}$$

where the cell problems for $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$ and $\boldsymbol{\psi}$ have been used. The next term in (2.82) is the first nonzero term:

$$
\begin{aligned}
F_4(\mathbf{x},t) &= -\frac{1}{Pe_l}\Delta T_1 + S_g\chi_l\frac{\partial^2 T_0}{\partial x_j\partial t} + V_j\,\chi_l\frac{\partial^2 T_0}{\partial x_l\partial x_j} + \frac{\partial^2 T_1}{\partial x_l\partial x_j}\left(v_j\,\chi_l - \frac{2}{Pe}\frac{\partial\chi_l}{\partial y_j}\right) \\
&+ \frac{\partial^2 T_0}{\partial x_l\partial x_j}\left(v_j\,\sigma_l - \frac{2}{Pe_l}\frac{\partial\sigma_l}{\partial y_j}\right) + \frac{\partial T_0}{\partial x_l}\left(v_j\frac{\partial\sigma_l}{\partial x_j} - \frac{2}{Pe_l}\frac{\partial^2\sigma_l}{\partial x_j\partial y_j}\right) \\
&+ V_k\left(\frac{\partial T_1}{\partial x_l}\frac{\partial\sigma_l}{\partial y_k} + \frac{\partial T_0}{\partial x_l}\frac{\partial\rho_l}{\partial y_k} + \frac{\partial^2 T_0}{\partial x_l\partial x_j}\frac{\partial\psi_{jl}}{\partial y_k}\right)
\end{aligned} \tag{2.86}
$$

Since the two components of the velocity field as well as the initial conditions are assumed to be smooth, the first two terms in the expansion T_0 and T_1 are smooth. Moreover, the solutions of the cell problems are also smooth functions. Consequently, $F_4(\mathbf{x},t)$ is a smooth, bounded function. The initial value problem for T_{err} becomes:

$$\mathcal{L}^\delta T_{err} = -\delta\, F_4(\mathbf{x},t) \;\; in\; \mathbb{R}^d \times (0,\infty) \tag{2.87a}$$

$$T_{err}(\mathbf{x}, t=0) = \delta f(\mathbf{x}) \;\; on\; \mathbb{R}^d \tag{2.87b}$$

By applying now lemma 2 to the initial value problem (2.87) we obtain the estimates

$$||T_{err}||_{L^\infty((0,t_0);L^2(\mathbb{R}^d))} \leq \delta\left(1+\frac{t_0}{S_g}\right)C \tag{2.88a}$$

$$||\nabla T_{err}||_{L^2((0,t_0)\times\mathbb{R}^d))} \leq \delta^{\frac{1}{4}}\sqrt{Pe_l\left(\frac{S_g}{2}+1+\frac{t_0}{S}\right)}\,C \tag{2.88b}$$

for every $t_0 > 0$. Let us now define the function $T^{1,\delta} := T_0 + \delta^{\frac{1}{2}} T_1$. From the above estimates on T_{err}, together with the triangle inequality, we can estimate the difference between T and T_{err} in the $X := L^\infty((0,t_0); L^2(\mathbb{R}^d))$ norm, for sufficiently small δ:

$$
\begin{aligned}
||T^\delta - T^{1,\delta}||_X &= ||\delta\, T_2 + \delta^{\frac{3}{2}}\, T_3 + \delta^2\, T_4 + T_{err}||_X \\
&\leq \delta||T_2||_X + \delta^{\frac{3}{2}}||T_3||_X + \delta^2||T_4||_X + ||T_{err}|| \\
&\leq C_1\,\delta + o(\delta)
\end{aligned} \tag{2.89}
$$

Combining now (2.56) and (2.66) we obtain an equation for $T^{1,\delta}$:

$$S_g \frac{\partial T^{1,\delta}}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla T^{1,\delta} = \frac{1}{Pe_l}\delta^{\frac{1}{2}}\nabla\cdot\left(\mathcal{K}^*\cdot\nabla T^{1,\delta}\right) + o(\delta^{\frac{1}{2}}) \;\; in\ \mathbb{R}^d\times(0,\infty) \tag{2.90a}$$

$$T^{1,\delta}(\mathbf{x},t=0) = T_{in}(\mathbf{x}) \;\; on\ \mathbb{R}^d \tag{2.90b}$$

Neglecting now higher order terms we obtain the effective equation:

$$S_g \frac{\partial \overline{T}}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla \overline{T} = \frac{1}{Pe_l}\delta^{\frac{1}{2}}\nabla\cdot\left(\mathcal{K}^*\cdot\nabla \overline{T}\right) \;\; in\ \mathbb{R}^d\times(0,\infty) \tag{2.91a}$$

$$\overline{T}(\mathbf{x},t=0) = T_{in}(\mathbf{x}) \;\; on\ \mathbb{R}^d \tag{2.91b}$$

Remark: From the above analysis we derived an effective advection—diffusion equation with a constant effective diffusion tensor which is independent of the mean flow. The effect of the mean flow upon the effective transport appears as a higher order effect. Thus, writing T_2 in the form:

$$T_2(\mathbf{x},t,\mathbf{y},\tau) = \boldsymbol{\chi}(\mathbf{y},\tau)\cdot\nabla_{\mathbf{x}}T_0(\mathbf{x},t) + \overline{T}_0(\mathbf{x},t) \tag{2.92}$$

and augmenting the expansion (2.47) with the $O(\delta^{\frac{5}{2}})$ term T_5 we obtain, from the solvability condition of the $O(\delta)$ equation, the following equation for $\overline{T}_2$:

$$\frac{\partial \overline{T}_2(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla\overline{T}_2(\mathbf{x},t) = -\nabla\cdot\left(\langle\mathbf{v}\otimes\boldsymbol{\chi}\rangle\cdot\nabla T_1(\mathbf{x},t)\right) - \nabla\cdot\left(\langle\mathbf{v}\otimes\boldsymbol{\sigma}\rangle(\mathbf{x},t)\cdot\nabla T_0(\mathbf{x},t)\right), \tag{2.93}$$

with zero initial conditions. Proceeding as before, we can obtain an equation for $T^{2,\delta} := T_0 + \delta^{\frac{1}{2}}\,T_1 + \delta\,T_2$:

$$S_g \frac{\partial T^{2,\delta}}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla T^{2,\delta} = \frac{1}{Pe_l}\delta^{\frac{1}{2}}\,\nabla\cdot\left(\mathcal{K}^*\cdot\nabla T^{2,\delta}\right) + \frac{1}{Pe_l}\delta\,\nabla\cdot\left(\Sigma(\mathbf{x},t)\cdot\nabla T^{2,\delta}\right) + o(\delta) \;\; in\ \mathbb{R}^d\times(0,\infty) \tag{2.94a}$$

$$T^{2,\delta}(\mathbf{x}, t = 0) = T_{in}(\mathbf{x}) \;\; on \; \mathbb{R}^d \tag{2.94b}$$

with $\Sigma(\mathbf{x}, t) = -\langle \mathbf{v} \otimes \boldsymbol{\sigma} \rangle(\mathbf{x}, t)$. We emphasize that $\boldsymbol{\sigma}$ depends on the macroscopic variables $\mathbf{x}, t$ through the mean flow which appears on the right hand side of the cell problem for $\boldsymbol{\sigma}$. Thus, we see that even weak mean flows, compared to the fluctuations, contribute to the structure of the effective equations, to higher orders. Continuing with this process, one can obtain higher order corrections that will involve higher order derivatives of the effective passive scalar field. We shall not pursue this issue further.

2.4.1.2 The Case $\gamma < \frac{3}{2}$

Now we wish to present the derivation of the effective equation as well as the cell problem for $\gamma \neq 1 - \alpha$. The presence of the operator $\delta^{-\gamma} \mathcal{R}_0$ will lead to terms of the form $\delta^{\frac{j}{2} - \gamma} \mathcal{R}_0 T_j$, $j = 0, 1, \ldots$ and we need to augment the expansion (2.47) with additional terms that will lead to well posed $O(\delta^{\frac{j}{2} - \gamma})$ equations. Consequently, we need to introduce terms of the form $\delta^{\frac{3+j}{2}} A_j$, $j = 0, 1, \ldots$. One can check, however, that this expansion will again lead to ill-posed equations unless more terms are introduced. The general form of the expansion that we shall need has to be augmented with terms of the form:

$$\delta^{\frac{3}{2} - \gamma} A_0 + \delta^{2-\gamma} A_1 + \delta^{\frac{5}{2} - \gamma} A_2 + \ldots \delta^{\gamma} B_0 + \delta^{\gamma + \frac{1}{2}} B_1 + \delta^{\gamma + 1} B_2 + \ldots, \tag{2.95}$$

the functions A_i, B_i, $i = 1, 2, \ldots$ being functions of both the fast and the slow variables. A discussion of a related problem is presented in [14, pp. 262—265]. For brevity of exposition we shall omit the details and present the derivation of the homogenized equation for the case where γ takes integer values. In this case, no further modification of (2.47) is necessary.

Thus, we set $\gamma = 1$. We shall further assume that the average of the velocity field over the spatial period is zero, $\langle \mathbf{v} \rangle_y = \mathbf{0}$, the spatial average being defined in equation (2.19a). This assumption is not necessary and we make it in order to simplify the calculations. At the end of this subsection we shall describe briefly how to proceed when this assumption is removed.

The equations (2.48) are still valid for the following differential operators:

$$\mathcal{L}_0 := \mathcal{R}_1 \tag{2.96a}$$

$$\mathcal{L}_1 := \mathcal{R}_0 + \mathcal{R}_2 \tag{2.96b}$$

$$\mathcal{L}_2 := \mathcal{R}_3 \tag{2.96c}$$

$$\mathcal{L}_3 := \mathcal{R}_4 \tag{2.96d}$$

$$\mathcal{L}_4 := \mathcal{R}_5 \tag{2.96e}$$

The main difference from the previous case is that $\mathcal{L}_0$ is now an elliptic, as opposed to parabolic, operator and the fast time τ enters merely as a parameter. Consequently, a variant of the solvability condition given at lemma 1, valid for elliptic operators with periodic boundary conditions has to be used. Namely, the inhomogeneous equation

$$\mathcal{R}_1 f(\mathbf{y}, \tau) = g(\mathbf{y}, \tau) \tag{2.97}$$

has a unique, up to a constant solution if and only if the average of g over the spatial period is equal to zero: $\langle g \rangle_y = 0$. Moreover, the only solutions to the homogeneous equation are constants with respect to $\mathbf{y}$ but they can depend on τ.

With the above discussion in mind, we proceed to analyze the $O(\delta^{-\frac{3}{2}})$ equation. From the solvability conditions we conclude that T_0 is independent of $\mathbf{y}$: $T_0 = T_0(\mathbf{x}, t, \tau)$. Hence, the solvability condition for the $O(\delta^{-1})$ equation becomes:

$$S_g \, \frac{\partial T_0}{\partial \tau} = 0 \tag{2.98}$$

from which we immediately deduce that $T_0 = T_0(\mathbf{x}, t)$. Now the $O(\delta^{-1})$ becomes $\mathcal{L}_0 \, T_1 = 0$ and we conclude that the second term in the expansion is independent of $\mathbf{y}$: $T_1 = T_1(\mathbf{x}, t, \tau)$.

The solvability condition for the $O(\delta^{-\frac{1}{2}})$ equation gives:

$$\langle \mathcal{L}_1 \, T_1 + \mathcal{L}_2 \, T_0 \rangle_y = 0 \tag{2.99}$$

Consequently:

$$S_g \frac{\partial T_1}{\partial \tau} = -\langle \mathbf{v} \rangle_y \cdot \nabla_{\mathbf{x}} T_0 = 0$$

and we conclude that, $T_1 = T_1(\mathbf{x}, t)$. The equation for T_2 becomes:

$$\mathcal{L}_0 T_2 = -\mathbf{v} \cdot \nabla_{\mathbf{x}} T_0 \tag{2.100}$$

We solve (2.100) through separation of variables. We look for a solution in the form (2.52) and we obtain an equation for the corrector field $\boldsymbol{\chi}$ which is an elliptic equation in $\mathbf{y}$ with periodic boundary conditions in which τ enters as a parameter:

$$\mathbf{v} \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} - \frac{1}{Pe_l} \Delta_{\mathbf{y}} \boldsymbol{\chi} = -\mathbf{v} \tag{2.101}$$

We emphasize that equation (2.101) is well-posed because we have assumed that the spatial average of the velocity field vanishes. We ensure uniqueness of solutions by requiring $\boldsymbol{\chi}$ to have zero average over the spatial period cell.

We now analyze the $O(1)$ equation. The solvability condition reads:

$$\langle \left(S_g \frac{\partial}{\partial \tau} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla_{\mathbf{y}} \right) \boldsymbol{\chi} \cdot \nabla_{\mathbf{x}} T_0 + \mathbf{v} \cdot \nabla_{\mathbf{x}} T_1 + \left(S_g \frac{\partial}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}} \right) T_0 \rangle_y = 0 \tag{2.102}$$

Since $\boldsymbol{\chi}$ and $\mathbf{v}$ have zero average in space and $\boldsymbol{\chi}$ is periodic in space the first two terms in the average vanish and we are left with:

$$\left(S_g \frac{\partial}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}} \right) T_0 = 0 \tag{2.103}$$

This is, as expected, a transport equation for T_0, together with the initial condition $T_0(\mathbf{x}, t = 0) = T_{in}$. We remark that the fact that we have set $\langle \boldsymbol{\chi} \rangle_y = 0$ is not necessary for this derivation: We could take the average in time of the equation that results after averaging in space.

We can now solve the equation for T_3 through separation of variables and obtain (2.59).

The cell problem for $\boldsymbol{\sigma}$ reads:

$$\mathcal{L}_0\, \sigma_l = -V_j(\mathbf{x},t)\frac{\partial \chi_l}{\partial y_j}(\mathbf{y},\tau) - S_g\frac{\partial \chi_l}{\partial t}, \quad l = 1,..,d \tag{2.104}$$

We shall obtain an equation for T_1 from the solvability condition for the $O(\delta^{\frac{3}{2}})$ equation:

$$\begin{aligned} \langle \quad & \left(S_g\,\frac{\partial}{\partial\,\tau} + \mathbf{V}(\mathbf{x},t)\cdot\nabla_{\mathbf{y}}\right)(\boldsymbol{\chi}\cdot\nabla T_1 + \boldsymbol{\sigma}\cdot\nabla T_0) + \left(\mathbf{v}(\mathbf{y},\tau)\cdot\nabla_{\mathbf{x}} - \frac{2}{Pe_l}\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}\right)\boldsymbol{\chi}\cdot\nabla_{\mathbf{x}}T_0 \\ + \quad & \left(S_g\,\frac{\partial}{\partial\,t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}\right)T_1 - \frac{1}{Pe_l}\Delta_{\mathbf{x}}T_0\rangle_y = 0 \end{aligned} \tag{2.105}$$

Consequently:

$$S_g\,\frac{\partial}{\partial\,\tau}(\boldsymbol{\chi}\cdot\nabla_{\mathbf{x}}T_1 + \boldsymbol{\sigma}\cdot\nabla T_0) + \left(S_g\,\frac{\partial}{\partial\,t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}\right)T_1 = \frac{1}{Pe_l}\Delta_{\mathbf{x}}T_0 - \nabla\cdot(\langle\mathbf{v}\,\boldsymbol{\chi}\rangle_y\nabla_{\mathbf{x}}T_0) \tag{2.106}$$

We now take temporal average (which is defined in equation 2.19b) to obtain an inhomogeneous transport equation for T_1:

$$\frac{\partial T_1(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}T_1(\mathbf{x},t) = \frac{1}{Pe_l}\nabla\cdot\left(\left(\mathcal{I} + \overline{\mathcal{K}}(\mathbf{x},t)\right)\nabla_{\mathbf{x}}T_0(\mathbf{x},t)\right), \tag{2.107}$$

together with the initial condition:

$$T_1(\mathbf{x},t) = 0 \tag{2.108}$$

The enhancement in the diffusivity is defined as:

$$\overline{\mathcal{K}}_{ij} = -Pe_l\,\langle v_i\,\chi_j\rangle, \tag{2.109}$$

and the effective diffusivity is:

$$\mathcal{K}^* = \mathcal{I} + \overline{\mathcal{K}} \tag{2.110}$$

In the above derivation we have used the fact that in order to compute the total average we can first take the spatial average and then the temporal average:

$$\langle\cdot\rangle = \langle\langle\cdot\rangle_y\rangle_\tau = \langle\langle\cdot\rangle_\tau\rangle_y \tag{2.111}$$

Proceeding now as in the case $\gamma = \frac{3}{2}$ we conclude that the homogenized equation is again (2.23) with the same formulas for the effective diffusivity and the corrector field satisfying the elliptic cell problem (2.101).

Let us now discuss what happens when $\langle\mathbf{v}\rangle_y \neq \mathbf{0}$. The $O(\delta^{-\frac{3}{2}})$ and $O(\delta^{-1})$ equations give us, as before, that $T_0 = T_0(\mathbf{x}, t)$ and $T_1(\mathbf{x}, t, \tau)$. However, the solvability condition for the $O(\delta^{-\frac{1}{2}})$ equation gives:

$$S_g \frac{\partial T_1}{\partial \tau} = -\langle\mathbf{v}\rangle_y \cdot \nabla_{\mathbf{x}} T_0$$

We solve this equation by setting $T_1(\mathbf{x}, t, \tau) = \overline{T}_1(\mathbf{x}, t) + \hat{T}_1(\mathbf{x}, t, \tau)$ with

$$\hat{T}_1(\mathbf{x}, t, \tau) = -\left(\frac{1}{S_g}\int_0^\tau \langle\mathbf{v}\rangle_y(s)\, ds\right) \cdot \nabla_{\mathbf{x}} T_0 \tag{2.112}$$

Now the equation for T_2 becomes:

$$\mathcal{L}_0\, T_2 = -(\mathbf{v} - \langle\mathbf{v}\rangle_y) \cdot \nabla_{\mathbf{x}} T_0 \tag{2.113}$$

We solve this equation using separation of variables to obtain:

$$T_2(\mathbf{x}, t, \mathbf{y}, \tau) = \boldsymbol{\chi}(\mathbf{x}, t, \mathbf{y}, \tau) \cdot \nabla_{\mathbf{x}} T_0(\mathbf{x}, t) + \hat{T}_2(\mathbf{x}, t, \tau) \tag{2.114}$$

and the corrector field $\boldsymbol{\chi}$ satisfies the cell problem:

$$\mathbf{v} \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} - \frac{1}{Pe_l} \Delta_{\mathbf{y}} \boldsymbol{\chi} = -(\mathbf{v} - \langle\mathbf{v}\rangle_y) \tag{2.115}$$

Now the solvability condition for the $O(1)$ equation gives an equation for $\hat{T}_2(\mathbf{x}, t, \tau)$:

$$S_g \frac{\partial \hat{T}_2}{\partial \tau} = -\langle\mathbf{v}\rangle_y \cdot \nabla_{\mathbf{x}} \overline{T}_1 - \nabla_{\mathbf{x}} \left(\left(\frac{1}{S_g} \langle\mathbf{v}\rangle_y(\tau) \int_0^\tau \langle\mathbf{v}\rangle_y(s)\, ds \right) \cdot \nabla_{\mathbf{x}} T_0 \right) + \mathcal{L}_3\, T_0 \tag{2.116}$$

The solvability condition for this equation gives the transport equation for T_0. We remark that, since $\langle \mathbf{v} \rangle_y(\tau)$ is a periodic function of τ, we have:

$$\langle \langle v_i \rangle_y(\tau) \int_0^\tau \langle v_j \rangle_y(s)\, ds \rangle_\tau = 0, \quad i, j = 1, \ldots d \tag{2.117}$$

Now we can solve for $\hat{T}_2$ to obtain:

$$\hat{T}_2(\mathbf{x}, t, \tau) = \boldsymbol{\rho}(\tau) \cdot \nabla_{\mathbf{x}} \overline{T}_1 + \nabla_{\mathbf{x}} \left(\frac{1}{S_g} \left(\int_0^\tau \langle \boldsymbol{\phi} \rangle_y(s)\, ds \right) \cdot \nabla_{\mathbf{x}} T_0 \right) \tag{2.118}$$

where $\boldsymbol{\rho}(\tau) := -\left(\frac{1}{S_g} \int_0^\tau \langle \mathbf{v} \rangle_y(s)\, ds \right)$ and $\boldsymbol{\phi}(\mathbf{x}, \tau) := \mathbf{v}(\mathbf{x}, \tau) \otimes \boldsymbol{\rho}(\tau)$.

The equation for T_3 becomes:

$$\mathcal{L}_0\, T_3 = -\left(S_g \frac{\partial \boldsymbol{\chi}}{\partial \tau} + \mathbf{V} \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} \right) \cdot \nabla_{\mathbf{x}} T_0 - (\mathbf{v} - \langle \mathbf{v} \rangle_y) \cdot \nabla_{\mathbf{x}} \overline{T}_1 - \nabla_{\mathbf{x}} \left((\boldsymbol{\phi} - \langle \boldsymbol{\phi} \rangle_y) \cdot \nabla_{\mathbf{x}} T_0 \right) \tag{2.119}$$

The solution of this equation is:

$$T_3(\mathbf{x}, t, \mathbf{y}, \tau) = \boldsymbol{\psi} \cdot \nabla_{\mathbf{x}} T_0 + \boldsymbol{\chi} \cdot \nabla_{\mathbf{x}} \overline{T}_1 + \nabla_{\mathbf{x}} (\boldsymbol{\sigma} \cdot \nabla_{\mathbf{x}} T_0) + \hat{T}_3(\mathbf{x}, t, \tau) \tag{2.120}$$

And the corrector fields $\boldsymbol{\psi}$, $\boldsymbol{\sigma}$ satisfy the following cell problems:

$$\cdot \nabla_{\mathbf{y}} \boldsymbol{\psi} - \frac{1}{Pe_l} \Delta_{\mathbf{y}} \boldsymbol{\psi} = -\left(S_g \frac{\partial \boldsymbol{\chi}}{\partial \tau} + \mathbf{V} \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} \right) \tag{2.121a}$$

$$\nabla_{\mathbf{y}} \boldsymbol{\sigma} - \frac{1}{Pe_l} \Delta_{\mathbf{y}} \boldsymbol{\sigma} = -\left(\boldsymbol{\phi} - \langle \boldsymbol{\phi} \rangle_y \right) \tag{2.121b}$$

We proceed with the $O(\delta^{\frac{1}{2}})$ equation. The solvability condition for this equation gives us an equation for $\hat{T}_3(\mathbf{x}, t, \tau)$:

$$S_g \frac{\partial \hat{T}_3}{\partial \tau} = -\langle \mathbf{v}\, \boldsymbol{\chi} \rangle_y \cdot \nabla_{\mathbf{x}} T_0 - \langle \mathbf{v}\, \boldsymbol{\rho} \rangle_y \cdot \nabla_{\mathbf{x}} T_0 - \boldsymbol{\rho}\, \mathcal{L}_3\, \nabla_{\mathbf{x}} T_0 - \mathcal{L}_3\, \overline{T}_1 - \mathcal{L}_4\, T_0 \tag{2.122}$$

Taking now the temporal average of this equation we obtain the equation (2.107) for $\overline{T}_1$. Continuing now as in the previous cases we obtain the effective equation for $\overline{T} = T_0 + \delta^{\frac{1}{2}} T_1$.

2.4.1.3 The Case $\gamma > \frac{3}{2}$

. We now discuss the case when $\gamma > 1 - \alpha$. For brevity we shall restrict ourselves to the case $\gamma = 2$. The operator $\mathcal{L}^{\delta}$ takes the form:

$$\mathcal{L}^{\delta} \quad = \quad \frac{1}{\delta^2}\mathcal{R}_0 + \frac{1}{\delta^{\frac{3}{2}}}\mathcal{L}_0 + \frac{1}{\delta}\mathcal{L}_1 + \delta^{\frac{1}{2}}\mathcal{L}_2 + \mathcal{L}_3 + \delta^{\frac{1}{2}}\mathcal{L}_4 \tag{2.123}$$

where:

$$\mathcal{L}_0 := \mathcal{R}_1 \tag{2.124a}$$

$$\mathcal{L}_1 := \mathcal{R}_2 \tag{2.124b}$$

$$\mathcal{L}_2 := \mathcal{R}_3 \tag{2.124c}$$

$$\mathcal{L}_3 := \mathcal{R}_4 \tag{2.124d}$$

$$\mathcal{L}_4 := \mathcal{R}_5 \tag{2.124e}$$

Substituting now the expansion (2.47) into the equation $\mathcal{L}^{\delta}\, T^{\delta} = 0$ we obtain the following sequence of equations:

$$O(\delta^{-2}) : \quad \mathcal{R}_0\, T_0 = 0 \tag{2.125a}$$

$$O(\delta^{-\frac{3}{2}}) : \quad \mathcal{R}_0\, T_1 + \mathcal{L}_0\, T_0 = 0 \tag{2.125b}$$

$$O(\delta^{-1}) : \quad \mathcal{R}_0\, T_2 + \mathcal{L}_0\, T_1 + \mathcal{L}_1\, T_0 = 0 \tag{2.125c}$$

$$O(\delta^{-\frac{1}{2}}): \quad \mathcal{R}_0\, T_3 + \mathcal{L}_0\, T_2 + \mathcal{L}_1\, T_1 + \mathcal{L}_2\, T_0 = 0 \tag{2.125d}$$

$$O(1): \quad \mathcal{R}_0\, T_4 + \mathcal{L}_0\, T_3 + \mathcal{L}_1\, T_2 + \mathcal{L}_2\, T_1 + \mathcal{L}_3\, T_0 = 0 \tag{2.125e}$$

$$O(\delta^{\frac{1}{2}}): \quad \mathcal{R}_0\, T_5 + \mathcal{L}_0\, T_4 + \mathcal{L}_1\, T_3 + \mathcal{L}_2\, T_2 + \mathcal{L}_3\, T_1 + \mathcal{L}_4\, T_0 = 0 \tag{2.125f}$$

From the $O(\delta^{-2})$ equation we get that T_0 is independent of τ: $T_0 = T_0(\mathbf{x}, t, \tau)$. In order for the $O(\delta^{-\frac{3}{2}})$ to be well posed we need:

$$\langle \mathcal{L}_0\, T_0 \rangle_\tau = 0 \Rightarrow \quad \langle \mathcal{L}_0 \rangle_\tau\, T_0 = 0 \tag{2.126}$$

where

$$\langle \mathcal{L}_0 \rangle_\tau := \langle \mathbf{v}(\mathbf{y}, \tau) \rangle_\tau \cdot \nabla_{\mathbf{y}} - \frac{1}{Pe_l} \Delta_{\mathbf{y}} \tag{2.127}$$

Clearly, (2.127) is a uniformly elliptic operator and consequently (2.126) implies that $T_0 = T_0(\mathbf{x}, t)$. The $O(\delta^{-\frac{3}{2}})$ becomes $\mathcal{R}_0\, T_1 = 0$ from which we deduce that $T_1 = T_1(\mathbf{x}, t, \mathbf{y})$. Since T_0 is independent of $\mathbf{y}$ the $O(\delta^{-1})$ equation becomes:

$$\mathcal{R}_0\, T_2 + \mathcal{L}_0\, T_1 = 0 \tag{2.128}$$

The analysis for this equation is the same as for the $O(\delta^{-\frac{3}{2}})$ and we conclude that $T_1 = T_1(\mathbf{x}, t)$ and $T_2 = T_2(\mathbf{x}, t, \mathbf{y})$.

The $O(\delta^{-\frac{1}{2}})$ equation has a solution for T_3 if and only if:

$$\langle \mathcal{L}_0\, T_2 + \mathcal{L}_1\, T_1 + \mathcal{L}_2\, T_0 \rangle_\tau = 0 \tag{2.129}$$

or:

$$\langle \mathcal{L}_0 \rangle_\tau\, T_2 = -\langle \mathbf{v}(\mathbf{y}, \tau) \rangle_\tau \cdot \nabla_{\mathbf{x}} T_0 \tag{2.130}$$

The solution to this equation has the form (2.52) $T_2(\mathbf{x}, t, \mathbf{y}) = \chi(\mathbf{y}) \cdot T_0(\mathbf{x}, t)$, with χ satisfying the following, cell problem:

$$\langle \mathbf{v}(\mathbf{y},\tau)\rangle_\tau \cdot \nabla_{\mathbf{y}}\boldsymbol{\chi} - \frac{1}{Pe_l}\Delta_{\mathbf{y}}\boldsymbol{\chi} = -\langle \mathbf{v}(\mathbf{y},\tau)\rangle_\tau \tag{2.131}$$

We remark that (2.131) is a well posed elliptic equation with periodic boundary conditions since the spatial average of the right hand side is 0. Now we can express T_3 in the form

$$T_3(\mathbf{x},t,\mathbf{y},\tau) = \hat{T}_3(\mathbf{x},t,\mathbf{y},\tau) + \tilde{T}_3(\mathbf{x},t,\mathbf{y}) \tag{2.132}$$

with

$$\hat{T}_3(\mathbf{x},t,\mathbf{y},\tau) = \frac{1}{S_g}\int_0^\tau \phi(\mathbf{x},t,\mathbf{y},s)\,ds \tag{2.133}$$

where:

$$\begin{aligned} \phi(\mathbf{x},t,\mathbf{y},\tau)) &= -(\mathcal{L}_0\,T_2 + \mathcal{L}_2\,T_0) \\ &= -(\mathbf{v} - \langle \mathbf{v}\rangle_\tau)\cdot(\nabla_{\mathbf{y}}\boldsymbol{\chi} - \mathcal{I})\cdot\nabla_{\mathbf{y}}T_0 \end{aligned} \tag{2.134}$$

Now we proceed with the $O(1)$ equation. The solvability condition reads:

$$\langle \mathcal{L}_0\,T_3 + \mathcal{L}_1\,T_2 + \mathcal{L}_2\,T_1 + \mathcal{L}_3\,T_0\rangle_\tau = 0 \tag{2.135}$$

This is an elliptic equation for $\tilde{T}_3$:

$$\langle \mathcal{L}_0\rangle_\tau\,\tilde{T}_3 = -\langle \mathcal{L}_0\,\hat{T}_3 + \mathcal{L}_1\,T_2 + \mathcal{L}_2\,T_1 + \mathcal{L}_3\,T_0\rangle_\tau \tag{2.136}$$

and it is well posed if and only if the spatial average of the right hand side is zero. We have:

$$\langle\langle \mathcal{L}_0\,\hat{T}_3\rangle_\tau\rangle_y = \langle\langle \mathcal{L}_0\,\hat{T}_3\rangle_y\rangle_\tau = 0 \tag{2.137}$$

By interchanging the order of taking averages we also obtain that $\langle\langle \mathcal{L}_1\,T_2\rangle_\tau\rangle_y = 0$. Using now the fact that T_1 is independent of the fast variables and that the spatiotemporal average of the velocity of field vanishes we obtain $\langle\langle \mathcal{L}_2\,T_1\rangle_\tau\rangle_y = 0$. Thus, we obtain that in order for

(2.136) to be well posed we should have:

$$S_g \frac{\partial T_0}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla T_0 = 0, \tag{2.138}$$

which is, as expected, the transport equation for the first term in the expansion. We can now solve (2.137) for $\hat{T}_3$ but since we won't need this for the derivation of the equation for T_1 we shall omit the details. Then, T_4 can be written as $T_4(\mathbf{x}, t, \mathbf{y}, \tau) = \hat{T}_4(\mathbf{x}, t, \mathbf{y}, \tau) + \tilde{T}_4(\mathbf{x}, t, \mathbf{y})$ with:

$$\hat{T}_4(\mathbf{x}, t, \mathbf{y}, \tau) = \frac{1}{S_g} \int_0^\tau (\mathcal{L}_0\, T_3 + \mathcal{L}_1\, T_2 + \mathcal{L}_2\, T_1 + \mathcal{L}_3\, T_0)\; d\tau \tag{2.139}$$

The $O(\delta^{\frac{1}{2}})$ equation can be solved for T_5 provided that:

$$\langle \mathcal{L}_0\, T_4 + \mathcal{L}_1\, T_3 + \mathcal{L}_2\, T_2 + \mathcal{L}_3\, T_1 + \mathcal{L}_4\, T_0 \rangle_\tau = 0 \tag{2.140}$$

which leads to:

$$\langle \mathcal{L}_0 \rangle_\tau\, \tilde{T}_4 = -\langle \mathcal{L}_0\, \hat{T}_4 + \mathcal{L}_1\, T_3 + \mathcal{L}_2\, T_2 + \mathcal{L}_3\, T_1 + \mathcal{L}_4\, T_0 \rangle_\tau = 0 \tag{2.141}$$

The solvability condition for (2.140) will give us the equation for T_1: by taking the spatial average of the right hand side and interchanging the order of spatial and temporal averages we obtain that $\langle\langle \mathcal{L}_0\, \hat{T}_4 \rangle_\tau \rangle_y = \langle\langle \mathcal{L}_1\, T_3 \rangle_\tau \rangle_y = 0$. We remark that in order to obtain this result we don't need the specific form of neither T_3 nor T_4 but only the fact that they are periodic in $\mathbf{y}$ and τ.

The third term gives:

$$\langle\langle \mathcal{L}_2\, T_2 \rangle_\tau \rangle_y = \nabla \cdot (\langle \mathbf{v}\, \boldsymbol{\chi} \rangle \cdot \nabla T_0) \tag{2.142}$$

The last two terms are independent of the fast variables and consequently they are equal to their average. Putting now everything together we obtain equation (2.66). The same formulas for the effective diffusivity hold, the difference now being that the cell problem is the elliptic equation (2.131).

2.4.2 The Case $\alpha \neq -\frac{1}{2}$, $\gamma = 1 - \alpha$

Before discussing the general case of $\alpha \in (-1, 0)$ let us review what has been accomplished so far. For $\alpha = -\frac{1}{2}$ we used a multiple scales expansion, treating fast and slow variables as independent, in order to derive a sequence of equations for the various terms in the expansion. We used the $O(\delta^{-1+\alpha})$ and $O(\delta^{2\,\alpha})$ equations to show that the first two terms in the expansion are indeed independent of the fast variables. We then used the solvability conditions for the higher order equations to derive the effective equations as well as the cell problems. We then used the higher order terms in the expansion in order to obtain the necessary estimates that enabled us to prove the homogenization theorem.

The highest order equation that we had to study was the $O(\delta^{1+\alpha})$ equation from which we obtained the effective equation for the second term in the expansion and the explicit expression for the highest order term that we had to consider in order to obtain the necessary estimates. Needles to say, the higher order equations that we neglected are not being satisfied, since they involve terms in the expansion that have already been determined.

The structure of the expansion in the general case should be such that all equations up to $O(\delta^{1+\alpha})$ be well posed. We may also need to consider also higher order equations in order to be able to obtain the error estimates. The general form of the expansion, for the case $\alpha \in (-1, 0)$ with $\alpha \neq -\frac{l}{m}$ with $l, m \in \mathbb{N}$ is:

$$T^{\delta} \sim \sum_{n=0}^{k} \delta^{n(1+\alpha)}\, B_n + \sum_{n=0}^{k} \delta^{1+n(1+\alpha)}\, \Gamma_n + \sum_{n=0}^{k} \delta^{2+n(1+\alpha)}\, \Delta_n + \sum_{n=1}^{p+1} \delta^{(1-n\alpha)}\, A_n \qquad (2.143)$$

with A_n, B_n, Γ_n, Δ_n being functions of both the slow and the fast variables. The values of the integers k, p depend upon the value of α. This will become more clear after we present the equations of various orders. Our goal is to obtain an equation for the first two terms $B_0 + \delta^{1+\alpha}\, B_1$. Substituting now (2.143) into (2.40) with $\mathcal{L}^{\delta}$ given by (2.42) with $\gamma = 1 - \alpha$ we obtain the following equations:

$$O(\delta^{(n-1)+(n+1)\alpha}): \quad \mathcal{L}_0\, B_n = 0, \quad n = 1, \dots, k \qquad (2.144\text{a})$$

$$O(\delta^{-1}): \quad \mathcal{L}_1\, B_0 = 0 \qquad (2.144\text{b})$$

$$O(\delta^{n(1+\alpha)+\alpha}): \quad \mathcal{L}_0\,\Gamma_n + \mathcal{L}_1\,B_{n+1} + \mathcal{L}_2\,B_n = 0, \quad n = 1, \ldots, k \tag{2.144c}$$

$$O(1): \quad \mathcal{L}_0\,A_1 + \mathcal{L}_1\,\Gamma_0 + \mathcal{L}_3\,B_0 = 0 \tag{2.144d}$$

$$O(\delta^{\alpha(1-n)}): \quad \mathcal{L}_0\,A_{n+1} + \mathcal{L}_1\,A_n = 0, \quad n = 1, \ldots, p+1 \tag{2.144e}$$

$$O(\delta^{(n+1)(1+\alpha)}): \quad \mathcal{L}_0\,\Delta_n + \mathcal{L}_1\,\Gamma_{n+1} + \mathcal{L}_2\,\Gamma_n + \mathcal{L}_3\,B_{n+1} + \mathcal{L}_4\,B_n = 0, \quad n = 1, \ldots, k \tag{2.144f}$$

We see that the equations for A_n decouple from the rest of the equations. This is natural since the number of A_n terms that we shall need is different than the number of the B_n, Γ_n, Δ_n terms. More specifically, as $\alpha \to 0$ we need to add more A_n terms. On the other hand, the $\alpha \to -1$ limit requires more B_n, Γ_n, Δ_n terms.

Let us now solve equations (2.144). The $O(\delta^{(n-1)+(n+1)\alpha})$ terms imply that the the functions B_n are independent of the fast variables: $B_n = B_n(\mathbf{x}, t)$, $n = 1, \ldots k$. The $O(\delta^{-1})$ equation is trivially satisfied. Since the functions B_n are independent of the fast variables the $O(\delta^{n(1+\alpha)+\alpha})$ equations become:

$$\mathcal{L}_0\,\Gamma_n + \mathcal{L}_2\,B_n = 0 \tag{2.145}$$

We solve these equations through separation of variables to obtain $\Gamma_n(\mathbf{x}, t, \mathbf{y}, \tau) = \boldsymbol{\chi}(\mathbf{y}, \tau) \cdot \nabla B_n(\mathbf{x}, t)$ where the corrector field $\boldsymbol{\chi}$ satisfies the cell problem (2.53).

The solvability condition for the $O(\delta^1)$ equation gives the transport equation (2.56) for B_0. Now we can solve for A_1 to obtain $A_0(\mathbf{x}, t, \mathbf{y}, \tau) = \boldsymbol{\sigma}^0(\mathbf{x}, t, \mathbf{y}, \tau) \cdot \nabla \Gamma_0(\mathbf{x}, t)$ where $\boldsymbol{\sigma}^0$ satisfies the cell problem (2.60). We can then proceed to solve the $O(\delta^{\alpha(1-n)})$ equations. The final result is:

$$A_{n+1}(\mathbf{x}, t, \mathbf{y}, \tau) = \boldsymbol{\sigma}^{n+1}(\mathbf{x}, t, \mathbf{y}, \tau) \cdot \nabla A_n(\mathbf{x}, t, \mathbf{y}, \tau) \tag{2.146a}$$

$$\mathcal{L}_0\,\sigma_l^{n+1} = -V_j(\mathbf{x}, t)\frac{\partial \sigma_l^n}{\partial y_j}(\mathbf{x}, t, \mathbf{y}, \tau), \quad l = 1, .., d \tag{2.146b}$$

Now we are ready to solve the $O(\delta^{(n+1)(1+\alpha)})$ equations. From the solvability conditions we get the inhomogeneous transport equations for B_n:

$$\frac{\partial B_{n+1}(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla B_{n+1}(\mathbf{x},t) = \frac{1}{Pe_l}\nabla\cdot\left(\left(\mathcal{I}+\overline{\mathcal{K}}(\mathbf{x},t)\right)\cdot\nabla B_n(\mathbf{x},t)\right), \tag{2.147}$$

with $\overline{K}_{ij} = -Pe_l\,\langle v_i\,\chi_j\rangle$

$$\mathcal{L}_0\Delta_n = -V_j\frac{\partial\chi_l}{\partial y_j}\frac{\partial B_{n+1}}{\partial x_l} - \left(v_j\,\chi_l - \langle v_j\,\chi_l\rangle - \frac{2}{Pe_l}\frac{\partial\chi_l}{\partial y_j}\right)\frac{\partial^2 B_n}{\partial x_l\partial x_j} \tag{2.148}$$

The solution of this equation is:

$$\Delta_n(\mathbf{x},t,\mathbf{y},\tau) = \psi_{jl}(\mathbf{y},\tau)\frac{\partial^2 B_n}{\partial x_l\partial x_j} + \sigma_l^0(\mathbf{x},t,\mathbf{y},\tau)\frac{\partial B_{n+1}}{\partial x_l} \tag{2.149}$$

with $\boldsymbol{\psi}$ given by equation (2.72a).

Now we can obtain the effective equation (2.23) for $\overline{T} := B_0 + \delta\,B_1$. The higher order terms can be used in order to obtain the necessary estimates.

CHAPTER 3
Strong Mean Flow

3.1 Introduction

In the previous chapter we discussed the nondimensionalization of the advection-diffusion equation and we identified the relevant local and global nondimensional numbers. We also derived the effective equation in the case where the mean flow is weak compared to the fluctuations. We saw that, to leading order, the effective diffusion tensor is constant, independent of the mean flow. The mean flow only determines the transport velocity at the large length and time scales. As we discussed in the first chapter, this result is not consistent with measurements and direct numerical simulations of passive tracers in the ocean and the atmosphere.

In this chapter we shall derive the effective equations for mean flows which are equal in strength with the fluctuations and explore the situation where the mean flow is stronger than the fluctuations. In section 3.2 we shall present the homogenization theorem for the strong mean flow case and make various comments. We shall see that in this case the effective diffusion tensor is a function of space and time and that this results, apart from an enhancement in the diffusivity, to an effective drift. In section 3.3 we compute the effective diffusivity for two types of velocity fields and discuss their physical significance. The proof of the homogenization theorem is presented in section 3.4.

An alternative approach to the problem of periodic homogenization for advection—diffusion equations, namely the use of mean Lagrangian coordinates, is presented in section 3.5. The formalism developed in this section is the starting point of our asymptotic and numerical study of the homogenization problem for mean flows which are stronger than the fluctuations. This material is presented in section 3.6. Finally, we close this chapter in section 3.7 with a description of the numerical method that we are using in order to solve the cell problem .

3.2 Statement of the Main Result

We pose the problem in a slightly different setting than the one used in the previous chapter and consider the advection—diffusion equation (2.1) with a velocity field $\mathbf{u}$ active on two length and time scales and periodic in the fast variables $\frac{\mathbf{x}}{\delta}, \frac{t}{\delta}$. The characteristic velocity is denoted by V_0 and we identify the mean flow with the average of $\mathbf{u}$ over the period cell. We also use the notation S, Pe for the nondimensional numbers with the understanding that they are defined with respect to the spatial and temporal period of oscillations. Our goal now is to derive an effective equation that governs the evolution of $T(\mathbf{x}, t)$ for small δ. We have the following theorem:

THEOREM 3.1 *Let* $T^{\delta}(\mathbf{x}, t)$ *be the solution of*

$$S\,\frac{\partial T^{\delta}(\mathbf{x}, t)}{\partial t} + \mathbf{u}(\mathbf{x}, t, \frac{\mathbf{x}}{\delta}, \frac{t}{\delta}) \cdot \nabla T^{\delta}(\mathbf{x}, t) = \frac{\delta}{Pe} \Delta T^{\delta}(\mathbf{x}, t) \;\; \textit{in } \mathbb{R}^d \times (0, \infty) \tag{3.1a}$$

$$T^{\delta}(\mathbf{x}, t = 0) = T_{in}(\mathbf{x}) \;\; \textit{on } \mathbb{R}^d, \tag{3.1b}$$

where $\mathbf{u}$ *is incompressible and smooth and the initial conditions* T_{in} *are also smooth. Further, let* $\mathbf{V}(\mathbf{x}, t) := \langle \mathbf{u}(\mathbf{x}, t, \frac{\mathbf{x}}{\delta}, \frac{t}{\delta}) \rangle$*. Then, for every* $t_0 > 0$ *and* δ *sufficiently small there exists a constant* C*, independent of* δ*, such that*

$$||T^{\delta} - T^{1,\delta}||_{L^{\infty}((0,t_0);L^2(\mathbb{R}^d))} \leq \delta^2 \left(1 + \frac{t_0}{S}\right) C(Pe, S) \tag{3.2a}$$

$$||\nabla T^{\delta} - \nabla T^{1,\delta}||_{L^2((0,t_0)\times\mathbb{R}^d))} \leq \delta^{\frac{3}{2}} \sqrt{Pe \left(\frac{S}{2} + 1 + \frac{t_0}{S}\right)}\, C(Pe, S) \tag{3.2b}$$

where the average of $T^{1,\delta}$ $\overline{T} = \langle T^{1,\delta} \rangle$ *satisfies the following initial value problem:*

$$S\,\frac{\partial \overline{T}(\mathbf{x}, t)}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla \overline{T}(\mathbf{x}, t) = \frac{\delta}{Pe} \nabla \cdot (\mathcal{K}^{*}(\mathbf{x}, t) \cdot \nabla \overline{T}(\mathbf{x}, t)) \;\; \textit{in } \mathbb{R}^d \times (0, \infty) \tag{3.3a}$$

$$\overline{T}(\mathbf{x}, t = 0) = T_{in}(\mathbf{x}) \;\; \textit{on } \mathbb{R}^d \tag{3.3b}$$

where the effective diffusion tensor is:

$$\begin{aligned}\mathcal{K}^*_{ij}(\mathbf{x},t) &:= \delta_{ij} + \overline{\mathcal{K}}_{ij}(\mathbf{x},t) \\ &= \delta_{ij} - Pe\,\langle (u_i - V_i)\,\chi_j \rangle \end{aligned} \tag{3.4}$$

and $\boldsymbol{\chi} = \{\chi_j\}_{j=1}^d$ *satisfies the cell problem:*

$$S\,\frac{\partial \boldsymbol{\chi}}{\partial \tau} + \mathbf{u}(\mathbf{x},t,\mathbf{y},\tau)\cdot\nabla_{\mathbf{y}}\boldsymbol{\chi} - \frac{1}{Pe}\Delta_{\mathbf{y}}\boldsymbol{\chi} = -(\mathbf{u}(\mathbf{x},t,\mathbf{y},\tau) - \mathbf{V}(\mathbf{x},t)) \tag{3.5}$$

where the spatial differential operators are with respect to the auxiliary variable $\mathbf{y}$*.*

We remark that since our goal is to obtain a coarse grained description for the evolution of the passive scalar field only the average of $T^{1,\delta}$ is of interest to us.

The following corollary is an immediate consequence of the above theorem:

COROLLARY 3.1 *Let* $T^\delta(\mathbf{x},t)$ *be the solution of (2.21) with* $\alpha = 0$, $\gamma = 1$ *and* $\mathbf{V}$, $\mathbf{v}$ *being smooth and incompressible velocity fields and* T_{in} *is also smooth. Then for sufficiently small* δ *there exists a constant* C *independent of* δ *such that estimates (3.2) hold.* $\overline{T}(\mathbf{x},t)$ *satisfies (3.3), the effective diffusion tensor is given by (3.4) and the cell problem is:*

$$S\,\frac{\partial \boldsymbol{\chi}}{\partial \tau} + (\mathbf{V}(\mathbf{x},t) + \mathbf{v}(\mathbf{y},\tau))\cdot\nabla_{\mathbf{y}}\boldsymbol{\chi} - \frac{1}{Pe}\Delta_{\mathbf{y}}\boldsymbol{\chi} = -\mathbf{v}(\mathbf{y},\tau) \tag{3.6}$$

That the case $\tau = \frac{t}{\delta}$ corresponds to the case $\gamma = 1 - \alpha = 1$ and the local Strouhal number is an $O(1)$ quantity. The cases $\gamma > 1$ and $\gamma < 1$ for $\alpha = 0$ are similar to the cases discussed in the previous chapter for weak mean flows and will not be discussed further. Since we consider the case $\gamma = 1 - \alpha = 1$, both global and local Strouhal numbers are $O(1)$ quantities and no distinction between them need to be made.

The above result deserves various comments. First, we observe that the $O(1)$ effect in the effective equation is that of transport due to mean flow. The effect of the fluctuations on the large scale properties of the passive scalar field is an $O(\delta)$ correction. This was to be expected, since the global Peclet number is an $O(\frac{1}{\delta})$ quantity and advection dominates. However, we emphasize that it is important to keep the $O(\delta)$ term in the effective equation: letting $\delta \to 0$ would lead one to conclude that the passive scalar field is advected by the mean flow and that the fluctuations in the velocity field have no effect on the homogenized transport. This is in contrast to what has been observed in physical oceanography where the tracers are advected by an effective transport velocity which is not equal to the mean flow [46]. In this context the parameter δ, taken to represent the scale separation between mesoscale and large—scale structures, is a small but finite quantity, $\delta \sim 10^{-1}$, [32]. Thus, it is questionable whether the mathematical limit of infinite scale separation expressed through $\delta \to 0$ is of real interest for the physical problem under investigation. Moreover, as we shall see later, the effective drift resulting from the spatiotemporally dependent mean flow becomes dominant near the regions in space—time where the mean flow vanishes. Thus, keeping the $O(\delta)$ term is important.

Moreover, we see that the effective diffusion tensor is now a function of space and time. This spatiotemporal dependence of $\mathcal{K}^*(\mathbf{x}, t)$ is due to both the presence of a nontrivial mean flow as well as of the slow modulations of the periodic fluctuations. This is a strikingly different situation than the one in which the mean flow is either constant or weak compared to the fluctuations and the fluctuations are independent of the macroscopic variables in which case the effective diffusion tensor is constant and the effect of the fluctuations on the effective transport is purely diffusive.

We write the effective equation in non-divergence form:

$$S\,\frac{\partial T(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla T(\mathbf{x},t) = \frac{\delta}{Pe}\sum_{i,j=1}^{d}\mathcal{K}^*_{ij}(\mathbf{x},t)\frac{\partial^2 T(\mathbf{x},t)}{\partial x_i \partial x_j} + \frac{\delta}{Pe}\sum_{i=1}^{d}U_i(\mathbf{x},t)\frac{\partial T(\mathbf{x},t)}{\partial x_i} \tag{3.7a}$$

$$T(\mathbf{x}, t=0) = T_{in}(\mathbf{x}), \tag{3.7b}$$

where $U_i := -\sum_{j=1}^{d}\frac{\partial \mathcal{K}^*_{ij}(\mathbf{x},t)}{\partial x_j}$. We use the formalism that we introduced in chapter 1: we split the effective diffusion tensor into its symmetric part and antisymmetric part:

$$\mathcal{K}^* = \mathcal{S} + \mathcal{A} \tag{3.8}$$

where:

$$\mathcal{S}_{ij} = \frac{1}{2}(\mathcal{K}^*_{ij} + \mathcal{K}^*_{ji}) \tag{3.9a}$$

$$\mathcal{A}_{ij} = \frac{1}{2}(\mathcal{K}^*_{ij} - \mathcal{K}^*_{ji}) \tag{3.9b}$$

It is easy to show [69, p. 252] that the symmetric parts can be rewritten in the form $\mathcal{S}_{ij} = \delta_{ij} + Pe \langle \nabla_{\mathbf{y}}\chi_i \cdot \nabla_{\mathbf{y}}\chi_j \rangle$. Consequently, $\mathcal{S}$ is positive definite: diffusion in always enhanced.

Similarly, the effective drift can be decomposed into two parts, corresponding to the divergence of the symmetric and the antisymmetric parts of $\mathcal{K}^*$:

$$\mathbf{U}^s = -\nabla \cdot \mathcal{S}, \quad \mathbf{U}^a = -\nabla \cdot \mathcal{A} \tag{3.10}$$

The effective equation can be written in the form:

$$S\,\frac{\partial \overline{T}(\mathbf{x},t)}{\partial t} + (\mathbf{V}(\mathbf{x},t) + \frac{\delta}{Pe}\,\mathbf{U}^s + \frac{\delta}{Pe}\,\mathbf{U}^a) \cdot \nabla \overline{T}(\mathbf{x},t) = \frac{\delta}{Pe} \sum_{i,j=1}^{d} \mathcal{S}_{ij}(\mathbf{x},t) \frac{\partial^2 \overline{T}(\mathbf{x},t)}{\partial x_i \partial x_j} \tag{3.11a}$$

$$\overline{T}(\mathbf{x}, t = 0) = T_{in}(\mathbf{x}), \tag{3.11b}$$

The effective drift due to the antisymmetric part of the effective diffusivity tensor is solenoidal:

$$\begin{aligned} \sum_{i=1}^{d} \frac{\partial U^a_i}{\partial x_i} &= -\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial \mathcal{A}_{ij}}{\partial x_i \partial x_j} \\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial \mathcal{A}_{ji}}{\partial x_i \partial x_j} \\ &= -\sum_{i=1}^{d} \frac{\partial U^a_i}{\partial x_i}, \end{aligned} \tag{3.12}$$

and thus $\nabla \cdot \mathbf{U}^a = 0$. In particular, for two dimensional flows we can write $\mathbf{U}^a = \nabla^{\perp}\mathcal{A}_{21}$ with $\nabla^{\perp} = (\partial/\partial x_2, -\partial/\partial x_1)$ and $\mathcal{A}_{21}$ is the stream function of the eddy induced transport velocity. In the general d–dimensional case $\mathcal{A}$ is the *stream matrix* of $\mathbf{U}^a$.

On the other hand, $\mathbf{U}^s$ consists of both a solenoidal and a potential part. A sufficient condition for $\mathbf{U}^s$ to be potential is for $\mathcal{S}$ to be of the form [81]:

$$\mathcal{S}_{ii} = K(\mathbf{x}, t), \quad \mathcal{S}_{ij} = 0, \; i \neq j \tag{3.13}$$

General necessary and sufficient conditions that ensure that $\mathbf{U}^s$ is potential can be obtained by transforming $\mathcal{S}$ to its principal axes and studying the problem in this coordinate frame. We shall not pursue this issue further.

From the above comments we conclude that an alternative form of the effective equation is:

$$S\,\frac{\partial \overline{T}(\mathbf{x}, t)}{\partial t} + \mathbf{V}^{eff}(\mathbf{x}, t) \cdot \nabla \overline{T}(\mathbf{x}, t) = \frac{1}{Pe_g} \sum_{i,j=1}^{d} \mathcal{S}_{ij}(\mathbf{x}, t) \frac{\partial^2 \overline{T}(\mathbf{x}, t)}{\partial x_i \partial x_j} \tag{3.14a}$$

$$\overline{T}(\mathbf{x}, t = 0) = T_{in}(\mathbf{x}), \tag{3.14b}$$

with $\mathbf{V}^{eff}(\mathbf{x}, t) = \mathbf{V}(\mathbf{x}, t) + \frac{\delta}{Pe} \mathbf{U}^s(\mathbf{x}, t) + \frac{\delta}{Pe} \mathbf{U}^a(\mathbf{x}, t)$ and the global Peclet number Pe_g being an $O(\frac{1}{\delta})$ quantity.

From the above discussion it becomes apparent that the antisymmetric part of the effective diffusion tensor is important for the effective transport of the passive scalar field. The study of the symmetry properties of the effective diffusion tensor will be studied in the next chapter.

3.3 Examples

In this section we exhibit the complicated dependence of the effective diffusivity on the mean flow and the slowly modulated fluctuations for some simple examples. We start with a two–dimensional example where the fluctuations have the form of a shear flow whose amplitude depends on the macroscopic variables:

$$\mathbf{v}(\mathbf{x}, t, \mathbf{y}, \tau) = (f(x_2, t)\, v(y_2, \tau), \, 0) \tag{3.15}$$

The fluctuations are obviously incompressible. No specific form of the mean flow is assumed at this point: $\mathbf{V}(\mathbf{x},t) = (V_1(\mathbf{x},t),\, V_2(\mathbf{x},t))$ with $\nabla \cdot \mathbf{V} = 0$. Oscillatory shear velocity fields have been used by various authors as simple models for internal waves or tidal currents in the ocean [21, 66, 116]. The simplicity of the velocity field enables us to solve the cell problem in closed form. From (3.15) it is clear that the second component of the corrector field $\boldsymbol{\chi}$ vanishes. We solve the first cell problem by looking for solutions of the form:

$$\chi^1 = \chi^1(y_2, \tau\,; x_1, x_2, t) \tag{3.16}$$

The macroscopic variables x_1, x_2, t enter as parameters. The cell problem becomes:

$$S\,\frac{\partial \chi^1}{\partial \tau} + V_2(x_1,t)\frac{\partial \chi^1}{\partial y_2} - \frac{1}{Pe}\frac{\partial^2 \chi^1}{\partial y_2^2} = -f(x_2,t)\,v(y_2,\tau) \tag{3.17}$$

We assume that $v(y_2,\tau)$ has the following Fourier representation:

$$v(y_2,\tau) = \sum_{k^2+l^2\neq 0} \hat{v}_{k,l} e^{2\pi i(ky_2 + l\tau)} \tag{3.18}$$

We solve equation (3.17) using Fourier series. χ^1 is:

$$\chi^1(y_2,\tau\,;x_1,x_2,t) = -Pe \sum_{k^2+l^2\neq 0} \frac{f(x_2,t)\,\hat{v}_{k,l}\,e^{2\pi i(ky_2+l\tau)}}{4\pi^2k^2 + 2\pi i Pe\,(l\,S + k\,V_2(x_1,x_2,t))} \tag{3.19}$$

The diffusivity is enhanced only along the $\hat{e}_1$ direction:

$$\begin{aligned} \overline{\mathcal{K}}_{11}(x_1,x_2,t) &= -Pe\,\langle f\,v\,\chi^1\rangle \\ &= Pe^2 \sum_{k^2+l^2\neq 0} \frac{k^2\,f^2(x_2,t)\,|\hat{v}_{k,l}|^2}{4\pi^2k^4 + Pe^2(l\,S + k\,V_2(x_1,x_2,t))^2} \end{aligned} \tag{3.20}$$

We observe that the enhancement in the diffusivity is independent of the V_1 component of the mean flow.

We consider now the following specific example. The fluctuations are taken to be of the form:

$$v_1(y_2,\tau\,;\,x_2,t) = 2f(x_2,t)\,\cos(2\pi\tau)\cos(2\pi y_2) \tag{3.21}$$

Now the enhancement in the diffusivity along the $\hat{e}_1$ direction is:

$$\overline{\mathcal{K}}_{11}(x_1, x_2, t) = \frac{Pe^2\, f^2(x_2, t)}{2} \left(\frac{1}{4\pi^2 + Pe^2(S + V_2(x_1, x_2, t))^2} + \frac{1}{4\pi^2 + Pe^2(S + V_2(x_1, x_2, t))^2} \right) \tag{3.22}$$

The only non vanishing component of the effective drift is:

$$\begin{aligned} U_1^s(x_1, x_2, t) &= -\frac{\partial \overline{\mathcal{K}}_{11}}{\partial x_1} \\ &= Pe^4\, f^2(x_2, t) \frac{\partial V_2(x_1, x_2, t)}{\partial x_1} \Big(\frac{S + V_2(x_1, x_2, t)}{(4\pi^2 + Pe^2(S + V_2(x_1, x_2, t))^2)^2} \\ &+ \frac{S - V_2(x_1, x_2, t)}{(4\pi^2 + Pe^2(S + V_2(x_1, x_2, t))^2)^2} \Big) \end{aligned} \tag{3.23}$$

Taking now the mean flow to be a shear flow along the $\hat{e}_2$ direction $\mathbf{V} = (0,\, V_2(x_1, t))$ we see that the effective drift $\mathbf{V}^{eff}$ has the form:

$$\mathbf{V}^{eff}(\mathbf{x}, t) = \left(\frac{\delta}{Pe}\, U_1^s(x_1, x_2, t),\; V_2(x_1, t) \right) \tag{3.24}$$

Taking δ to be small but finite, for example $\delta = 10^{-1}$ as is the case for the mesoscale eddies, we see that U_1^s will be weak compared to V_2. Despite this, the velocity V^{eff} with which the passive scalar field is advected at the large scales is substantially different than the mean flow in that a component in the $\hat{e}_1$ direction appears. This specific example of shear layers demonstrates the importance of keeping the $O(\delta)$ term in the effective equation.

For our second example we choose a two–dimensional steady mean flow which is a linear function of space and for the fluctuations we choose time dependent perturbations of cellular flows. More specifically, the mean flow is:

$$\mathbf{V}(\mathbf{x}) = \mathcal{A}\,\mathbf{x} \tag{3.25}$$

where $\mathcal{A} \in L(\mathbb{R}^2, \mathbb{R}^2)$. [8] The incompressibility of the flow implies that $\mathcal{A}$ is traceless: $\mathrm{Tr}(\mathcal{A}) = 0$. One can show [65, ch. 1] that for a traceless matrix $\mathcal{A}$ in two dimensions there

[8] We are using the same notation as the one we used for the antisymmetric part of the effective diffusivity. We hope that no confusion will appear.

exists a rotation matrix $\mathcal{O} \in SO(\mathbb{R}^2)$ such that $\mathcal{O}\mathcal{A}\mathcal{O}^{-1} = \mathcal{M}_{\gamma,\omega_0}$ with

$$\mathcal{M}_{\gamma,\omega_0} = \begin{pmatrix} -\gamma & \frac{1}{2}\omega_0 \\ -\frac{1}{2}\omega_0 & \gamma \end{pmatrix} \tag{3.26}$$

where $\gamma,\, \omega_0 \in \mathbb{R}$. In the examples that follow we shall take $\mathcal{O} = \mathcal{I}$ and, thus, identify $\mathcal{A}$ with $\mathcal{M}_{\gamma,\omega_0}$. We shall also take $\gamma,\, \omega_0 \geq 0$.

The linear mean flow will enable us to experiment with different types of large scale flows. In the examples that follow we shall consider mean flows in the form of shear, strain and rotation.

For the fluctuations we choose time dependent perturbations of cellular flows [16]:

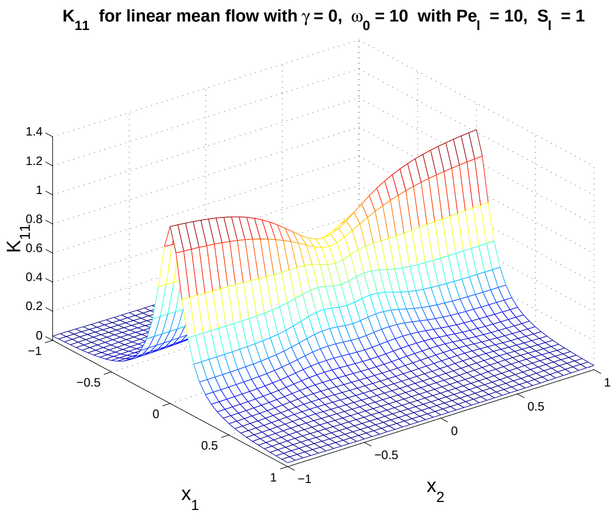
$$\begin{aligned} \mathbf{v}(y_1, y_2, t) &= (v_1, v_2) \\ &= (\cos(2\pi y_2) + \sin(2\pi y_2)\cos(2\pi\tau),\, \cos(2\pi y_1) + \sin(2\pi y_1)\cos(2\pi\tau)) \end{aligned} \tag{3.27}$$

The time dependent perturbation destroys all regular islands and the Hamiltonian system $\frac{d\mathbf{y}}{d\tau} = \mathbf{v}(\mathbf{y}, \tau)$ exhibits irregular behavior.

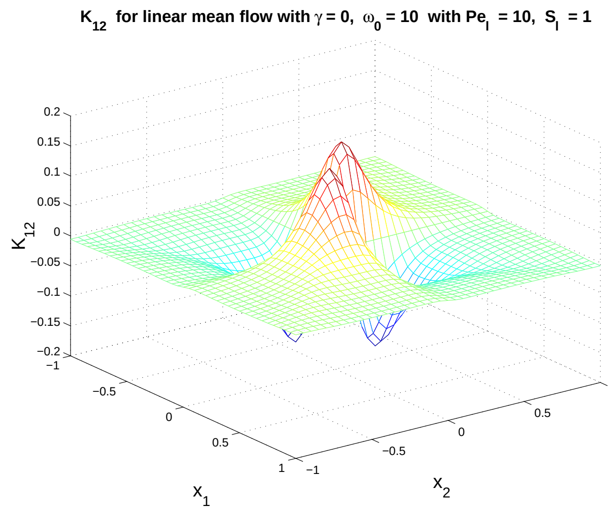
We solve now the cell problem for the velocity field $\mathbf{u}(\mathbf{x}, \mathbf{y}, \tau) = \mathcal{A}\mathbf{x} + \mathbf{v}(\mathbf{y}, \tau)$ in the domain $[-1, 1] \times [-1, 1]$. We fix $Pe_l = 10$, $S_l = 1$ and consider three different types of mean flows: a pure strain $\gamma = 10, \omega_0 = 0$, a pure rotation $\gamma = 0, \omega_0 = 10$ and a shear flow $\gamma = 5, \omega_0 = 10$. We plot the effective diffusion tensor as a function of the large–scale spatial variables x_1, x_2. Due to the symmetries of the fluctuations we have that $\overline{\mathcal{K}}_{11} = \overline{\mathcal{K}}_{22}$, $\overline{\mathcal{K}}_{12} = \overline{\mathcal{K}}_{21}$ and consequently it is enough to plot only the components $\overline{\mathcal{K}}_{11}$, $\overline{\mathcal{K}}_{12}$. For the numerical solution of the cell problem we are using the algorithm presented in section 3.7.

The numerical results are presented in figure 3.1. From these plots we clearly see that the effective diffusion tensor exhibits a complicated dependence on the properties of the mean flow.

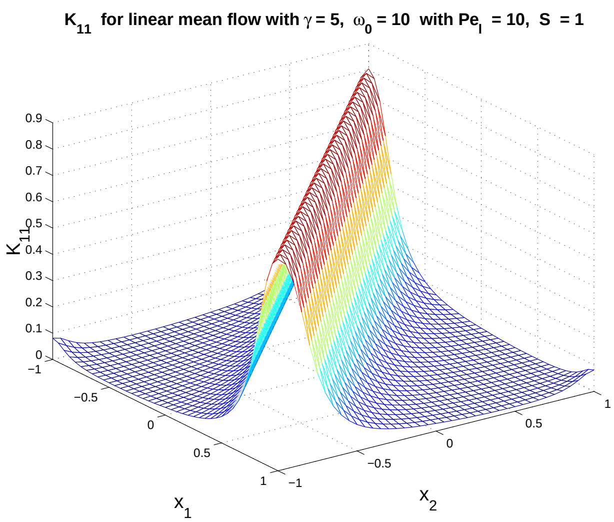
We emphasize that the numerical simulations were completed in a few minutes on a sun workstation. Moreover, the velocity fields that we have considered, as well as other types of periodically fluctuating velocity fields with very few non zero Fourier modes, have been used by various authors as simple models for geophysical and astrophysical flows [20, 21,



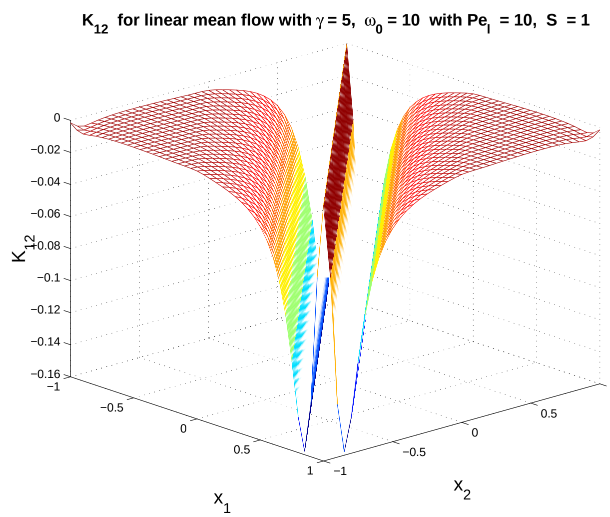
a. $\overline{\mathcal{K}}_{11}$ for $\gamma = 0.0$, $\omega_0 = 10.0$



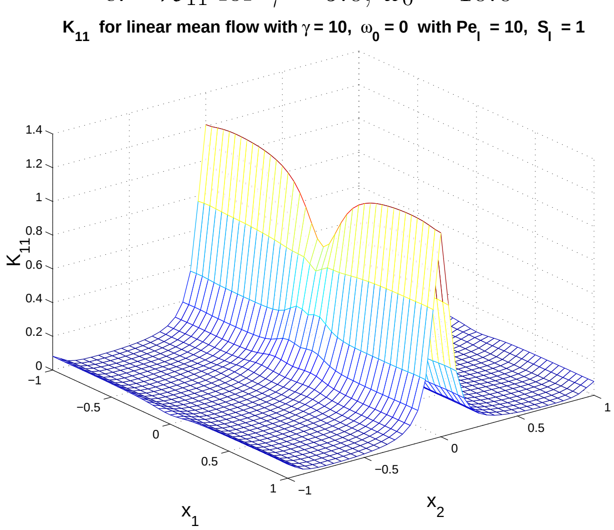
b. $\overline{\mathcal{K}}_{12}$ for $\gamma = 0.0$, $\omega_0 = 10.0$



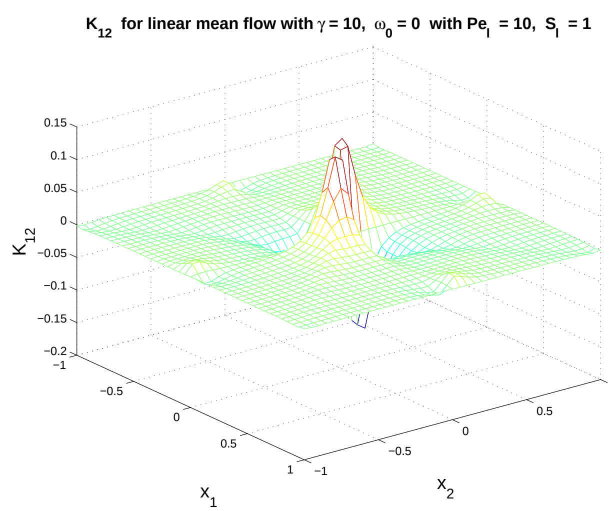
c. $\overline{\mathcal{K}}_{11}$ for $\gamma = 5.0$, $\omega_0 = 10.0$



d. $\overline{\mathcal{K}}_{12}$ for $\gamma = 5.0$, $\omega_0 = 10.0$



a. $\overline{\mathcal{K}}_{11}$ for $\gamma = 10.0$, $\omega_0 = 0.0$



b. $\overline{\mathcal{K}}_{12}$ for $\gamma = 10.0$, $\omega_0 = 0.0$

Figure 3.1: $\overline{\mathcal{K}}(\mathbf{x})$ for rotation, strain and shear linear mean flows and $Pe_l = 10$, $S_l = 1$ **.**

60, 66, 116, 117]. We believe that, despite the simplicity of our model for the velocity field, numerical computations of the effective diffusion tensor and the effective drift for simple model flows might be of interest to the ocean/atmosphere science community.

3.4 Proof of the Homogenization Theorem

In this section we shall prove theorem 3.1. We shall derive and justify the effective equation (3.3) using similar techniques to those of the previous chapter. The only new feature is that in order to obtain the necessary estimates we have to build the initial layer that it is created since the first two terms in the expansion satisfy the initial conditions of the original problem only to $O(\delta)$. Roughly speaking, we shall solve this problem by augmenting the $O(\delta)$ and $O(\delta^2)$ terms in the expansion with terms which satisfy the appropriate initial conditions and decay exponentially fast away from the origin. Our discussion will be brief. For a general discussion about the problem of initial and boundary layers in higher order periodic homogenization we refer to the book [9] and the papers [18, 103].

We decompose the velocity field into its mean and fluctuating parts:

$$\mathbf{u}(\mathbf{x}, t, \mathbf{y}, \tau) \quad = \quad \mathbf{V}(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t, \frac{\mathbf{x}}{\delta}, \frac{t}{\delta}) \tag{3.28}$$

with $\mathbf{V}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t, \frac{\mathbf{x}}{\delta}, \frac{t}{\delta}) \rangle$. Apart from the slow variables $\mathbf{x}, t$ we introduce the fast ones $\mathbf{y} = \frac{\mathbf{x}}{\delta}$, $\tau = \frac{t}{\delta}$. The differential operators transform as:

$$\nabla \to \nabla_x + \frac{1}{\delta}\nabla_{\mathbf{y}}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{1}{\delta}\frac{\partial}{\partial \tau} \tag{3.29}$$

For the parabolic differential operator which appears in equation (3.1) we have:

$$\begin{aligned} \mathcal{L}^\delta \quad &:= \quad S\frac{\partial}{\partial t} + (\mathbf{V} + \mathbf{v}) \cdot \nabla - \frac{1}{Pe}\Delta \\ &= \quad \frac{1}{\delta}\mathcal{L}_0 + \mathcal{L}_1 + \delta\, \mathcal{L}_2, \end{aligned} \tag{3.30}$$

where

$$\mathcal{L}_0 := S\frac{\partial}{\partial \tau} + (\mathbf{V} + \mathbf{v}) \cdot \nabla_{\mathbf{y}} - \frac{1}{Pe}\Delta_{\mathbf{y}} \tag{3.31a}$$

$$\mathcal{L}_1 := S\frac{\partial}{\partial t} + (\mathbf{V}+\mathbf{v})\cdot\nabla_{\mathbf{x}} - 2\frac{1}{Pe}\nabla_{\mathbf{x}}\nabla_{\mathbf{y}} \tag{3.31b}$$

$$\mathcal{L}_2 := -\frac{1}{Pe}\Delta_{\mathbf{x}} \tag{3.31c}$$

We now look for an approximate solution of (3.1) in the form of a multiple scales expansion:

$$T^{\delta}(\mathbf{x},t) \approx T_0(\mathbf{x},t,\mathbf{y},\tau) + \delta\, T_1(\mathbf{x},t,\mathbf{y},\tau) + \delta^2\, T_2(\mathbf{x},t,\mathbf{y},\tau) + \dots \tag{3.32}$$

where the functions T_i, $i = 0, 1, 2, \dots$ are periodic in $\mathbf{y}$ and τ. We substitute (3.32) into (3.1) and, by equating the coefficients of powers of δ to 0 we obtain a sequence of equations to be solved:

$$O(\delta^{-1}): \quad \mathcal{L}_0\, T_0 = 0 \tag{3.33a}$$

$$O(1): \quad \mathcal{L}_0\, T_1 + \mathcal{L}_1\, T_0 = 0 \tag{3.33b}$$

$$O(\delta): \quad \mathcal{L}_0\, T_2 + \mathcal{L}_1\, T_1 + \mathcal{L}_2\, T_0 = 0, \tag{3.33c}$$

The $O(\delta^{-1})$ equation gives, by lemma 2.1 in chapter 2, that $T_0 = T_0(\mathbf{x},t)$, independent of the fast variables. The solvability condition for the $O(1)$ equation gives:

$$\langle \mathcal{L}_1\, T_0 \rangle = 0, \tag{3.34}$$

from which we obtain the equation for T_0 [9] :

$$S\,\frac{\partial T_0}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla T_0 = 0, \tag{3.35}$$

together with the initial condition $T_0(\mathbf{x}, t=0) = T_{in}(\mathbf{x})$. This is, as expected, a transport

[9] When writing the effective equations we shall use the notation ∇ as opposed to $\nabla_{\mathbf{x}}$.

equation. Using now (3.35) into the $O(1)$ equation we obtain the equation for $T_1(\mathbf{x}, t)$:

$$\mathcal{L}_0 \, T_1 = -\mathbf{v}(\mathbf{x}, t, \mathbf{y}, \tau) \cdot \nabla_{\mathbf{x}} T_0(\mathbf{x}, t) \tag{3.36}$$

To proceed further, we decompose $T_1(\mathbf{x}, t)$ into its fluctuating and mean part:

$$T_1(\mathbf{x}, t, \mathbf{y}, \tau) = \boldsymbol{\chi}(\mathbf{x}, t, \mathbf{y}, \tau) \cdot \nabla_{\mathbf{x}} T_0(\mathbf{x}, t) + \overline{T}_1(\mathbf{x}, t) \tag{3.37}$$

where $\langle \chi \rangle = \mathbf{0}$. Substituting now (3.37) into equation (3.36) we obtain the cell problem for $\boldsymbol{\chi}$:

$$\mathcal{L}_0 \, \boldsymbol{\chi}(\mathbf{x}, \mathbf{y}, t, \tau) = -\mathbf{v}(\mathbf{x}, t, \mathbf{y}, \tau), \tag{3.38}$$

Since the average of the right hand side of equation (3.38) vanishes and we have also set $\langle \boldsymbol{\chi} \rangle = \mathbf{0}$, existence and uniqueness of solutions to (3.38) is ensured.

We now wish to obtain an equation for $\overline{T}_1(\mathbf{x}, t)$. We derive this equation from the solvability condition for the $O(\delta)$ equation:

$$\langle \mathcal{L}_1 T_1 + \mathcal{L}_2 T_0 \rangle = 0, \tag{3.39}$$

from which, after some algebra, we obtain:

$$\frac{\partial \overline{T}_1(\mathbf{x}, t)}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla \overline{T}_1(\mathbf{x}, t) = \frac{1}{Pe} \nabla \cdot \left(\left(\mathcal{I} + \overline{\mathcal{K}}(\mathbf{x}, t) \right) \cdot \nabla T_0(\mathbf{x}, t) \right), \tag{3.40}$$

together with homogeneous initial conditions $\overline{T}_1(\mathbf{x}, t = 0) = 0$. The enhancement in the diffusivity is:

$$\overline{\mathcal{K}}_{ij}(\mathbf{x}, t) = -Pe \, \langle v_i \, \chi_j \rangle, \tag{3.41}$$

Now we can solve the $O(\delta)$ equation for T_2. We substitute (3.40) in the $O(\delta)$ equation to obtain the equation for T_2:

$$\begin{aligned}\mathcal{L}_0 T_2 &= -\frac{\partial}{\partial x_j}\left(\left((V_j+v_j)\chi_l - \frac{2}{Pe}\frac{\partial \chi_l}{\partial y_j} + \frac{1}{Pe}\overline{\mathcal{K}}_{jl}\right)\frac{\partial T_0}{\partial x_l}\right) - v_l\frac{\partial \overline{T}_1}{\partial x_l} - S\,\frac{\partial}{\partial t}\left(\chi_l\frac{\partial T_0}{\partial x_l}\right)\\ &= -\rho_{jl}\frac{\partial^2 T_0}{\partial x_j \partial x_l} - \left(\frac{\partial \rho_{jl}}{\partial x_j} + S\,\frac{\partial \chi_l}{\partial t}\right)\frac{\partial T_0}{\partial x_l} - v_l\frac{\partial \overline{T}_1}{\partial x_l} - S\chi_l\frac{\partial^2 T_0}{\partial x_l \partial t} \qquad (3.42)\end{aligned}$$

where

$$\rho_{jl} = (V_j + v_j)\chi_l - 2\frac{\partial \chi_l}{\partial y_j} + \frac{1}{Pe}\overline{\mathcal{K}}_{jl} \qquad (3.43)$$

We introduce the auxiliary functions $\{\psi_{jl}\}_{j,l=1}^{d}$, $\{\theta_l\}_{l=1}^{d}$, $\{\sigma_l\}_{l=1}^{d}$ that satisfy the following higher order cell problems:

$$\mathcal{L}_0\,\psi_{jl} = -\rho_{jl}, \quad j,l = 1,..,d \qquad (3.44a)$$

$$\mathcal{L}_0\,\sigma_l = -\frac{\partial \rho_{jl}}{\partial x_j} - S\frac{\partial \chi_l}{\partial t}, \quad l = 1,..,d \qquad (3.44b)$$

$$\mathcal{L}_0\,\theta_l = -S\chi_l, \quad l = 1,..,d \qquad (3.44c)$$

We note that since $\overline{\mathcal{K}}_{ij} = -Pe\,\langle v_i\,\chi_j\rangle$ and $\langle \chi_j \rangle = 0$ the right hand sides of the above equations have zero average and consequently equations (3.44) have smooth and unique, up to a constant, solutions. We ensure uniqueness by requiring the solutions to have zero average. T_2 has the form:

$$T_2(\mathbf{x},t,\mathbf{y},\tau) = \psi_{jl}(\mathbf{x},t,\mathbf{y},\tau)\frac{\partial^2 T_0}{\partial x_l \partial x_j} + \sigma_l(\mathbf{x},t,\mathbf{y},\tau)\frac{\partial T_0}{\partial x_l} + \chi_l(\mathbf{x},t,\mathbf{y},\tau)\frac{\partial \overline{T}_1}{\partial x_l} + \theta_l(\mathbf{x},t,\mathbf{y},\tau)\frac{\partial^2 T_0}{\partial x_l \partial t} \qquad (3.45)$$

Now we wish to estimate the error that we are doing by neglecting higher order terms. We denote by T_{err} the discrepancy between the first three terms in the multiple scales expansion and the solution T^δ of (3.1):

$$T^\delta(\mathbf{x},t) = T_0(\mathbf{x},t) + \delta\, T_1(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta}) + \delta^2\, T_2(\mathbf{x},t,\frac{\mathbf{x}}{\delta},\frac{t}{\delta}) + T_{err} \tag{3.46}$$

Our goal is to obtain uniform estimates for the error term in (3.46). We shall accomplish this by obtaining a parabolic equation for T_{err} and using energy estimates.

We apply $\mathcal{L}^\delta$ to (3.46) to obtain:

$$\mathcal{L}^\delta T_{err} = -\mathcal{L}^\delta T_0 - \delta\, \mathcal{L}^\delta T_0 - \delta^2 \mathcal{L}^\delta T_2 \tag{3.47a}$$

$$T_{err}(\mathbf{x}, t=0) = -\delta\, \chi_l \frac{\partial T_0}{\partial x_l}|_{t=0} - \delta^2\, T_2|_{t=0} \tag{3.47b}$$

We recall that the auxiliary functions $\boldsymbol{\chi}$, $\boldsymbol{\theta}$, $\boldsymbol{\psi}$, $\boldsymbol{\sigma}$ are functions of both $\mathbf{x}$, t and $\frac{\mathbf{x}}{\delta}$, $\frac{t}{\delta}$. Bearing this in mind, we compute the right hand side of the above equation. We have:

$$\mathcal{L}^\delta T_0 = \mathbf{v}\cdot\nabla T_0 - \frac{\delta}{Pe}\Delta T_0 \tag{3.48}$$

Now we want to compute the second term in the right hand side of (3.47a). We have:

$$\mathcal{L}^\delta \overline{T}_1 = \frac{1}{Pe}\Delta T_0 + \frac{1}{Pe}\nabla\cdot\left(\overline{\mathcal{K}}\cdot\nabla T_0\right) + \mathbf{v}\cdot\nabla\overline{T}_1 - \frac{\delta}{Pe}\Delta\overline{T}_1 \tag{3.49}$$

and:

$$\begin{aligned}
\mathcal{L}^\delta\left(\chi_l\frac{\partial T_0}{\partial x_l}\right) &= \frac{1}{\delta}(\mathcal{L}_0\chi_l)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta}}\frac{\partial T_0}{\partial x_l} \\
&+ \left(S\frac{\partial\chi_l}{\partial t} + (V_j+v_j)\frac{\partial\chi_l}{\partial x_j} - \frac{2}{Pe}\frac{\partial^2\chi_l}{\partial x_j\partial y_j}\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta}}\frac{\partial T_0}{\partial x_l} \\
&+ \left((V_j+v_j)\chi_l - \frac{2}{Pe}\frac{\partial\chi_l}{\partial y_j}\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta}}\frac{\partial^2 T_0}{\partial x_l\partial x_j} + S\chi_l\frac{\partial^2 T_0}{\partial x_0\partial t} \\
&- \frac{\delta}{Pe}\left(\frac{\partial T_0}{\partial x_l}\Delta_{\mathbf{x}}\chi_l + 2\frac{\partial\chi_l}{\partial x_j}\frac{\partial^2 T_0}{\partial x_l\partial x_j} + \chi_l\frac{\partial^3 T_0}{\partial x_l\partial x_j\partial x_j}\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta}}
\end{aligned} \tag{3.50}$$

Now we compute $\mathcal{L}^\delta T_2$. We have:

$$\begin{aligned}\mathcal{L}^\delta T_2 &= \frac{1}{\delta}\left(\mathcal{L}^\delta \psi_{jl}\frac{\partial^2 T_0}{\partial x_l \partial x_j} + \mathcal{L}^\delta \sigma_l|\frac{\partial T_0}{\partial x_l} + \mathcal{L}^\delta \chi_l \frac{\partial \overline{T}_1}{\partial x_l} + \mathcal{L}^\delta \theta_l \frac{\partial^2 T_0}{\partial x_l \partial t}\right)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta}} \\ &+ \mathcal{L}_1 T_2 + \delta\, \mathcal{L}_2 T_2 \end{aligned} \tag{3.51}$$

Putting the above equations together we get:

$$\begin{aligned}\mathcal{L}^\delta T_{err} &= -\left(\mathcal{L}^\delta T_0 + \mathcal{L}^\delta \overline{T}_1 + \mathcal{L}^\delta\left(\chi_l \frac{\partial T_0}{\partial x_l}\right) + \delta^2 \mathcal{L}^\delta T_2\right) \\ &= -(\mathcal{L}^\delta \chi_l + v_l)|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta}} \frac{\partial T_0}{\partial x_j} \\ &- \delta((\mathcal{L}^\delta \psi_{jl} + \rho_{jl})\frac{\partial^2 T_0}{\partial x_l \partial x_j} + (\mathcal{L}^\delta \sigma_l + \frac{\partial \rho_{jl}}{\partial x_j} + S\frac{\partial \chi_l}{\partial t})\frac{\partial T_0}{\partial x_l} + (\mathcal{L}^\delta \chi_l + v_l)\frac{\partial \overline{T}_1}{\partial x_l} \\ &+ (\mathcal{L}^\delta \theta_l + S\chi_l)\frac{\partial^2 T_0}{\partial x_l \partial t})|_{\mathbf{y}=\frac{\mathbf{x}}{\delta},\tau=\frac{t}{\delta}} - \delta^2 F(\mathbf{x},t) \\ &= -\delta^2 F(\mathbf{x},t) \end{aligned} \tag{3.52}$$

where

$$F(\mathbf{x},t) = -\frac{1}{Pe}\left(\Delta_{\mathbf{x}}\overline{T}_1 + \frac{\partial T_0}{\partial x_l}\Delta_{\mathbf{x}}\chi_l + 2\frac{\partial \chi_l}{\partial x_j}\frac{\partial^2 T_0}{\partial x_l \partial x_j} + \chi_l \frac{\partial^3 T_0}{\partial x_l \partial x_j \partial x_j}\right) + \mathcal{L}_1 T_2 + \delta\, \mathcal{L}_2 T_2$$

For the initial conditions we have:

$$f(\mathbf{x}) = \chi_l \frac{\partial T_0}{\partial x_l}|_{t=0} + \delta\, T_2|_{t=2} \tag{3.53}$$

Since we have assumed that the velocity field as well as the initial conditions are smooth, the solutions of (3.36) and (3.40) as well as of the cell problems are smooth and bounded. Consequently, the functions $F(\mathbf{x},t)$, $f(\mathbf{x},t)$ are also smooth and bounded. Thus, neglecting higher order terms, T_{err} satisfies the following initial value problem problem:

$$\mathcal{L}^\delta T_{err} = -\delta^2 F(\mathbf{x},t) \tag{3.54a}$$

$$T_{err}(x, t=0) = -\delta f(\mathbf{x}) \tag{3.54b}$$

Now we can apply lemma 2.2 to the initial value problem (3.54) to estimate T_{err}. However, it is clear that the bound that we shall obtain is not sharp. The reason for this is that the initial condition (3.54b) for T_{err} is $O(\delta)$ and not $O(\delta^2)$ that we need in order to be able to obtain a sharper bound. Remembering that $f(\mathbf{x}) = \chi_l \frac{\partial T_0}{\partial x_l}|_{t=0} + o(\delta)$ we see that in order to solve this problem we have to augment the multiple scales expansion with terms that will compensate for the $O(\delta)$ term in (3.54b) and will vanish exponentially fast away from the origin so that they will not introduce $O(\delta)$ terms in (3.54a). Following [18] we introduce the following multiple scales expansion:

$$\begin{aligned} T^{\delta}(\mathbf{x},t) &\approx T_0(\mathbf{x},t) + \delta\, T_1(\mathbf{x},t,\mathbf{y},\tau) + \delta^2\, T_2(\mathbf{x},t,\mathbf{y},\tau) + \\ &+ \delta\, T_1^{in}(\mathbf{x},t,\mathbf{y},\tau,\tau^*) + \delta^2\, T_2^{in}(\mathbf{x},t,\mathbf{y},\tau,\tau^*) + \dots \end{aligned} \tag{3.55}$$

with T_1^{in}, T_2^{in} being periodic in $\mathbf{y}$ and τ. We have also introduced the new time–like variable $\tau^* = \frac{t}{\delta}$. We remark that these two terms depend on $\frac{t}{\delta}$ in two ways and we have explicitly taken this into account. T_0, T_1 satisfy equations (3.36) and (3.40), respectively and T_2 is given by formula (3.45). Substituting now the expansion (3.55) into equation (3.1) we obtain equations for the terms T_1^{in}, T_2^{in}:

$$\mathcal{L}^{in}\, T_1^{in} = 0 \tag{3.56a}$$

$$T_1^{in}(\mathbf{x}, t=0, \mathbf{y}, \tau, \tau^*=0) = -\boldsymbol{\chi}(\mathbf{x},\mathbf{y},\tau,t=0)\cdot\nabla_{\mathbf{x}} T_{in}(\mathbf{x}), \tag{3.56b}$$

and

$$\mathcal{L}^{in}\, T_2^{in} = -\mathcal{L}_1\, T_1^{in} \tag{3.57a}$$

$$T_2^{in}(\mathbf{x}, t=0, \mathbf{y}, \tau, \tau^*=0) = -\left(\psi_{jl}\frac{\partial^2 T_0}{\partial x_l \partial x_j} + \sigma_l \frac{\partial T_0}{\partial x_l} + \chi_l \frac{\partial \overline{T}_1}{\partial x_l} + \theta_l \frac{\partial^2 T_0}{\partial x_l \partial t}\right)|_{t=0} \tag{3.57b}$$

where

$$\mathcal{L}_{in} := S\frac{\partial}{\partial\,\tau^*} + S\frac{\partial}{\partial\,\tau} + (\mathbf{V}(\mathbf{x},t) + \mathbf{v}(\mathbf{x},t,\mathbf{y},\tau))\cdot\nabla_{\mathbf{y}} - \frac{1}{Pe}\Delta_{\mathbf{y}} \tag{3.58}$$

We emphasize that the system of equations (3.56) is an initial value problem for T_1^{in}. Using now a variant of theorem 3 from [18, p. 13], together with the fact that $\langle \boldsymbol{\chi} \rangle = \mathbf{0}$ we can prove that there exist constants c_1, $c_2 > 0$ independent of $\mathbf{x}$, t, $\mathbf{y}$, τ, τ^* such that:

$$T_1^{in}(\mathbf{x}, t, \mathbf{y}, \tau, \tau^*) \leq c_1 e^{-c_2 \tau^*} \tag{3.59}$$

We remark that we can solve the initial value problem (3.56) using separation of variables and express the solution in the form $T_1^{in}(\mathbf{x}, t, \mathbf{y}, \tau, \tau^*) = \boldsymbol{\chi}^{in}(\mathbf{x}, t, \mathbf{y}, \tau, \tau^*) \cdot \nabla_{\mathbf{x}} T_{in}(\mathbf{x})$. We can obtain similar results for T_2^{in} but we shall not need this.

Going back to the initial value problem for T_{err} we now have:

$$\mathcal{L}^{\delta} T_{err} = -\delta^2 F(\mathbf{x}, t) \tag{3.60a}$$

$$T_{err}(x, t = 0) = -\delta^2 f^*(\mathbf{x}) \tag{3.60b}$$

with $f^* = T_2|_{t=0}$. Using lemma 2.2 we obtain the estimates:

$$||T_{err}||_X \leq \delta^2 \left(1 + \frac{t_0}{S}\right) C \tag{3.61a}$$

$$||\nabla T_{err}||_X \leq Pe\, \delta^{\frac{3}{2}} \sqrt{\left(\frac{S}{2} + 1 + \frac{t_0}{S}\right)}\, C \tag{3.61b}$$

where the notation $X := L^{\infty}((0, t_0); L^2(\mathbb{R}^d))$ was used. We define now $T^{1,\delta} = T_0 + \delta\, T_1$ and use the triangle inequality to obtain:

$$||T^{\delta} - T^{1,\delta}||_X \leq C_1\, \delta^2 \tag{3.62}$$

We define now $\overline{T}$ as the average of $T^{1,\delta}$: $\overline{T} = \langle T^{1,\delta} \rangle$. From the above analysis it follows that, neglecting higher order terms, $\overline{T}$ satisfies the initial value problem (3.3). The proof of the homogenization theorem 3.1 is now complete.

3.5 Homogenization for Spatially Independent Mean Flow

From theorem (3.1) we know that for the model velocity field that we have been studying in this chapter and at the length and time scales determined by the the mean flow the behavior of the passive scalar field is described, to leading order with respect to the small parameter δ, by a transport equation with the velocity field determined by the mean flow alone. The diffusive correction, due to the fluctuations, is an $O(\delta)$ correction to transport due to the mean flow. This is to be expected since, in the framework that we have developed, the global Peclet number is an $O(\frac{1}{\delta})$ quantity and consequently the behavior of the passive scalar field at the length and time scales of the mean flow should be described by an advection dominated advection—diffusion equation.

The analysis presented so far was performed in Eulerian coordinates. However, the fact that to leading order the passive scalar is being transported by the mean flow suggests another approach to the problem, namely to introduce appropriate coordinates that will enable us to distinguish between the effect of the mean flow and the fluctuations in a transparent fashion. We shall accomplish this by introducing *mean Lagrangian coordinates*, [65, ch. 3]. These coordinates α are being defined through the mean flow map $\bar{\mathbf{X}}(\boldsymbol{\alpha}(\mathbf{x},t),t) = \mathbf{x}$ where $\bar{\mathbf{X}}(\boldsymbol{\alpha},t)$ solves the equation:

$$\frac{d\,\bar{\mathbf{X}}(\boldsymbol{\alpha},t)}{d\,t} = \mathbf{V}(\bar{\mathbf{X}}(\alpha,t),t) \tag{3.63a}$$

$$\bar{\mathbf{X}}(\boldsymbol{\alpha},t=0) = \boldsymbol{\alpha} \tag{3.63b}$$

By formally inverting the solution of this equation we obtain a reparametrization of space-time $(\mathbf{x},t)$ by the new set of variables $(\boldsymbol{\alpha},t)$. Using these coordinates we follow the motion of a fluid element evolving under the mean flow alone. In these new coordinates the $O(1)$ effect of transport due to the mean flow is automatically taken into account and we are only left with the effect of the fluctuations to the large scale evolution of the passive scalar field.

Let us now try to study the problem of homogenization in mean Lagrangian coordinates. Since the effect of the mean flow is "hidden" in the transformation to the new coordinate system, we want to study the problem at the time scale where the diffusion becomes an $O(1)$ phenomenon [10]. For this approach it is more convenient to nondimensionalize with respect to the characteristic length and time scales of the fluctuations. We have:

$$S_l \frac{\partial T(\mathbf{x},t)}{\partial t} + \left(\frac{1}{a}\mathbf{V}(\delta\,\mathbf{x},\eta\,t) + \mathbf{v}(\mathbf{x},t)\right)\cdot\nabla T(\mathbf{x},t) = \frac{1}{Pe_l}\Delta T(\mathbf{x},t) \tag{3.64a}$$

$$T(\mathbf{x},t=0) = T_{in}(\delta\mathbf{x}) \tag{3.64b}$$

We shall assume that the local nondimensional parameters are $O(1)$. Moreover, we shall only consider the distinguished limit $\eta = \delta^2$ [11]. We also consider the specific case where the mean flow is a function only of time, $\mathbf{V} = \mathbf{V}(t)$. At the end of this section we shall make some remarks concerning the general case.

The transformation to mean Lagrangian coordinates will be obtained by inverting the solution to the system of ODEs (3.63) (with the difference that now $\mathbf{V}$ depends on t through $\tau = \delta^2\,t$):

$$\begin{aligned}\bar{\mathbf{X}}(\alpha,t) &= \frac{1}{a}\int_0^t \mathbf{V}(\delta^2\,s)\,d\,s + \boldsymbol{\alpha} \\ &= \frac{1}{a\,\delta^2}\int_0^{\delta^2 t}\mathbf{V}(s)\,d\,s + \boldsymbol{\alpha}\end{aligned} \tag{3.65}$$

We now define $\mathbf{x} = \bar{\mathbf{X}}(\boldsymbol{\alpha},t)$ as well as:

$$\tilde{T}(\boldsymbol{\alpha},t) = T(\frac{1}{a\,\delta^2}\int_0^{\delta^2 t}\mathbf{V}(s)\,d\,s + \boldsymbol{\alpha},t) \tag{3.66}$$

The advection—diffusion equation for $\tilde{T}(\boldsymbol{\alpha},t)$ becomes:

$$S_l \frac{\partial \tilde{T}(\boldsymbol{\alpha},t)}{\partial t} + \mathbf{v}(\frac{1}{a\,\delta^2}\int_0^{\delta^2 t}\mathbf{V}(s)\,d\,s + \boldsymbol{\alpha},t)\nabla_{\boldsymbol{\alpha}}\tilde{T}(\boldsymbol{\alpha},t) = \frac{1}{Pe_l}\Delta_{\boldsymbol{\alpha}}\tilde{T}(\boldsymbol{\alpha},t) \tag{3.67a}$$

[10] In Eulerian coordinates at the diffusive time scale advection due to the mean flow becomes $O(\frac{1}{\delta})$ and this leads to a singularity in the multiple scale expansion. In the mean Lagrangian coordinates this singularity disappears and we can obtain a homogenization theorem.

[11] We expect to see only diffusive behavior in mean lagrangian coordinates and consequently the natural scaling is $t \sim x^2 \Rightarrow \eta \sim \delta^2$. On the contrary, in Eulerian coordinates the first order effect is transport due to the mean flow and the proper scaling is $t \sim x \Rightarrow \eta \sim \delta$

$$\tilde{T}(\boldsymbol{\alpha}, t=0) = T_0(\delta\boldsymbol{\alpha}), \tag{3.67b}$$

where the spatial derivatives in (3.67) are with respect to $\boldsymbol{\alpha}$. Upon rescaling, $\boldsymbol{\alpha} \to \delta\,\boldsymbol{\alpha}$, $t \to \delta^2\, t$, we get:

$$S_l\,\frac{\partial \tilde{T}^\delta(\boldsymbol{\alpha},t)}{\partial t} + \frac{1}{\delta}\mathbf{v}\Big(\frac{1}{a\,\delta^2}\int_0^t \mathbf{V}(s)\,d\,s + \frac{\boldsymbol{\alpha}}{\delta}, \frac{t}{\delta^2}\Big)\Big)\nabla_{\boldsymbol{\alpha}}\tilde{T}^\delta(\boldsymbol{\alpha},t) = \frac{1}{Pe_l}\Delta_{\boldsymbol{\alpha}}\tilde{T}^\delta(\boldsymbol{\alpha},t) \tag{3.68a}$$

$$\tilde{T}(\boldsymbol{\alpha}, t=0) = T_0(\boldsymbol{\alpha}) \tag{3.68b}$$

We introduce the variables $\mathbf{z} = \frac{1}{a\,\delta^2}\int_0^t \mathbf{V}(s)\,d\,s + \frac{\boldsymbol{\alpha}}{\delta}$, $\tau = \frac{t}{\delta^2}$. The differential operators transform as:

$$\frac{\partial}{\partial\,t} \to \frac{\partial}{\partial\,t} + \frac{1}{a\,\delta^2}\mathbf{V}(t)\cdot\nabla_{\mathbf{z}} + \frac{1}{\delta^2}\frac{\partial}{\partial\,\tau} \tag{3.69a}$$

$$\nabla_{\boldsymbol{\alpha}} \to \nabla_{\boldsymbol{\alpha}} + \frac{1}{\delta}\nabla_{\mathbf{z}} \tag{3.69b}$$

$$\Delta_{\boldsymbol{\alpha}} \to \Delta_{\boldsymbol{\alpha}} + \frac{2}{\delta}\nabla_{\mathbf{z}}\nabla_{\boldsymbol{\alpha}} + \frac{1}{\delta^2}\Delta_{\mathbf{z}} \tag{3.69c}$$

The multiple scales expansion has the form:

$$\tilde{T}^\delta(\boldsymbol{\alpha},t) \approx \tilde{T}_0(\boldsymbol{\alpha},t,\mathbf{z},\tau) + \delta\tilde{T}_1(\boldsymbol{\alpha},t,\mathbf{z},\tau) + \delta^2\tilde{T}_2(\boldsymbol{\alpha},t,\mathbf{z},\tau) + \ldots \tag{3.70}$$

Performing now the standard calculations, we arrive at the effective equation:

$$S_l\,\frac{\partial\overline{\tilde{T}}(\boldsymbol{\alpha},t)}{\partial t} = \frac{1}{Pe_l}\nabla_{\boldsymbol{\alpha}}(\mathcal{K}^*(t)\nabla_{\boldsymbol{\alpha}}\overline{\tilde{T}}(\boldsymbol{\alpha},t)) \tag{3.71}$$

The cell problem reads:

$$S_l\,\frac{\partial\boldsymbol{\chi}}{\partial\tau} + \left(\frac{1}{a}\mathbf{V}(t) + \mathbf{v}(\mathbf{z},t)\right)\cdot\nabla_{\mathbf{z}}\boldsymbol{\chi} - \frac{1}{Pe_l}\Delta_{\mathbf{z}}\boldsymbol{\chi} = -\mathbf{v}(\mathbf{z},\tau), \tag{3.72}$$

The effective diffusion tensor is defined to be:

$$\mathcal{K}^*_{ij} := \delta_{ij} - \frac{1}{Pe_l}\langle v_i\,\chi_j\rangle \tag{3.73}$$

[12] Let us rewrite the homogenized equation in the original Eulerian coordinates for small, finite δ. Upon performing the inverse transformation, from $\boldsymbol{\alpha}$ to $\mathbf{x}$, we obtain:

$$S_l\,\frac{\partial \overline{T}(\mathbf{x},t)}{\partial t} + \frac{1}{\delta}\mathbf{V}(t)\cdot\nabla\overline{T}(\mathbf{x},t) = \frac{1}{Pe_l}\cdot\nabla\left(\mathcal{K}^*(t)\nabla\cdot\overline{T}(\mathbf{x},t)\right) \tag{3.74}$$

The cell problem remains the same. Equation (3.74) is in accordance with our previous findings: the global Peclet number is $O(\frac{1}{\delta})$ and the cell problem is given by equation (3.72). The difference between equations (3.3) and (3.74) is that the later is valid at longer time scales, $O(\frac{1}{\delta})$ compared to those of the former.

Remark It would be very interesting if we could apply the technique described in this section to spatiotemporally dependent mean flows. However, the spatial dependence of the mean flow distorts the spatial partial differential operators and breaks the periodic dependence of the coefficients of equation (3.67) on the fast variables. To see this, let us consider the case of a linear mean flow $\mathbf{V} = \mathcal{A}\,\mathbf{x}$. In this case the mean flow map is:

$$\mathbf{x} = \bar{\mathbf{X}}(\boldsymbol{\alpha},t) = e^{\delta\,\mathcal{A}\,t}\boldsymbol{\alpha} \tag{3.75}$$

and the passive scalar field in mean Lagrangian coordinates is $\tilde{T}(\alpha,t) = (e^{\delta\,\mathcal{A}\,t}\boldsymbol{\alpha},t)$. Now the advection—diffusion equation reads:

$$S_l\,\frac{\partial \tilde{T}(\boldsymbol{\alpha},t)}{\partial t} + e^{-\delta\,\mathcal{A}\,t}\mathbf{v}(e^{\delta\,\mathcal{A}\,t}\boldsymbol{\alpha},t)\nabla_{\boldsymbol{\alpha}}\tilde{T}(\boldsymbol{\alpha},t) = \frac{1}{Pe_l}\nabla_{\boldsymbol{\alpha}}\cdot\left(e^{-\delta\,\mathcal{A}\,t}\cdot e^{-\delta\,\mathcal{A}^{\dagger}\,t}\cdot\nabla_{\boldsymbol{\alpha}}\tilde{T}(\boldsymbol{\alpha},t)\right) \tag{3.76a}$$

$$\tilde{T}(\boldsymbol{\alpha},t=0) = T_0(\delta\boldsymbol{\alpha}), \tag{3.76b}$$

[12]It would be more natural do write the effective equation as $\frac{\partial\overline{\tilde{T}}(\boldsymbol{\alpha},t)}{\partial t} = \nabla_{\boldsymbol{\alpha}}\cdot(\mathcal{K}^*(t)\cdot\nabla_{\boldsymbol{\alpha}}\overline{\tilde{T}}(\boldsymbol{\alpha},t))$ and define the effective diffusion tensor to be $\mathcal{K}^*_{ij} := S_l\left(\frac{1}{Pe_l}\delta_{ij} - \langle v_i\,\chi_j\rangle\right)$. We shall not do this in order to be consistent with the definition of the effective diffusion tensor and the effective equation that we have been using so far

where $\mathcal{A}^\dagger$ denotes the transpose of $\mathcal{A}$ The rescaling $\boldsymbol{\alpha} \to \delta\, \boldsymbol{\alpha},\ t \to \delta^2\, t$ leads to the equation:

$$S_l \frac{\partial \tilde{T}(\alpha,t)}{\partial t} + \frac{1}{\delta} e^{-\mathcal{A}\frac{t}{\delta}} \mathbf{v}(e^{\mathcal{A}\frac{t}{\delta}} \frac{\boldsymbol{\alpha}}{\delta}, \frac{t}{\delta^2}) \nabla_{\boldsymbol{\alpha}} \tilde{T}(\boldsymbol{\alpha},t) = \frac{1}{Pe_l} \nabla_{\boldsymbol{\alpha}} \cdot \left(e^{-\mathcal{A}\frac{t}{\delta}} \cdot e^{-\mathcal{A}^\dagger \frac{t}{\delta}} \cdot \nabla_{\boldsymbol{\alpha}} \tilde{T}(\boldsymbol{\alpha},t) \right) \tag{3.77a}$$

$$\tilde{T}(\boldsymbol{\alpha}, t=0) = T_0(\boldsymbol{\alpha}), \tag{3.77b}$$

In this case the spatial dependence of the mean flow has led to coefficients that are non periodic functions of the variable $\frac{t}{\delta}$. Hence, the method of periodic homogenization does not apply. It might be possible to prove that an effective diffusion equation does exist and to obtain some estimates on the effective diffusion tensor through more sophisticated techniques that are appropriate for non periodic homogenization problems such as H–convergence [28, ch. 13], [35]. However, these methods will not enable us to obtain a cell problem and we will not be able to compute the effective diffusion explicitly. We think though that the study of the mixing and spreading properties of passive tracers in mean Lagrangian coordinates is a very interesting problem and should be explored further.

3.6 The Case Where the Mean Flow is Stronger than the Fluctuations

3.6.1 Introduction

In our work so far we have been studying mean flows that are either weak or of equal strength compared to the fluctuations. This is natural since, as explained in section 1.2, this is the most interesting case from the perspective of physical oceanography. On the other hand, one could consider the problem of the motion of a particle in a velocity field which is fluctuating weakly about its mean. In subsection 1.4.4 we discussed about a similar problem with random fluctuations, as opposed to periodic. We saw there that the limiting behavior of the concentration of the passive tracer is diffusive with an $O(\delta^2)$ global Peclet number. The effective diffusion tensor is computed through the Kubo formula [69][pp. 294-296]. We would like to see wether one could obtain a similar result for the case of periodic fluctuations.

To simplify matters, let us consider steady velocity fields and assume that the fluctuations are $O(\delta)$-weak compared to the mean flow. Under these assumptions equation (2.21) becomes:

$$\frac{\partial T(\mathbf{x},t)}{\partial t} + \Big(\mathbf{V}(\mathbf{x}) + \delta\, \mathbf{v}(\frac{\mathbf{x}}{\delta})\Big) \cdot \nabla T(\mathbf{x},t) = \frac{\delta^2}{Pe_l} \Delta T(\mathbf{x},t) \;\text{ in } \mathbb{R}^d \times (0,\infty) \tag{3.78a}$$

$$T(\mathbf{x}, t=0) = T_{in}(\mathbf{x}) \;\text{ on } \mathbb{R}^d, \tag{3.78b}$$

Unfortunately the method of multiple scales that we have been using so far is no longer applicable. Indeed, by expanding the solution of (3.78) in a multiple scales expansion and equating the coefficients of the various powers of δ to zero we end up with equations of the form:

$$\mathbf{V}(\mathbf{x}) \cdot \nabla_{\mathbf{y}} T_0 = 0 \tag{3.79}$$

and

$$\mathbf{V}(\mathbf{x}) \cdot \nabla_{\mathbf{y}} T_1 = -\mathbf{v} \cdot \nabla_{\mathbf{x}} T_0 \tag{3.80}$$

From equation (3.79) we cannot conclude that the first term in the expansion is independent of the fast variables since the equation is not in general well-posed. We shall study this problem in chapter 5 using the method of two-scale convergence where we shall show that, for $\alpha \in (0,1)$, the first term in the expansion satisfies a transport equation. However, the second term, from which we would hope to obtain a cell problem, is coupled with another two terms in a system of three equations that involve both the fast and the slow scales. We will not be able, in general, to decouple the system and obtain an effective equation for the second term in the expansion, together with a cell problem. This situation is similar to the one we are faced with when studying homogenization for transport equations [36, 57].

In this section we shall try to understand the problem in two dimensions for the special case of a constant mean flow through a numerical and asymptotic analysis of the cell problem.

3.6.2 Asymptotic Analysis of the Cell Problem

It will be more convenient for our purposes to use the formalism developed in the previous paragraph. Our starting point is the cell problem (3.72):

$$S_l \frac{\partial \boldsymbol{\chi}}{\partial \tau} + \left(\frac{1}{a}\mathbf{V} + \mathbf{v}(\mathbf{y},t)\right) \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} - \frac{1}{Pe_l}\Delta_{\mathbf{y}} \boldsymbol{\chi} = -\mathbf{v}(\mathbf{y},\tau), \tag{3.81}$$

We shall perform some formal asymptotic analysis of (3.81) treating a as a small parameter. We reiterate that the following computations are only formal. They can be justified only a posteriori from the numerical examples and the two-scale convergence arguments. We shall also restrict ourselves to the study of the symmetric part of the effective diffusion tensor. Similar results hold for the antisymmetric part.

We assume that the mean flow is strong enough so that we can neglect the fluctuating part in the left hand side of (3.81):

$$S_l \frac{\partial \boldsymbol{\chi}}{\partial \tau} + \frac{1}{a}\mathbf{V} \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} - \frac{1}{Pe_l}\Delta_{\mathbf{y}} \boldsymbol{\chi} = -\mathbf{v}(\mathbf{y},\tau), \tag{3.82}$$

Under this assumption the cell problem can be solved exactly. We assume that the fluctuations have the following Fourier representation:

$$v_j = \sum_{l^2+m^2+n^2 \neq 0} \hat{v}^j_{n,m,l} e^{2\pi i (n y_1 + m y_2 + l\tau)}, \quad j = 1, 2 \tag{3.83}$$

The Fourier coefficients of the corrector field χ are:

$$\hat{\chi}^j_{n,m,l} = -\frac{\hat{v}^j_{n,m,l}}{2\pi i \left(S_l\, l + \frac{1}{a}V^1\, n + \frac{1}{a}V^2\, m\right) + \frac{4\pi^2}{Pe_l}(n^2+m^2)}, \quad j = 1, 2 \tag{3.84}$$

Straightforward computations give us the following expression for the diagonal components of the effective diffusion tensor (we shall only be concerned with the enhancement $\overline{\mathcal{K}}$):

$$\overline{\mathcal{K}}_{jj} = 2\, Pe_l^2 \sum_{l^2+m^2+n^2 \neq 0} \frac{|\hat{v}^j_{n,m,l}|^2 (n^2+m^2)}{Pe_l^2 \left(S_l\, l + \frac{1}{a}V^1\, n + \frac{1}{a}V^2\, m\right)^2 + 4\pi^2 (n^2+m^2)^2}, \quad j = 1, 2 \tag{3.85}$$

Now we want to study the limit of $\overline{\mathcal{K}}_{jj}$ as $a \to 0$. We shall also assume that the fluctuations that we consider have a finite number of nonzero Fourier modes: $\hat{v}^j_{n,m,l} \neq 0$ iff $\{n, m, l\} \in$

$\mathcal{J}^3$, $\mathcal{J}^3$ being a subset of $\mathbb{Z}^3$ with a finite number of elements. We shall also use the notation $\{n, m\} \in \mathcal{J}^2$ for the nonzero spatial Fourier modes. We make this assumption since we are interested in comparing the predictions of the formal asymptotics to numerical experiments.

From (3.85) we see that there are two possible limiting behaviors, depending on the properties of the mean flow:

1. $nV_1 + mV_2 \neq 0 \; \forall \; \{n^*, m^*\} \in \mathcal{J}^2$

   In this case every component of the sum in the right hand side of equation (3.85) tends to zero, as $a \to 0$, and consequently we have:

$$\lim_{a \to 0} \overline{\mathcal{K}}_{jj} = 0 \tag{3.86}$$

   We see that in this case there is no enhancement in the diffusivity and that, in fact, the fluctuations do not affect the large scale behavior of the passive scalar field. If we were to remove the assumption that the velocity field has only a finite number of nonzero Fourier modes this absence of resonance phenomena would occur when the ratio of the two components of the mean flow is an irrational number. In chapter 5 we shall obtain rigorously this result for $\alpha \in (0, 1)$.

2. $\exists \; \{n^*, m^*\} \in \mathcal{J}^2 \; : \; n^* V_1 + m^* V_2 = 0$

Now all components of the sum in (3.85) tend to 0 apart from the $\{n^*,\, m^*,\, l\}$ ones:

$$\overline{\mathcal{K}}_{jj}^{lim} = 2 \, Pe_l^2 \sum_l \frac{|\hat{v}^j_{n^*, m^*, l}|^2 ((n^*)^2 + (m^*)^2)}{S_l \, l^2 \, Pe_l^2 + 4\pi^2((n^*)^2 + (m^*)^2)^2}, \quad j = 1, 2 \tag{3.87}$$

The limiting effective diffusion tensor $\overline{\mathcal{K}}_{jj}^{lim}$ depends on both nondimensional numbers S_l, Pe_l. From equation (3.87) we can immediately compute the asymptotic behavior of $\overline{\mathcal{K}}_{jj}^{lim}$ with respect to these two parameters:

$$\lim_{S_l \to \infty} \overline{\mathcal{K}}_{jj}^{lim} = 0, \quad \text{for fixed } Pe_l \tag{3.88a}$$

$$\lim_{S_l \to \infty} \overline{\mathcal{K}}_{jj}^{lim} = \frac{Pe_l^2}{2\pi^2 (n^*)^2 + (m^*)^2} \sum_l |\hat{v}^j_{n^*, m^*, l}|^2, \quad \text{for fixed } Pe_l \tag{3.88b}$$

$$\lim_{Pe_l \to \infty} \overline{\mathcal{K}}_{jj}^{lim} = 2\,\frac{(n^*)^2 + (m^*)^2}{S_l^2} \sum_l \frac{|\hat{v}^j_{n^*,m^*,l}|^2}{l^2}, \quad \text{for fixed } S_l \tag{3.88c}$$

$$\lim_{Pe_l \to 0} \overline{\mathcal{K}}_{jj}^{lim} = 0, \quad \text{for fixed } S_l \tag{3.88d}$$

One can think of this result as an example of *resonant enhanced diffusion*: the mean flow interacts with the fluctuations in such a way that the diffusion is boosted far above its molecular value. In this way, the effective diffusion tensor reaches a constant, nonzero value which is independent of the mean flow.

From equation (3.85) we see that a different type of resonant enhanced diffusion is also possible, for finite a: for values of $\{n^*,\, m^*,\, l^*\} \in \mathcal{J}$ such that

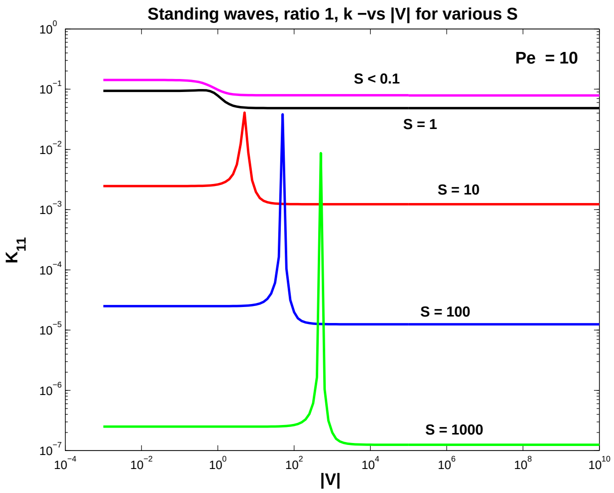
$$S_l\, l^* + \frac{1}{a} V^1\, n^* + \frac{1}{a} V^2\, m^* = 0 \tag{3.89}$$

the $\{n^*,\, m^*,\, l^*\}$ term in the right hand side of (3.85) will attain its maximum value and the diffusion will be greatly enhanced. We remark that this is a different type of resonance than the one that occurs at the limit $a \to 0$ and is related to synchronization between the mean flow and the temporal oscillations of the fluctuations. This type of resonance was discussed in [69, p. 268] and in [24]. It is possible that this type of resonances are related to the so called accelerator modes that were used by Mezic et al. in [80] in their study of maximal enhanced diffusion for time dependent flows. However we are not aware of any work that justifies the connection between these two concepts.

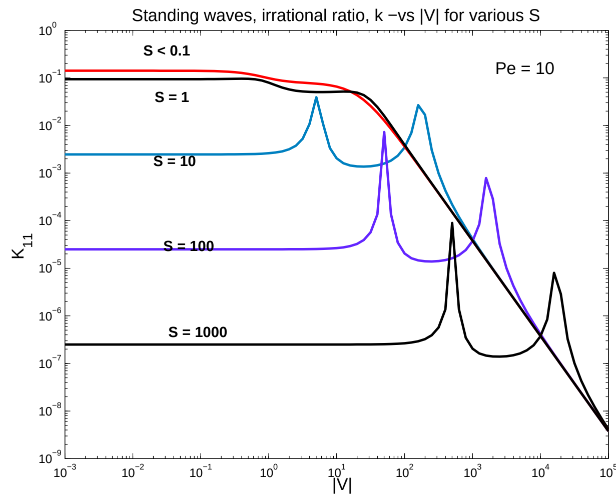
3.6.3 Numerical Experiments

In this subsection we shall solve the cell problem numerically for two types of fluctuating velocity fields and we shall compute the enhancement in the diffusivity as a function of the relative strength of the mean flow. Moreover, we shall analyze the second type of resonance phenomenon by solving the cell problem for various values of the local Strouhal number.

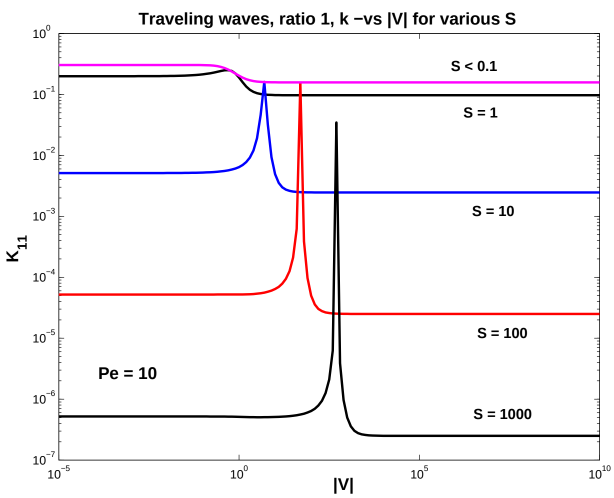
We choose to work with two types of two-dimensional time dependent fluctuations that were used by Knobloch and Merryfield in [60]: oscillatory flows in the form of a standing



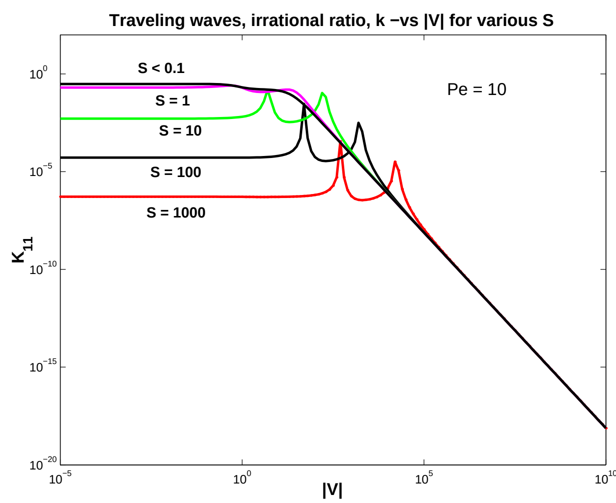
a. Standing waves, rational ratio



b. Standing waves, irrational ratio



c. Traveling waves, rational ratio



d. Traveling waves, irrational ratio

Figure 3.2: $\overline{\mathcal{K}}$ as a function of the relative strength of the mean flow for standing and traveling waves for $Pe_l = 10$ and various Strouhal numbers.

wave:

$$\begin{aligned} \mathbf{v}(y_1, y_2, t) &= (v_1, v_2) \\ &= (\sin(2\pi y_1)\cos(2\pi y_2)\cos(2\pi t), -\cos(2\pi y_1)\sin(2\pi y_2)\cos(2\pi t)) \end{aligned} \tag{3.90}$$

and a traveling wave

$$\begin{aligned}\mathbf{v}(y_1, y_2, t) &= (v_1, v_2) \\ &= (\sin(2\pi(y_1 - t))\cos(2\pi y_2),\ -\cos(2\pi(y_1 - t))\sin(2\pi y_2))\end{aligned} \tag{3.91}$$

[this is the nondimensional form of the fluctuating part of the velocity field].

We start with the standing waves. For strong mean flows the enhancement in the diffusivity in the $\hat{e}_1$ direction can be approximated by the formula:

$$\begin{aligned}\overline{\mathcal{K}}_{11} = \frac{Pe_l^2}{8} \quad (&\frac{1}{16\pi^2 + Pe_l^2(S_l + \frac{1}{a}V^1 + \frac{1}{a}V^2)^2} + \frac{1}{16\pi^2 + Pe_l^2(S_l + \frac{1}{a}V^1 - \frac{1}{a}V^2)^2} \\ + &\frac{1}{16\pi^2 + Pe_l^2(S_l - \frac{1}{a}V^1 + \frac{1}{a}V^2)^2} + \frac{1}{16\pi^2 + Pe_l^2(S_l - \frac{1}{a}V^1 - \frac{1}{a}V^2)^2} \ )\end{aligned} \tag{3.92}$$

From equation (3.92) we see that there are only two possible choices that lead to resonant enhanced diffusion: either $V_1 = V_2$ or $V_1 = -V_2$. In either case the limiting enhancement as a tends to zero is:

$$\overline{\mathcal{K}}_{11}^{lim} = \frac{Pe_l^2}{4}\frac{1}{16\pi^2 + Pe_l^2 S_l^2} \tag{3.93}$$

Similar computations can be performed for the traveling waves. We have:

$$\overline{\mathcal{K}}_{11} = \frac{Pe_l^2}{2}\left(\frac{1}{16\pi^2 + Pe_l^2(S_l - \frac{1}{a}V^1 - \frac{1}{a}V^2)^2} + \frac{1}{16\pi^2 + Pe_l^2(S_l - \frac{1}{a}V^1 + \frac{1}{a}V^2)^2}\right) \tag{3.94}$$

As for the standing waves, there are only two choices of mean flows that lead to resonant enhanced diffusion: either $V_1 = V_2$ or $V_1 = -V_2$. The limiting enhancement is:

$$\overline{\mathcal{K}}_{11}^{lim} = \frac{Pe_l^2}{2}\frac{1}{16\pi^2 + Pe_l^2 S_l^2} \tag{3.95}$$

Now we compare between the predictions of the formal asymptotics and the numerical solution of the cell problem. In figure 3.2 we plot $\overline{\mathcal{K}}_{11}$ as a function of the magnitude of the mean flow $|\mathbf{V}|$ (that is, as a function of $\frac{1}{a}$) for fixed $Pe_l = 10$ and for various choices of the

Strouhal number. We make two choices for V^1 and V^2. In figures 3.2.a and 3.2.c we choose $V^1 = V^2 = 1$, a choice which leads to resonant enhanced diffusion. In figures 3.2.b and 3.2.d we choose $V^1 = 1$, $V^2 = \frac{3\pi}{10}$ which, according to the foregoing computations, leads to a zero enhancement in the diffusivity as $|\mathbf{V}| \to 0$.

From figure 3.2 we see that $\overline{\mathcal{K}}_{11}$ depends upon $|\mathbf{V}|$ according to our theory: for $|\mathbf{V}| << 1$ the enhancement in the diffusivity is independent of the mean flow. This corresponds to the regime in which the mean flow is weaker than the fluctuations in which we already know that the mean flow does not enter into the cell problem. After some transient interval for which $|\mathbf{V}| \sim O(1)$ and both mean flow and fluctuations contribute to the computation of the effective diffusion tensor (in the sense that we have to retain both components of the velocity field on the left hand side of the cell problem), we reach the asymptotic behavior which is described by the results of this section. For values $V^1 = V^2 = 1$ the effective diffusion tensor reaches a constant value which is independent of the mean flow and is given by formulas (3.93) and (3.95) for standing and traveling waves, respectively. On the contrary, for the choice $V^1 = 1$, $V^2 = \frac{3\pi}{10}$ $\overline{\mathcal{K}}_{11}$ decreases as $|\mathbf{V}|$, in accordance to the predictions of the asymptotic theory.

We also observe the presence of resonant phenomena of the second type. These resonances are very sharp for the choice $V^1 = V^2 = 1$. They are less sharp for $V^1 = 1$, $V^2 = \frac{3\pi}{10}$, however in this case we observe that secondary resonances of this type are also present.

3.6.4 Discussion

In this section we used formal asymptotics and numerics in order to study the problem of homogenization of advection–diffusion equations with mean flow in the case where the mean flow is stronger than the fluctuations. We saw that in this regime the structure of the effective equation depends very sensitively on the properties of the mean flow. We emphasize that the arguments of this section are only formal in nature and have to be justified. This will be accomplished in chapter 5 using the method of two-scale convergence.

As we have already mentioned, the sensitive dependence of the effective diffusion tensor on the specific properties of the mean flow is due to the fact that the fluctuations are taken to be periodic. This sensitive dependence vanishes in the random setting. Indeed, according to the results of Kesten and Papanicolaou [59] velocity fields of the form $\mathbf{u} = \mathbf{V} + \delta\, \mathbf{v}(\mathbf{x})$ where $\mathbf{V}$ is constant and $\mathbf{v}$ is a random velocity field which is sufficiently mixing and has short range correlations lead, at long times and large scales, to an enhancement in the diffusivity which is $O(\delta^2)$ and which depends on the mean flow. In the periodic setting resonant enhanced diffusion leads to an $O(\delta^2)$ enhacement which is, however, independent of the mean flow. Moreover, for velocity fields with a finite number of nonzero Fourier modes, the generic situation is that of non resonance, since only a finite number of choices for V^1, $V^2 \in \mathbb{R}$ would lead to a non zero enhacement in the diffusivity.

3.7 Numerical Method

As we have already seen we can obtain exact solutions of the cell problem only for simple flow geometries of the fluctuations, in particular for shear flows. For more complicated flow geometries we had to solve the cell problem numerically. In this section we analyze the numerical method that we are using. To ease the notation we shall describe the method for two–dimensional time dependent flows.

We solve the cell problem using a Fourier spectral method [16, 17, 70]. The idea is that since the cell problem is either an elliptic or parabolic equation with periodic coefficients and periodic boundary conditions, we can expand its mean zero solution in a Fourier series:

$$\chi_j = \sum_{l^2+m^2+n^2\neq 0} \hat{\chi}^j_{n,m,l}(x_1, x_2, t) e^{2\pi i(ny_1+my_2+l\tau)} \tag{3.96}$$

We note that, since we are looking for real valued solutions of the cell problem (3.6), we have to require that $\left(\hat{\chi}^j_{n,m,l}(x_1, x_2, t)\right)^* = \hat{\chi}^j_{-n,-m,-l}(x_1, x_2, t)$ where x^* stands for the complex conjugate of x. Let us also remark that the macroscopic variables x_1, x_2, t enter into the problem through the spatiotemporally dependent mean flow as parameters. Assuming that the mean flow is smooth and that the fluctuations depend smoothly on x_1, x_2, t (which is the case for all of the examples we considered in this chapter), the Fourier coefficients in (3.96) depend continuously on the macroscopic variables.

Since the fluctuations in the velocity field are periodic, we can also expand them in a Fourier series whose coefficients are functions of the macroscopic variables. Substituting these expansions into the parabolic cell problem and using the orthogonality of the basis functions of the Fourier expansion we finally arrive at an infinite, nonhermitian in general, linear system of equations:

$$\mathcal{A}(x_1, x_2, t\,;\, S_l, Pe_l)\,\mathbf{X} = \mathbf{b}(x_1, x_2, t) \tag{3.97}$$

$\mathcal{A}$ is the coefficient matrix and is a function of the mean flow, the fluctuations and the nondimensional quantities Pe_l and S_l. $\mathbf{X}$ is the vector containing the Fourier coefficients of the auxiliary function χ_j and $\mathbf{b}$ corresponds to the Fourier coefficients of the fluctuations. We now have to address two issues. First, how to obtain solutions of (3.97) that are functions of the macroscopic variables and second how to truncate (3.96) and obtain a finite system of equations.

In order to solve the first problem, we discretize the domain in which we want to calculate the effective diffusivities as well as the time interval that we are interested in. To this end, we introduce a mesh $x_1^n = n\,h,\ x_2^m = m\,h,\ t^l = l\,k$ where h and k are the step sizes for the spatial and temporal discretizations, respectively. Thus, we have to solve a sequence of cell problems, one at each point in space and time.

Now we discuss the second issue. We truncate (3.96) and consequently (3.97) as follows: We start with only 5 Fourier modes. We solve the linear system of equations (details about the solution procedure follow) and calculate the enhancement in the diffusivity $\overline{\mathcal{K}}_{ij}^5$:

$$\overline{\mathcal{K}}_{ij}^5(x_l^n, x_2^m, t^l) = -Pe_l\,\langle v_i(y_1, y_2, \tau\,;\, x_l^n, x_2^m, t^l)\,\chi_j^5(y_1, y_2, \tau\,;\, x_l^n, x_2^m, t^l)\rangle \tag{3.98}$$

We solve the problem again with 6 modes and compute $\overline{\mathcal{K}}_{ij}^6$. We calculate the relative difference of these two solutions for the first, say component of $\overline{\mathcal{K}}$:

$$err = |\frac{\overline{\mathcal{K}}_{11}^5 - \overline{\mathcal{K}}_{ij}^6}{\overline{\mathcal{K}}_{ij}^5}| \tag{3.99}$$

where the superscipts refer to the number of Fourier modes that are being used. If the error is smaller than a desired tolerance TOL we stop. If not, we continue the same procedure until the relative error is smaller than TOL:

$$|\frac{\overline{\mathcal{K}}_{ij}^n - \overline{\mathcal{K}}_{ij}^{n+1}}{\overline{\mathcal{K}}_{ij}^n}| < TOL \tag{3.100}$$

The value at the n+1 iteration is the enhacement in the diffusivity: $\overline{\mathcal{K}}_{ij}^5 := \overline{\mathcal{K}}_{ij}^5$.

Since we will be dealing with fluctuating velocity fields with a few Fourier modes, the resulting linear system will be sparse. Consequently, we can solve it efficiently by an iterative method which is valid for complex, nonhermitian problems. We have chosen to use the Bi-conjugate Gradient Stabilized method [113] with incomplete LU factorization for preconditioning. Let us mention that since for the problems that we have studied the coefficient matrix is well conditioned, any Krylov subspace iterative method for complex nonhermitian problems will be sufficient. The efficiency of the numerical solution of the linear system was virtually the same for all the methods that we tried. For the iterative solver as well as the preconditioning routines we used the nag library routines.

A few remarks are in order. First, the problem, as it has been stated so far, not only is it not well conditioned but it is actually singular, the matrix $\mathcal{A}$ having a zero eigenvalue. This is due to the fact that we have excluded the coefficient $\chi_{0,0,0}^j$ from (3.96), since we are looking for solutions with zero average. However we can easily alleviate this difficulty by including $\chi_{0,0,0}^j$ in the expansion and setting the coefficient in the matrix $\mathcal{A}$ that multiplies $\chi_{0,0,0}^j$ equal to 1. Since the velocity field has also zero average, we will arrive at the trivial equation $1 \times \chi_{0,0,0}^j = 0$. In this way we both ensure the well posedness of the system of equations as well that the auxiliary functions will have zero average.

Moreover, preconditioning speeded up the computations enormously. In most cases the approximate solution was obtained after one or two iterations. Without preconditioning, convergence was achieved after hundreds of even thousands of iterations. On the other hand, there is some computer time spent for the construction of the preconditioner but it is small compared to the computer time spent when a great number of iterations was required. So, preconditioning is important.

The numerical solution of the cell problem depends on the nondimensional parameters of the problem as well as the relative strength of the fluctuations. As the Peclet number increases the problem becomes more advection dominated and hence more difficult to solve. On the other hand, as the mean flow becomes stronger compared to the fluctuations the resulting linear system of equations becomes diagonally dominant and consequently easier to solve.

In figure 3.3 we present the procedure for solving the cell problem in an algorithmic fashion.

Algorithm for solving the cell problem

Construct the system $\mathcal{A}(x_1, x_2, t\,;\, S_l, Pe_l)\,\mathbf{X} = \mathbf{b}(x_1, x_2, t)$ through Fourier series expansions.

Construct a mesh in $[-X_{max}, X_{max}] \times [-X_{max}, X_{max}] \times [0, T_{max}]$:

$$x_1^n = n\,h,\; x_2^m = m\,h,\; t^l = l\,k$$

$$N\,h = X_{max}, \quad L\,k = T_{max}$$

for $n = -N : N$ **do**
for $m = -N : N$ **do**
for $l = 0 : L$ **do**
for $i = 5 : FMAX$ **do** *
begin

1. Construct the system $\mathcal{A}_{ij}(x_1^n, x_2^m, t^l\,;\, S_l, Pe_l)\,X_j = b_i(x_1, x_2, t)$
2. Insert a 0 in the entry of $\mathcal{A}_{ij}$ corresponding to $\hat{\chi}^j_{000}$
3. Calculate the incomplete LU factorization of $\mathcal{A}$.
4. Solve the linear system using BCGS.
5. Compute $\overline{\mathcal{K}}_{ij}$ using (3.98).
6. Calculate the error using

$$err = \left|\frac{\overline{\mathcal{K}}^i_{11} - \overline{\mathcal{K}}^{i+1}_{11}}{\overline{\mathcal{K}}^i_{11}}\right|$$

7. **if** $err < tol$ then continue **else** go to *.

end

Figure 3.3: Algorithm for solving the cell problem

CHAPTER 4
STUDY OF THE ANTISYMMETRIC PART OF THE EFFECTIVE DIFFUSION TENSOR

4.1 Introduction

As we have already discussed, the effective diffusion tensor is not in general symmetric. Moreover, when $\mathcal{K}^*$ is a function of space, which will be the case when a nontrivial mean flow is present, the antisymmetric part of $\mathcal{K}^*$ becomes important since it results in an additional incompressible effective drift. We discussed in chapter 1 about the relevance of this effective drift to the problem of parametrization of passive tracers in the atmosphere and ocean. Consequently it is important to study the properties of the antisymmetric part of $\mathcal{K}^*$ in detail.

In this chapter we shall derive some necessary and sufficient conditions that ensure the symmetry of the effective diffusion tensor for steady velocity fields. We shall also study the dependence of the antisymmetric part of $\mathcal{K}^*$ upon the Peclet number. Finally, we shall present numerical examples for two types of velocity fields, one steady and one time dependent.

Before presenting our results, let us first review previous work on this problem. The symmetry properties of the effective diffusion tensor that results from the method of homogenized averaging have been discussed, to my knowledge, only in two papers: Koch and Brady in [62] and Fannjiang and Papanicolaou in [41]. In both papers the symmetry of the effective diffusion tensor was related to some sort of "parity invariance" of the velocity field. Koch and Brady argued that an antisymmetric part of the effective diffusion tensor is present only when the velocity field lacks a center of reflectional symmetry. More detailed sufficient conditions were given by Fannjiang and Papanicolaou for steady two–dimensional velocity fields. In particular, they derived the following sufficient conditions for the symmetry of $\mathcal{K}^*$ ($\psi(\mathbf{y})$ denotes the stream matrix associated with the fluctuations; we also denote by $\mathbb{T}^2$ the unit two–dimensional torus):

1. Translational antisymmetry of ψ: $\exists\, \mathbf{r} : \psi(\mathbf{y}+\mathbf{r}) = -\psi(\mathbf{y})\ \forall\, \mathbf{y} \in \mathbb{T}^2$
2. Reflectional antisymmetry of ψ with respect to an axis, for example the y_2 axis: $\psi(y_1, -y_2) = -\psi(\mathbf{y})\ \forall\, \mathbf{y} \in \mathbb{T}^2$

3. 180°-Rotational antisymmetry of ψ with respect to a point, for example the origin: $\psi(-y_1, -y_2) = -\psi(\mathbf{y}) \; \forall \, \mathbf{y} \in \mathbb{T}^2$

The dependence of the antisymmetric part of the diffusion tensor with respect to the Peclet number was discussed by Koch and Brady. They argued, based on numerical examples and physical reasoning, that the antisymmetric part should scale like Pe^3 for small Peclet numbers and like Pe for large Peclet numbers.

We shall study the symmetry properties of the effective diffusion tensor by obtaining an explicit expression for the solution of the steady cell problem as a Fourier expansion in the orthonormal basis defined by the eigenfunctions of an appropriate compact, skew-symmetric operator $\mathcal{A}^v$. Using this formula we shall be able to relate the symmetry properties of the effective diffusion tensor to the properties of the Fourier coefficients of the fluctuating part of the velocity field with respect to the eigenfunctions of $\mathcal{A}^v$. From this expression we shall also be able to study the dependence of the antisymmetric part of $\mathcal{K}^*$ on Pe.

4.2 Spectral Representation of the Antisymmetric Part of the Effective Diffusion Tensor

First we start with some general remarks. The cell problem for time dependent fluctuations is:

$$S_g \frac{\partial \chi_l}{\partial \tau} + (\mathbf{V} + \mathbf{v}) \cdot \nabla \chi_l - \frac{1}{Pe_l} \Delta \chi_l = -v_l, \quad l = 1 \ldots d \tag{4.1}$$

The effective diffusion tensor is:

$$\mathcal{K}^*_{lk} = \delta_{lk} - Pe_l \, \langle v_l \, \chi_k \rangle \tag{4.2}$$

We multiply (4.1) by χ_k and integrate over the period cell. We perform the same calculation with the indices k and l interchanged. After an integration by parts we find that the antisymmetric part of the effective diffusivity is:

$$\begin{aligned} \mathcal{A}_{lk} &:= \frac{1}{2} \left(\overline{\mathcal{K}}_{lk} - \overline{\mathcal{K}}_{kl} \right) \\ &= Pe_l \, \langle \chi_k \left(S_l \frac{\partial \chi_l}{\partial \tau} + (\mathbf{V} + \mathbf{v}) \cdot \nabla \chi_l \right) \rangle \end{aligned} \tag{4.3}$$

Thus, the necessary and sufficient condition for the effective diffusion tensor to be symmetric is:

$$\langle \chi_k \left(S_l \frac{\partial \chi_l}{\partial \tau} + (\mathbf{V} + \mathbf{v}) \cdot \nabla \chi_l \right) \rangle = 0 \tag{4.4}$$

For the case of steady velocity fields equations (4.3) and (4.4) become:

$$\mathcal{A}_{lk} = Pe_l \, \langle \chi_k (\mathbf{V} + \mathbf{v}) \cdot \nabla \chi_l) \rangle \tag{4.5}$$

and

$$\langle \chi_k (\mathbf{V} + \mathbf{v}) \cdot \nabla \chi_l) \rangle = 0, \tag{4.6}$$

respectively. As a simple example, consider that of a shear flow in two dimensions:

$$\mathbf{v}(y_1, y_2, t) = (0, v_2(y_1, t)), \tag{4.7}$$

with $\mathbf{V} = 0$. In this case we have $\chi_1(y_1, y_2, t) = 0$ and we immediately conclude from (4.3) that $\mathcal{A}_{12} = 0$ (in fact, for shear flows the off-diagonal elements of the effective diffusion tensor are equal to 0).

Now we restrict ourselves to steady velocity fields. We wish to express $\mathcal{A}_{lk}$ in terms of the Fourier coefficients of the velocity field in an appropriate basis. In the next section we shall show that $\mathcal{A}_{lk}$ is given by the following formula:

$$\mathcal{A}_{lk} = Pe_l^3 \sum_{n=1}^{n=\infty} \frac{\mu_n Im(\beta_{ln}^* \beta_{kn})}{1 + Pe_l^2 \mu_n^2} \tag{4.8}$$

where $\{\beta_{ln}\}_{n=1}^{\infty}$ are the Fourier coefficients of $\Delta^{-1} v_l$ in the basis defined by the eigenfunctions of a compact, skew-symmetric operator $\mathcal{A}^v$ and $\{\mu_n\}_{n=1}^{\infty}$ are related to the eigenvalues of $\mathcal{A}^v$ and have the property that $\lim_{n \to \infty} \mu_n = 0$. Precise definitions, together with the proof of (4.8) will be given in the next section.

We can now use the above formula to draw several conclusions. First, we see that a necessary and sufficient condition for the effective diffusivity to be symmetric can be derived:

$$\mathcal{A}_{lk} = 0 \Leftrightarrow \sum_{n=1}^{n=\infty} \frac{\mu_n Im(\beta_{ln}^* \beta_{kn})}{1 + Pe_l^2 \mu_n^2} = 0 \tag{4.9}$$

Of course, (4.9) is not very useful for determining whether a given velocity field gives rise to a symmetric effective diffusion tensor since it requires knowledge of the complete spectrum of $\mathcal{A}^v$. However, it might be possible to prove the validity of (4.9) by taking advantage of the specific properties of the velocity fluctuations. For example, it is easy to show that for velocity fields with the property $\mathbf{v}(-\mathbf{y}) = -\mathbf{v}(\mathbf{y})$ equation (4.9) is indeed satisfied. The relationship between the antisymmetry properties of the velocity fluctuations and condition (4.9) is yet to be explored.

Moreover, from (4.8) we can easily derive the asymptotic behavior of σ_{kl} for large and small values of the Peclet number:

$$\mathcal{A}_{lk} \sim Pe_l^3 \;\text{ for }\; Pe_l \to 0 \tag{4.10}$$

and

$$\mathcal{A}_{lk} \sim Pe_l \;\text{ for }\; Pe_l \to \infty \tag{4.11}$$

The asymptotic behavior of $\mathcal{A}_{lk}$ for small Peclet numbers can also be derived through regular perturbation theory of the cell problem. In order to study large Peclet number asymptotics we need a more refined method such as the one we will present.

What is particularly interesting is that the asymptotic behavior of $\mathcal{A}_{lk}$ for large Pe_l is independent of the presence and properties of the mean flow. This result is in contrast with the asymptotic behavior of the symmetric part which depends strongly upon the mean flow [70]. As will become clear in the next section, the reason for this difference is that the expression for $\mathcal{A}_{lk}$ is independent of the orthogonal projection of $\Delta^{-1}\,\mathbf{v}$ onto the null space of the compact, skew–symmetric operator $\mathcal{A}^v$ that we shall introduce in the next section. On the other hand, the asymptotic behavior of the symmetric part of the effective diffusion tensor depends exactly on this orthogonal projection which, in turn, depends sensitively on the properties of the mean flow.

We also mention that formula (4.8) can be generalized to time-dependent velocity fields.

In this case the operator $\mathcal{A}^v := \Delta^{-1}(\mathbf{V}+\mathbf{v})\nabla$ that we shall use in the next section has to be replaced by $\mathcal{A}_t^v := S_g\,\Delta^{-1}\frac{\partial}{\partial\tau} + \Delta^{-1}(\mathbf{V}+\mathbf{v})\nabla$. A formula similar to (4.8) can be obtained, with the Fourier coefficients being functions of the Strouhal number. Consequently, we can reach the same conclusions as before, valid for *fixed* Strouhal number. On the other hand, this formula will not provide us with information concerning the dependence of $\mathcal{A}_{lk}$ on the Strouhal number. We shall study this problem through numerical simulations for a time dependent velocity field later in this chapter.

4.3 Derivation of Formula (4.8)

The techniques that we shall use in this section are based upon earlier work of Bhattacharya and coworkers, for example [13, 15]. A slightly different version of the method was introduced by Avellanada and Majda in [6, 7]. Similar methods have also been used in the context of the theory of composite materials [48]. For background on the spectral theory of compact, skew–symmetric operators we refer to the books [94, 95].

We consider the cell problem for steady velocity fields:

$$-Pe_l(\mathbf{V}+\mathbf{v})\cdot\nabla\chi_l + \Delta\chi_l = Pe_l\, v_l, \quad l = 1\dots d \tag{4.12}$$

We want to rewrite (4.12) as an integral equation:. To this end, we apply the operator Δ^{-1} to both sides of the equation to obtain:

$$(\mathcal{I} - Pe_l\,\mathcal{A}^v)\chi_l = Pe_l\,\Delta^{-1}v_l, \tag{4.13}$$

where $\mathcal{I}$ is the identity operator and

$$\mathcal{A}^v := \Delta^{-1}(\mathbf{V}+\mathbf{v})\nabla \tag{4.14}$$

We introduce now the complex Hilbert space $\mathcal{H}^1$ which consists of all functions in the Sobolev space $H^1(\mathbb{T}^d)$ with mean zero:

$$\mathcal{H}^1 := \{f \in H^1(\mathbb{T}^d)\ ;\ \langle f\rangle = 0\} \tag{4.15}$$

We equip $\mathcal{H}^1$ with the norm

$$||f||_1 = \sqrt{\langle |\nabla f|^2 \rangle} \tag{4.16}$$

and the inner product

$$(f,g)_1 = \langle \nabla f \cdot \nabla g^* \rangle \tag{4.17}$$

where g^* denotes the complex conjugate of g. One can show that the operator $\mathcal{A}^v$ is compact and skew–symmetric in $\mathcal{H}^1$ [13, pp. 966-968].

Using now $\mathcal{A}^v$ the enhancement in the diffusivity can be expressed as:

$$\begin{aligned}
\overline{\mathcal{K}}_{kl} &= -Pe_l \int_{\mathbb{T}^d} v_k\, \chi_l\, dy \\
&= -Pe_l \int_{\mathbb{T}^d} \Delta\Delta^{-1} v_k\, \chi_l\, dy \\
&= Pe_l \int_{\mathbb{T}^d} \nabla\Delta^{-1} v_k \cdot \nabla\chi_l\, dy \\
&= Pe_l\, (\Delta^{-1} v_k, \chi_l)_1 \\
&= (\chi_k, \chi_l)_1 - Pe_l\, (\mathcal{A}^v \chi_k, \chi_l)_1
\end{aligned} \tag{4.18}$$

In the above derivation we have used the fact that χ_k, $k = 1 \dots d$ are real. At the last step the integral formulation of the cell problem was used. For the antisymmetric part of the effective diffusivity we have:

$$\begin{aligned}
\mathcal{A}_{lk} &= \frac{1}{2}\left(\overline{\mathcal{K}}_{lk} - \overline{\mathcal{K}}_{kl}\right) \\
&= \frac{1}{2} Pe_l\, \left((\mathcal{A}^v \chi_l, \chi_k)_1 - (\mathcal{A}^v \chi_k, \chi_l)_1\right) \\
&= \frac{1}{2} Pe_l\, \left((\mathcal{A}^v \chi_l, \chi_k)_1 + (\chi_k, \mathcal{A}^v \chi_l)_1\right) \\
&= Pe_l\, (\mathcal{A}^v \chi_l, \chi_k)_1,
\end{aligned} \tag{4.19}$$

where the skew–symmetry of $\mathcal{A}^v$ and the reality of χ_k, $k = 1 \dots d$ were used. The necessary and sufficient condition for the effective diffusivity to be symmetric (4.6) can now be expressed as:

$$(\mathcal{A}^v \chi_l, \chi_k)_1 = 0 \tag{4.20}$$

We shall use the spectral decomposition of $\mathcal{A}^v$ in order to study $\mathcal{A}_{lk}$. Since $\mathcal{A}^v$ is compact and skew–symmetric, we know that it can be written in the form $\mathcal{A}^v = i\,\mathcal{G}$, $\mathcal{G}$ being compact and self-adjoint, [94, p. 200]. From the spectral theorem for compact, self-adjoint operators [94, p. 203] we conclude that $\mathcal{A}^v$ has eigenfunctions $\{\phi_n\}$ with purely imaginary eigenvalues that come in conjugate pairs and can be indexed as $\{\pm i\mu_n\}_{n=1}^{n=\infty}$ with the property $\lim_{n\to\infty}\mu_n = 0$. Moreover, $\phi_{-n} = \phi_n^*$.

The Hilbert space $\mathcal{H}^1$ admits the decomposition

$$\mathcal{H}^1 = \mathcal{N} \oplus \mathcal{N}^\perp \tag{4.21}$$

where $\mathcal{N}$ is the null space of $\mathcal{A}^v$. $\mathcal{N}^\perp$ is spanned by the eigenfunctions $\{\phi_n, \overline{\phi_n}\}_{n=1}^{n=\infty}$. Let us now compute the expansion of χ_l with respect to this basis.

To begin, let $\Delta^{-1}v_l$ have the following representation:

$$\begin{aligned}
\Delta^{-1}v_l &= (\Delta^{-1}v_l)_N + \sum_{n\neq 0}(\Delta^{-1}v_l, \phi_n)_1\,\phi_n \\
&:= (\Delta^{-1}v_l)_N + \sum_{n\neq 0}^{n=+\infty} \beta_{ln}\,\phi_n,
\end{aligned} \tag{4.22}$$

where $(\Delta^{-1}v_l)_N$ is the projection of $\Delta^{-1}v_l$ onto the null space of $\mathcal{A}^v$ and $\beta_{ln} := (\Delta^{-1}v_l, \phi_n)_1$. Similarly, for the corrector field χ_l we have:

$$\chi_l = (\chi_l)_N + \sum_{n\neq 0} \hat{\chi}_{ln}\,\phi_n \tag{4.23}$$

From the cell problem we have:

$$
\begin{aligned}
(\mathcal{I} - Pe_l\,\mathcal{A}^v)\chi_l &= (\mathcal{I} - Pe_l\,\mathcal{A}^v)\left((\chi_l)_N + \sum_{n\neq 0}\hat{\chi}_{ln}\,\phi_n\right) \\
&= (\chi_l)_N + \sum_{n\neq 0}(1 - Pe_l\lambda_n)\hat{\chi}_{ln}\,\phi_n \\
&= Pe_l\left((\Delta^{-1}v_l)_N + \sum_{n\neq 0}\beta_{ln}\,\phi_n\right) \qquad (4.24)
\end{aligned}
$$

where $\lambda_n = i\mu_n$, $\lambda_{-n} = \lambda_n^*$, $n = 1, \ldots, \infty$. We obtain the following expressions for the coefficients of χ_l:

$$(\chi_l)_N = Pe_l\,(\Delta^{-1}v_l)_N \qquad (4.25a)$$

$$\hat{\chi}_{ln} = Pe_l\frac{\beta_{ln}}{1 - Pe_l\lambda_n} \qquad (4.25b)$$

Consequently:

$$\chi_l = Pe_l\left((\Delta^{-1}v_l)_N + \sum_{n\neq 0}\frac{\beta_{ln}}{1 - Pe_l\lambda_n}\,\phi_n\right) \qquad (4.26)$$

$$\mathcal{A}^v\chi_l = Pe_l\sum_{n\neq 0}\frac{\lambda_n\beta_{ln}}{1 - Pe_l\lambda_n}\,\phi_n \qquad (4.27)$$

Substituting (4.26 and 4.27) into (4.19) we obtain:

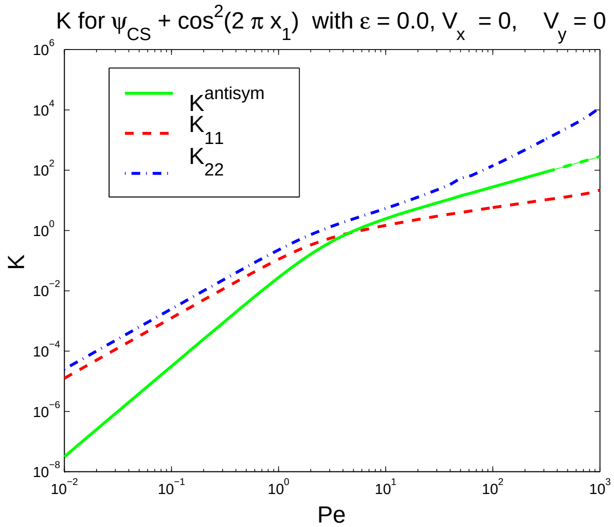
$$
\begin{aligned}
\mathcal{A}_{lk} &= Pe_l(\mathcal{A}^v\chi_l, \chi_k)_1 \\
&= Pe_l^3\left(\sum_{n\neq 0}\frac{\lambda_n\beta_{ln}}{1-Pe_l\lambda_n}\,\phi_n, (\Delta^{-1}v_k)_N + \sum_{n\neq 0}\frac{\beta_{kn}}{1-Pe_l\lambda_n}\,\phi_n\right)_1 \\
&= Pe_l^3\sum_{n=-\infty\, n\neq 0}^{n=+\infty}\frac{\lambda_n\beta_{ln}\beta_{kn}^*}{|1-Pe_l\lambda_n|^2} \\
&= Pe_l^3\left(\sum_{n=-\infty}^{n=-1}\frac{\lambda_n\beta_{ln}\beta_{kn}^*}{|1-Pe_l\lambda_n|^2} + \sum_{n=1}^{n=\infty}\frac{\lambda_n\beta_{ln}\beta_{kn}^*}{|1-Pe_l\lambda_n|^2}\right) \\
&= Pe_l^3\sum_{n=1}^{n=\infty}\left(\frac{\lambda_{-n}\beta_{l\,-n}\beta_{k\,-n}^*}{|1-Pe_l\lambda_{-n}|^2} + \frac{\lambda_n\beta_{ln}\beta_{kn}^*}{|1-Pe_l\lambda_n|^2}\right) \\
&= Pe_l^3\sum_{n=1}^{n=\infty}\left(\frac{-i\mu_n\beta_{ln}^*\beta_{kn}}{|1+iPe_l\mu_n|^2} + \frac{i\mu_n\beta_{ln}\beta_{kn}^*}{|1-iPe_l\mu_n|^2}\right) \\
&= Pe_l^3\sum_{n=1}^{n=\infty}\frac{i\mu_n(\beta_{ln}\beta_{kn}^* - \beta_{ln}^*\beta_{kn})}{1+Pe_l^2\mu_n^2} \\
&= Pe_l^3\sum_{n=1}^{n=\infty}\frac{\mu_n Im(\beta_{ln}^*\beta_{kn})}{1+Pe_l^2\mu_n^2} \qquad (4.28)
\end{aligned}
$$

Now the derivation of formula (4.8) is complete.

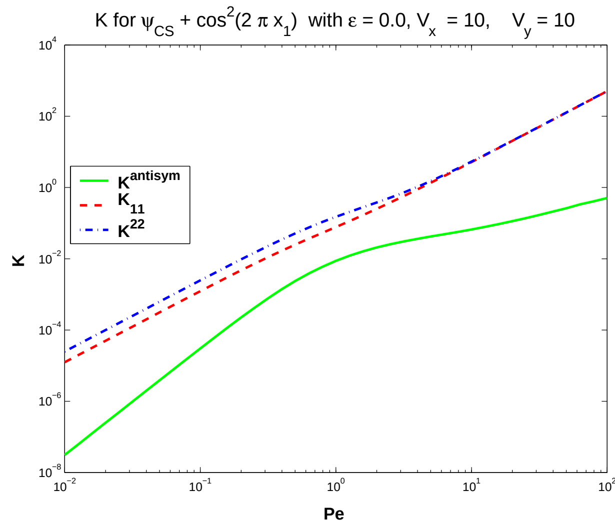
4.4 Numerical Examples

In this section we study numerically the properties of the antisymmetric part of the effective diffusivity for two types of velocity fields. The first example is a steady velocity field and, as we shall see, the numerical results are in complete accordance with the theoretical predictions of the previous sections. For the second example we choose a time dependent velocity field. In this case the analysis of the previous sections is no longer valid. In fact, we shall see that for this specific example the antisymmetric part of the effective diffusion tensor depends only on the temporal part of the fluctuations. Let us start with the Childress-Soward flow [27, 69] perturbed by a higher order harmonic:

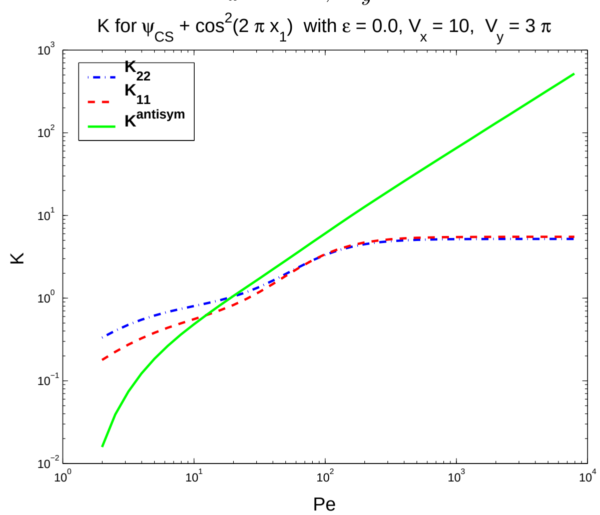
$$
\begin{aligned}
\psi(y_1, y_2) &= \psi_{cs}(y_1, y_2) + \cos^2(2\pi y_1) \\
&= \sin(2\pi y_1)\sin(2\pi y_2) + \epsilon\cos(2\pi y_1)\cos(2\pi y_2) + \cos^2(2\pi y_1) \qquad (4.29)
\end{aligned}
$$



a. $V_x = 0.0$, $V_y = 0.0$



b. $V_x = 10.0$, $V_y = 10.0$



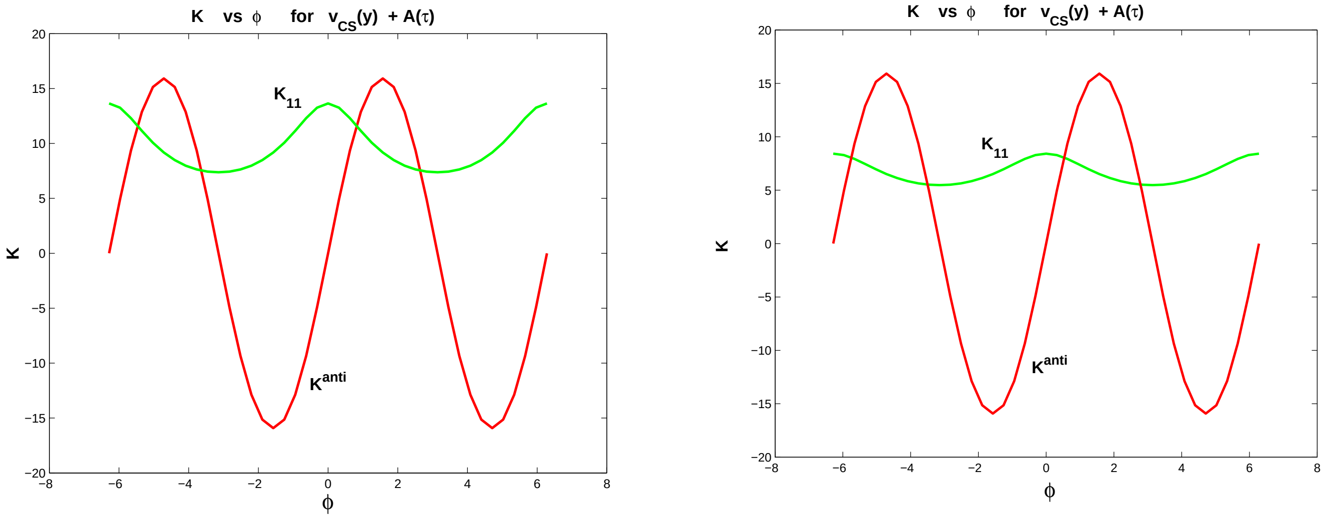
c. $V_x = 10.0$, $V_y = 3\pi$

Figure 4.1: $\overline{\mathcal{K}}$ vs the Peclet number for $\psi_{CS} + \cos^2(2\pi y_1)$ with $\epsilon = 0.0$, for different constant mean flows .

The Childress-Soward stream function has the following antisymmetry property:

$$\psi_{cs}(y_1, y_2 + \frac{1}{2}) = -\psi_{cs}(y_1, y_2) \tag{4.30}$$

Thus, from the results of Fannjiang and Papanicolaou we know that the effective diffusion tensor corresponding to ψ_{CS} is symmetric. However, the addition of the higher order harmonic term $\cos^2(2\pi y_1)$ breaks all possible antisymmetries of the stream function ψ and the



a. $V_1 = 10.0$, $V_2 = 10.0$ b. $V_1 = 0.0$, $V_2 = 0.0$

Figure 4.2: $\overline{\mathcal{K}}$ vs the phase ϕ for $\mathbf{A}(\tau) + \mathbf{v}_{CS}(\mathbf{y})$ with $\epsilon = 0.5$, $Pe = 10$, $S = 1, \delta_x = \delta_y = 1$, for different constant mean flows.

resulting effective diffusion tensor is not symmetric.

Now we wish to study how the antisymmetric part of the effective diffusivity depends on the Peclet number, in particular in connection to the presence and proeprties of the mean flow. In figure 4.1 we plot the symmetric and antisymmetric parts of the effective diffusivity for $\psi(\mathbf{y})$ and various choices of the mean flow. We see that the scaling of the symmetric parts of $\mathcal{K}^*$ for large Pe_l depends very sensitively on the presence of the mean flow (for small Pe_l the scaling is always quadratic). In particular, in the absence of mean flow, figure 4.1a, the diagonal components $\mathcal{K}_{11}$, $\mathcal{K}_{22}$ scale quadratically and like the square root of Pe_l, respectively. The choice $V_1 = V_2 = 10$, figure 4.1b, leads to maximally enhanced diffusion along the direction of the mean flow and both $\mathcal{K}_{11}$, $\mathcal{K}_{22}$ scale quadratically with the Peclet number. On the other hand, the slightly different mean flow $V_1 = 10$, $V_2 = 3\pi$, figure 4.1c leads to minimally enhanced diffusion and both $\mathcal{K}_{11}$, $\mathcal{K}_{22}$ approach constants, independent of the Peclet number for $Pe \gtrsim 10^{2.5}$.

On the other hand, the antisymmetric part of $\mathcal{K}$ has the same scaling, independently of the mean flow. It scales like Pe_l^3 for small Pe_l and like Pe_l for large values of the Peclet number.

Let us now consider a different type of perturbation to the Childress-Soward flow which breaks the symmetry of $\mathcal{K}$. Following [17] we add one time harmonic to the basic, steady flow:

$$\mathbf{v}(\mathbf{y}, \tau) = \mathbf{A}(\tau) + \mathbf{v}_{CS}(\mathbf{y}) \tag{4.31}$$

where $\mathbf{v}_{CS}(\mathbf{y}) = \nabla^{\perp}\psi_{CS}$ and

$$\mathbf{A}(\tau) = (\delta_x \sin(2\,\pi\,\tau),\, \delta_y \sin(2\,\pi\tau + \phi)) \tag{4.32}$$

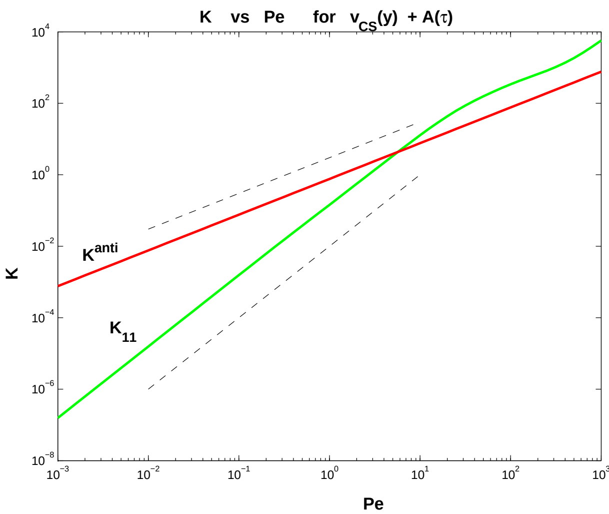
Now, since we already know that the Childress-Soward flow leads to a symmetric $\mathcal{K}^*$, the antisymmetric part of the effective diffusion tensor is due to the time dependent part of the fluctuations and in particular the phase ϕ. The antisymmetric part vanishes for $\phi = 0$ and $\phi = \pm\pi$. We will study the dependence of the effective diffusion tensor on the phase ϕ and the Strouhal and Peclet numbers. We start by fixing the nondimensional numbers of the problem as well as the strength of the time dependent part of the fluctuations and vary ϕ. In figure 4.2 we plot the symmetric and antisymmetric parts of the effective diffusion tensor (in this example the values of $\mathcal{K}_{11}$ and $\mathcal{K}_{22}$ are the same so it is enough to plot only $\mathcal{K}_{11}$) as a function of the phase ϕ. We fix $\delta_x = \delta_y = 1$, $Pe = 10$, $S = 1$, $\epsilon = 0.5$ and solve the cell problem in the absence of a mean flow and for $V_1 = V_2 = 10$. We observe that the effective diffusion tensor is a periodic function of ϕ, as expected. We also observe that, for these choices of the parameters of the problem, the antisymmetric part can become larger, in absolute value, than the symmetric part. We also observe that the antisymmetric part is independent of the presence of a mean flow. We shall come back to this point later.

In figure 4.3 we plot the symmetric as well as the antisymmetric part of the effective diffusivity versus the Peclet number for various choices of the mean flow as well as the Strouhal number. We observe that the antisymmetric part scales linearly with the Peclet number, for fixed Strouhal number. The scaling of the symmetric part as a function of the Peclet number depends upon the properties of the mean flow.

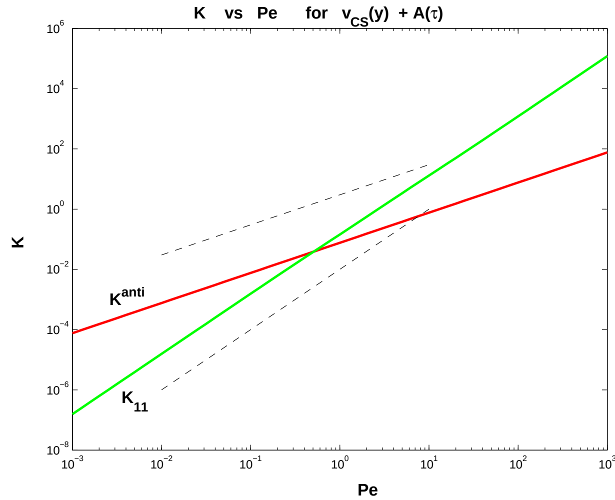
In figure 4.4 we plot the effective diffusion tensor as a function of the Strouhal number, for fixed Peclet number. We observe a nonmonotonic dependence of $\text{symm}(\overline{\mathcal{K}})$ on the Strouhal number as has already been reported in [17]. On the other hand, the antisymmetric part is inversely proportional to the Strouhal number and $\lim_{S\to\infty} \mathcal{K}^{anti} = 0$. One can show that the antisymmetric part vanishes at the limit $S \to \infty$ through asymptotic analysis of the cell problem: for large Strouhal numbers the effective diffusivity tensor is determined from the steady part of the velocity field which leads to a symmetric diffusion tensor[13].

Remark: From the numerical results that we presented it is clear that the antisymmetric part of $\mathcal{K}^*$ is independent of the mean flow. Consequently, there will not be an effective drift due to the antisymmetric part. However, one could consider a generalization of the velocity field (4.31) in which the amplitude of the temporal part is a function of the large scales, for example $\delta_x = \delta_x(x_2, t)$, $\delta_y = \delta_y(x_1, t)$. For such a velocity field with slowly modulated fluctuations the antisymmetric part will be a function of the large scale variables and this will lead to a non zero eddy induced transport velocity.

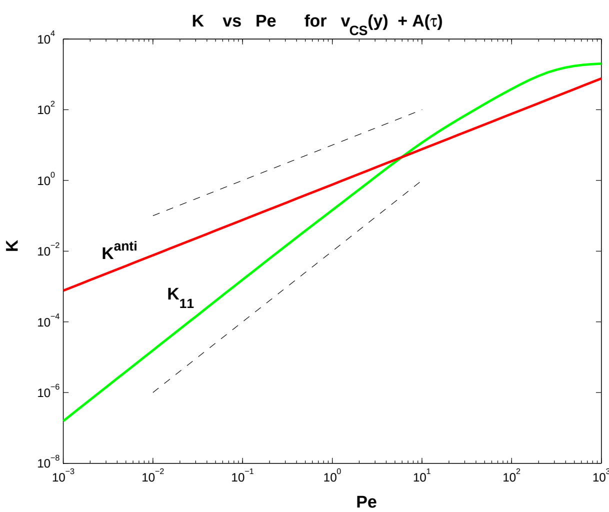
[13]The conclusions regarding the dependence of antisymmetric part of $\mathcal{K}^*$ on the non dimensional parameters of the problem for this particular time dependent field are based on our numerical results and formal asymptotics of the cell problem, as opposed to a rigorous theory. We do not expect the scalings presented in figures 4.1 and 4.2 to be valid for arbitrary time dependent fields.



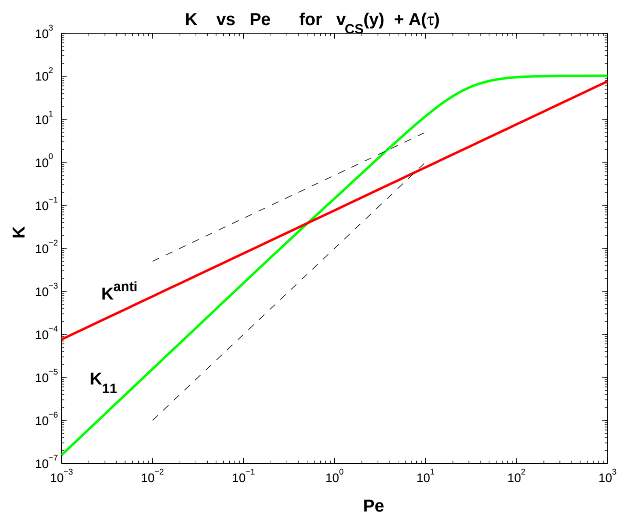
a. $S = 0.1,\ V_1 = 10.0,\ V_2 = 10.0$



b. $S = 1,\ V_1 = 10.0,\ V_2 = 10.0$

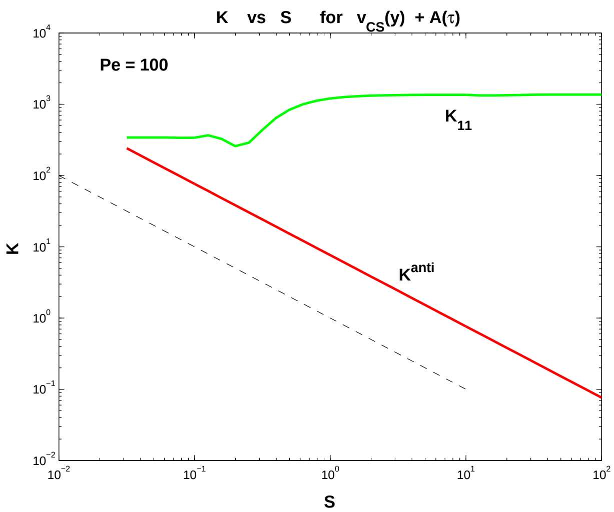


c. $S = 0.1,\ V_1 = 10.0,\ V_2 = 9.55$

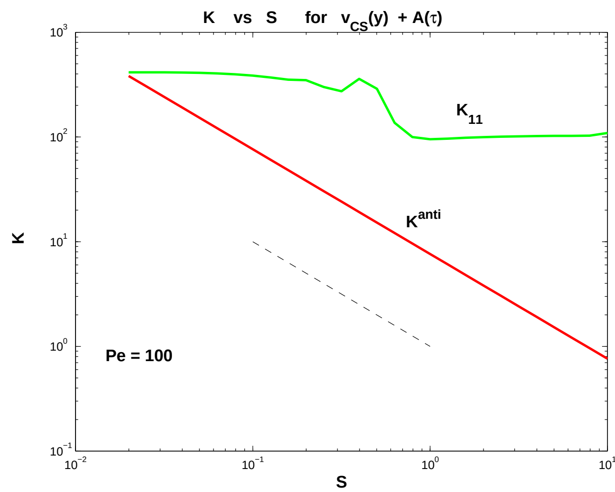


d. $S = 1,\ V_1 = 10.0,\ V_2 = 9.55$

Figure 4.3: $\overline{\mathcal{K}}$ **vs the Peclet number for** $\mathbf{A}(\tau)+\mathbf{v}_{CS}(\mathbf{y})$ **with** $\epsilon = 0.5,\ \phi = \frac{\pi}{2}, \delta_x = \delta_y = 1$**, for different steady mean flows. The lines with slopes 1 and 2 are also drawn for comparison**



a. $Pe = 100,\ V_1 = 10.0,\ V_2 = 10.0$



b. $Pe = 100,\ V_1 = 10.0,\ V_2 = 9.55$

Figure 4.4: $\overline{\mathcal{K}}$ vs the Strouhal number for $\mathbf{A}(\tau) + \mathbf{v}_{CS}(\mathbf{y})$ with $\epsilon = 0.5$, $\phi = \frac{\pi}{2}, \delta_x = \delta_y = 1$, for different constant mean flows. The (broken) line with slope -1 is also drawn for comparison.

CHAPTER 5
TWO–SCALE CONVERGENCE

5.1 Introduction

In chapters 2 and 3 we derived the effective equations for various choices of the parameters α and γ. We used a multiple scales expansion to derive formally equations for the first two terms in the expansion and then used energy estimates to prove their validity. We saw that in the parameter range $\alpha \in (-1, 0)$, $\gamma > 0$ one has to use elaborate forms of the multiple scales expansion in order to make sure that all the equations that we need to consider are well posed and to derive the necessary estimates. Moreover, we saw that these formal multiple scales expansions failed to provide us with the effective equations in the case where the mean flow is stronger than the fluctuations.

In this chapter we shall present alternative definitions of the homogenized equations using the method of two scale convergence. For brevity we shall restrict ourselves to the case of steady velocity fields. At the end of the chapter we shall make comments concerning the case of time dependent velocity fields.

In the first section we shall present the relevant definitions and theorems from the method of two-scale convergence, in a form that is suitable for our purposes. Later on we shall apply this method to the problem of homogenization for advection-diffusion equations with mean flow.

5.2 Two–Scale Convergence

The method of two scale convergence is a powerful method for studying homogenization problems for partial differential equations with periodically oscillating coefficients. It was devised by Nguetseng in [83] and later popularized by Allaire in [1] and [2]. It was used, for example, to study problems of fluid flow through inhomogeneous porous media in [29] and [56]. More importantly for us, this method was applied to the problem of homogenization of transport equations with incompressible velocity fields in [36] and [57]. The method of two-scale convergence was extended in various ways: periodic oscillating coefficients in both space and time—in particular, in connection to the problem of homogenization for parabolic partial differential equations with oscillating coefficients in both space and time— were treated in [53]. Non periodic oscillations using two-scale convergence were studied in [73]. Finally, the concept of stochastic two-scale convergence was developed in [19] as a tool to study random homogenization.

We recall briefly the basic definitions and theorems. For details and proofs we refer to the papers of Allaire [2] and Holmbom [53]. We shall use the notation of W. E from [36]. We first define the space of test functions that we shall need:

$$J_p = \{\phi \ : \ \mathbb{R}^d \times \mathbb{R}^+ \times [0,1]^d \to \mathbb{R},\ \phi(\mathbf{x},t,\mathbf{y}) \text{ is smooth and periodic in } \mathbf{y} \text{ with period } Y=[0,1]^d \text{ and has compact support}\}$$

We have the following definition:

DEFINITION 5.1 *A sequence* $T^\delta \in L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ *two- scale converges to* $T_0(\mathbf{x},t,\mathbf{y}) \in L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d \times Y)$ *if for any test function* $\psi(\mathbf{x},t,\mathbf{y}) \in J_p$

$$\lim_{\delta\to 0}\int_{\mathbb{R}^+\times\mathbb{R}^d} T^\delta(\mathbf{x},t)\psi\left(\mathbf{x},t,\frac{\mathbf{x}}{\delta}\right)\,dxdt = \int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y T_0(\mathbf{x},t,\mathbf{y})\psi(\mathbf{x},t,\mathbf{y})\,dydxdt \tag{5.1}$$

We shall use the notation $T^\delta \stackrel{2}{\rightharpoonup} T_0$ to mean that the sequence T^δ two-scale converges to T_0. The concept of two scale convergence is useful because of the following compactness theorem:

THEOREM 5.1 *Let* T^δ *be a uniformly bounded sequence in* $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$. *Then there exists a function* $T_0(x,t,y)$ *in* $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d \times Y)$ *and a subsequence, still denoted by* T^δ, *such that* T^δ *two-scale converges to* $T_0(x,t,y)$. *Moreover, the subsequence converges weakly in* $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ *to* $T(x,t) = \int_Y T_0(x,t,y)\,dy$.

The two scale limit T_0 is essentially the first term in the multiple scales expansion. We see that in general it will depend on the oscillations through the auxiliary variable y. This is the case for example in the homogenization of transport equations, [36, 57]. This due to the fact that the above theorem does not assume any control over the L^2 norm of the gradient of T^δ. Since the gradient can become unbounded, the oscillations will persist and appear in the two-scale limit. Uniform bounds in better spaces provide us with more detailed information about the two-scale limit. We shall use the notation $H^1(Y)/\mathbb{R} := \{f \in H^1(Y);\ \langle f\rangle = 0\}$

which is more commonly used in this setting than the notation $\mathcal{H}^1$ that was used in chapter 4.

THEOREM 5.2 *(i) Let T^δ be a uniformly bounded sequence in $L^2_{loc}(\mathbb{R}^+; H^1_{loc}(\mathbb{R}^d))$. Then T^δ two-scale converges to a function $T(\mathbf{x},t) \in L^2_{loc}(\mathbb{R}^+; H^1_{loc}(\mathbb{R}^d))$ and there exists a function $T_1(\mathbf{x},t,\mathbf{y})$ in $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d; H^1(Y)/\mathbb{R})$ such that, up to a subsequence, $\nabla_x T^\delta$ two-scale converges to $\nabla_x T(\mathbf{x},t) + \nabla_y T_1(\mathbf{x},t,\mathbf{y})$. $T(\mathbf{x},t)$ is the strong $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ limit of T^δ.*
(ii) Let T^δ and $\delta\,\nabla T^\delta$ be uniformly bounded sequences in $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$. Then there exists a function $T_0(\mathbf{x},\mathbf{y},t) \in L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d; H^1(Y))$ such that, up to a subsequence, T^δ and $\delta\nabla T^\delta$ two-scale converge to $T_0(\mathbf{x},\mathbf{y},t)$ and to $\nabla_y T_0(\mathbf{x},\mathbf{y},t)$, respectively.

From the first part of theorem 5.2 we see that a uniform bound over the gradient of T^δ is enough to justify the second term in the multiple scales expansion $T_1(x,t,y)$. We shall denote the type of convergence described by the first part of theorem 5.2 by $T^\delta \xrightarrow{2} T(\mathbf{x},t)$ and say that T^δ two-scale converges **strongly** to T. An immediate corollary of the second part of theorem 5.2 that we shall need later on is:

COROLLARY 5.1 *Let T^δ and $\delta^\gamma\,\nabla T^\delta$ be uniformly bounded sequences in $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$ with $\gamma \in (0,1)$. Then the two-scale limit T_0 of T^δ is independent of $\mathbf{y}$. The two-scale limit $T_0(\mathbf{x},t)$ is the weak -L^2 limit of T^δ. Moreover, there exists a function $T_1(\mathbf{x},t,\mathbf{y})$ in $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d; H^1(Y)/\mathbb{R})$ such that $\delta^\gamma\,\nabla T^\delta$ two-scale converges to $\nabla_y T_1(t,\mathbf{x},\mathbf{y})$.*

Proof:
From the second part of theorem (5.2) we know that $\delta\,\nabla T^\delta \overset{2}{\rightharpoonup} \nabla_y T_0(\mathbf{x},\mathbf{y},t)$. On the other hand, the uniform bound on $\delta^\gamma\,\nabla T^\delta$, $\gamma \in (0,1)$ implies that $\delta\,\nabla T^\delta \overset{2}{\rightharpoonup} \mathbf{0}$. Consequently, $\nabla_y T_0(\mathbf{x},\mathbf{y},t) = 0$ (in the weak sense), and $T_0 = T_0(\mathbf{x},t)$. The fact that T_0 is the weak L^2 limit of T^δ follows from theorem (5.2).

Now, the uniform bound on $\delta^\gamma\,\nabla T^\delta$ implies that, for every $\Psi \in J^d_p$ there exists a function $\Phi(\mathbf{x},t,\mathbf{y}) \in (L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d \times Y))^d$ such that:

$$\lim_{\delta\to 0} \delta^\gamma \int_{\mathbb{R}^+ \times \mathbb{R}^d} \nabla T^\delta(\mathbf{x},t) \cdot \Psi\left(\mathbf{x},t,\frac{\mathbf{x}}{\delta}\right) dxdt = \int_{\mathbb{R}^+ \times \mathbb{R}^d} \int_Y \Phi(\mathbf{x},t,\mathbf{y}) \cdot \Psi(\mathbf{x},t,\mathbf{y})\, dydxdt \quad (5.2)$$

We now consider a test function Ψ such that $\nabla_y \cdot \Psi = \mathbf{0}$. We integrate by parts to obtain:

$$\begin{aligned}
\lim_{\delta\to 0} \delta^\gamma \int_{\mathbb{R}^+\times\mathbb{R}^d} \nabla T^\delta(\mathbf{x},t)\cdot\Psi\left(\mathbf{x},t,\frac{\mathbf{x}}{\delta}\right) dxdt &= \lim_{\delta\to 0}\left(-\delta^\gamma \int_{\mathbb{R}^+\times\mathbb{R}^d} T^\delta(\mathbf{x},t)\nabla\cdot\Psi\left(\mathbf{x},t,\frac{\mathbf{x}}{\delta}\right) dxdt\right) \\
&= \lim_{\delta\to 0}\left(-\delta^\gamma \int_{\mathbb{R}^+\times\mathbb{R}^d} T^\delta(\mathbf{x},t)\nabla_x\cdot\Psi\left(\mathbf{x},t,\frac{\mathbf{x}}{\delta}\right) dxdt\right) \\
&= 0, \qquad (5.3)
\end{aligned}$$

on account of the uniform bound on T^δ. Combining now the above two equations we deduce:

$$\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \Phi(\mathbf{x},t,\mathbf{y})\cdot\Psi(\mathbf{x},t,\mathbf{y})\, dydxdt = 0 \qquad (5.4)$$

for all test functions with $\nabla_y \cdot \Psi = \mathbf{0}$. Since now the orthogonal complement to divergence free functions is the space of the gradients, (see for example [58, ch. 1]), we conclude that there exists a function $T_1(\mathbf{x},t,\mathbf{y}) \in L^2_{loc}(\mathbb{R}^+\times\mathbb{R}^d; H^1(Y)/\mathbb{R})$ such that $\delta^\gamma\,\nabla T^\delta \stackrel{2}{\rightharpoonup} \nabla_y T_1(t,\mathbf{x},\mathbf{y})$ and the proof is complete. We remark that the same conclusion is valid for $\gamma > 1$. However, in this case the two-scale limit to T^δ will, in general, depend on $\mathbf{y}$.

To understand intuitively the meaning of this corollary, let us consider a multiple scales expansion of the form $T^\delta \sim T_0(\mathbf{x},t) + \delta^{1-\gamma}\,T_1(\mathbf{x},t,\frac{\mathbf{x}}{\delta})$ with $\gamma \in (0,1)$ (in the next section we shall see why this is relevant for our problem). We have:

$$\begin{aligned}
\delta^\gamma\,\nabla T^\delta &\sim \delta^\gamma\,\nabla_x T_0(\mathbf{x},t) + (\nabla_x + \frac{1}{\delta}\nabla_y)\,\delta\,T_1(\mathbf{x},t,\mathbf{y}=\frac{\mathbf{x}}{\delta}) \\
&\sim \delta^\gamma\,\nabla_x T_0 + \delta\,\nabla_x T_1 + \nabla_y T_1 \\
&\to \nabla_y T_1 \qquad (5.5)
\end{aligned}$$

Thus, in this setting, the function $T_1(\mathbf{x},t,\mathbf{y})$ is exactly the higher order term in the expansion.

Finally, in order to study the two scale limit of the given incompressible field $\mathbf{v}$ we shall need the following [2]:

THEOREM 5.3 *(i) Let $\mathbf{v}(\mathbf{x},\mathbf{y})$ be a smooth vector field, periodic in $\mathbf{y}$. Then the associated sequence $\mathbf{v}^\delta(\mathbf{x},\frac{\mathbf{x}}{\delta})$ two-scale converges to $\mathbf{v}$.*
(ii) Let $\mathbf{v}^\delta$ be a divergence-free bounded sequence in $[L^2_{loc}(\mathbb{R}^d)]^d$, which two-scale converges to $\mathbf{v}(\mathbf{x},\mathbf{y}) \in [L^2_{loc}(\mathbb{R}^d\times Y)]^d$. Then the two-scale limit satisfies $\nabla_y\cdot\mathbf{v}(\mathbf{x},\mathbf{y}) = 0$, $\int_Y \nabla_x\cdot\mathbf{v}(\mathbf{x},\mathbf{y})\,dy = 0$.

We remark that, since we merely assume that $\mathbf{v}(\mathbf{x},\mathbf{y}) \in [L^2_{loc}(\mathbb{R}^d \times Y)]^d$, we have to interpret the divergence of $\mathbf{v}(\mathbf{x},\mathbf{y})$ in the appropriate sense, namely in the H^{-1} sense. However, in the sequel we shall assume that the velocity field is smooth, so the divergence free conditions can be interpreted in the strong sense.

We now wish to apply the method of two-scale convergence to the initial value problem for the advection diffusion equation. Our goal is to justify rigorously the equations obtained for the first two terms in the formal multiple scales expansion.

5.3 The Homogenization Theorem

We consider the initial value problem for the advection diffusion equation with mean flow for steady, incompressible velocity fields written in nondimensional form:

$$\frac{\partial T^\delta(\mathbf{x},t)}{\partial t} + \Big(\mathbf{V}(\mathbf{x}) + \delta^\alpha \mathbf{v}(\frac{\mathbf{x}}{\delta})\Big) \cdot \nabla T^\delta(\mathbf{x},t) = \delta^{\alpha+1} \Delta T^\delta(\mathbf{x},t) \;\; in\, \mathbb{R}^d \times (0,\infty) \tag{5.6a}$$

$$T^\delta(\mathbf{x}, t=0) = T_{in}(\mathbf{x}) \;\; on\, \mathbb{R}^d, \tag{5.6b}$$

where, to simplify the notation, we have set $Pe_l = 1$. Since we consider only steady velocity fields there is no need to introduce the Strouhal number. Before proceeding with the statement and proof of the homogenization theorem, let us outline the procedure that we shall use for the case $\alpha \in (-1,0)$. We expect a solution of the form $T^\delta \sim T_0 + \delta^{1+\alpha} T_1 + \delta T_2$. We first obtain estimates on T^δ that enable us to pass to the two scale limit and show that T_0 satisfies a transport equation, the transport velocity being the mean flow. We then subtract T_0 from T^δ and "renormalize", obtaining an equation for $T^{1,\delta} = \frac{T^\delta - T_0}{\delta^{1+\alpha}}$. We then use energy estimates that enable us, using corollary (5.1), to obtain an equation for T_1, as well as the cell problem. A similar methodology will be used when $\alpha \geq 0$.

The parameter range that we shall consider is $\alpha \in (-1,1)$ (the —easier— case $\alpha = -1$ will be treated in the appendix). Apart from the incompressibility assumption we shall also assume that the fluctuations are periodic with period Y and have zero average and that the velocity field as well as the initial conditions are smooth (in fact, all we really need is $T_{in} \in L^2(\mathbb{R})$, $\mathbf{V} \in (\mathbf{L}^\infty(\mathbb{R}^d))^d$, $\mathcal{H} \in (\mathbf{L}^\infty(Y))^{d\times d}$, $\mathcal{H}$ being the stream matrix defined below, but we shall not concern ourselves with such technicalities).

The incompressibility assumption implies that there exists an antisymmetric matrix $\mathcal{H}(\mathbf{y})$, the *stream matrix* with the following properties:

$$\mathcal{H}_{ij} = -\mathcal{H}_{ji}, \;\; i,j = 1\dots d \; :$$
$$v_j(y) = -\frac{\partial \mathcal{H}_{ij}}{\partial y_i}, \; j = 1, \dots d$$
$$< \mathcal{H}_{ij} >= 0, \;\; i,j = 1\dots d \; :$$

where we have used the summation convention. We present a proof of the above result in the appendix. We remark that, using theorem (5.3), we have that $\mathbf{v}(\frac{\mathbf{x}}{\delta}) \stackrel{2}{\rightarrow} \mathbf{v}(\mathbf{y})$ and $\mathcal{H}(\frac{\mathbf{x}}{\delta}) \stackrel{2}{\rightarrow} \mathcal{H}(\mathbf{y})$. The strong convergence will enable us to pass to the two scale limits in various terms of the equations later on .

It will prove useful to rewrite the initial value problem (5.6) using the stream matrix:

$$\frac{\partial T^\delta(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x}) \cdot \nabla T^\delta(\mathbf{x},t) = \delta^{\alpha+1} \nabla(\mathcal{K}^\delta \nabla T^\delta(\mathbf{x},t)) \;\; \text{in } \mathbb{R}^d \times (0,\infty) \tag{5.7a}$$

$$T^\delta(\mathbf{x}, t=0) = T_{in}(\mathbf{x}) \;\; \text{on } \mathbb{R}^d, \tag{5.7b}$$

Using now the techniques presented in the proof of lemma (2.2), we obtain the following estimates:

$$||T^\delta(x,t)||_{L^2_{loc}(\mathbb{R}^+;L^2(\mathbb{R}^d))} \leq C \tag{5.8a}$$

$$\delta^{\frac{1+\alpha}{2}} ||\nabla T^\delta(x,t)||_{L^2_{loc}(\mathbb{R}^+;L^2(\mathbb{R}^d))} \leq C \tag{5.8b}$$

From these estimates we deduce that T^δ two-scale converges to $T_0(\mathbf{x},t)$. We multiply now (5.6) by $\phi^\delta \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ and integrate by parts on $\mathbb{R}^+ \times \mathbb{R}^d$ to obtain:

$$\begin{aligned} -\int_{\mathbb{R}^+ \times \mathbb{R}^d} T^\delta \phi^\delta_t \, dx\, dt \;\; &- \;\; \int_{\mathbb{R}^d} T_{in}(x) \phi^\delta(x,0)\, dx - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \nabla \phi^\delta T^\delta \, dx\, dt \\ &+ \;\; \delta^{1+\alpha} \int_{\mathbb{R}^+ \times \mathbb{R}^d} \nabla \phi^\delta \cdot \mathcal{K}^\delta \nabla T^\delta \, dx\, dt = 0 \end{aligned} \tag{5.9}$$

In order to obtain the homogenized equation for T_0 it is enough to consider a test function

$\phi(x,t) \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$. We immediately pass to the two scale limit to obtain:

$$\int_{\mathbb{R}^+\times\mathbb{R}^d} T_0\,\phi_t\,dx\,dt \;+\; \int_{\mathbb{R}^d} T_{in}(x)\phi(x,0)\,dx \;+\; \int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\nabla\phi T_0\,dx\,dt = 0 \tag{5.10}$$

which is the weak formulation of the transport equation for the first term in the expansion:

$$\frac{\partial T_0}{\partial t} + \mathbf{V}(\mathbf{x})\cdot\nabla T_0 = 0 \tag{5.11a}$$

$$T_0(\mathbf{x},t) = T_{in}(\mathbf{x}) \tag{5.11b}$$

Now we wish to obtain an equation for the second term in the expansion. We define:

$$T^{1,\delta} = \frac{T^\delta - T_0}{\delta^{1+\alpha}} \tag{5.12}$$

Using now (5.6) and (5.11) we get and equation for $T^{1,\delta}$:

$$\begin{aligned}\frac{\partial T^{1,\delta}(\mathbf{x},t)}{\partial t} \;&+\; \mathbf{V}(\mathbf{x})\cdot\nabla T^{1,\delta}(\mathbf{x},t) - \delta^{\alpha+1}\nabla(\mathcal{K}^\delta\nabla T^{1,\delta}(\mathbf{x},t))\\ &=\; -\frac{1}{\delta}\mathbf{v}(\frac{\mathbf{x}}{\delta})\cdot\nabla T_0(\mathbf{x},t) + \Delta T_0(\mathbf{x},t)\;\; in\,\mathbb{R}^d\times(0,\infty)\end{aligned} \tag{5.13a}$$

$$T^{1,\delta}(\mathbf{x},t=0) = 0\;\; on\,\mathbb{R}^d, \tag{5.13b}$$

or:

$$\frac{\partial T^{1,\delta}(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x})\cdot\nabla T^{1,\delta}(\mathbf{x},t) - \delta^{\alpha+1}\nabla(\mathcal{K}^\delta\nabla T^{1,\delta}(\mathbf{x},t)) = \nabla(\mathcal{K}^\delta\nabla T_0(\mathbf{x},t))\;\; in\,\mathbb{R}^d\times(0,\infty) \tag{5.14a}$$

$$T^{1,\delta}(\mathbf{x},t=0) = 0\;\; on\,\mathbb{R}^d, \tag{5.14b}$$

We multiply (5.14) by $T^{1,\delta}$, integrate by parts and use the Cauchy-Schwarz inequality to obtain the estimates:

$$||T^{1,\delta}(x,t)||_{L^2_{loc}(\mathbb{R}^+;L^2(\mathbb{R}^d))} \leq C \tag{5.15a}$$

$$\delta^{1+\alpha}||\nabla T^{1,\delta}(x,t)||_{L^2_{loc}(\mathbb{R}^+;L^2(\mathbb{R}^d))} \leq C \tag{5.15b}$$

There are three different cases to consider, depending on α:

1. $\alpha \in (-1, 0)$:

$$T^{1,\delta} \overset{2}{\rightharpoonup} T_1(\mathbf{x}, t) \tag{5.16a}$$

$$\delta^{1+\alpha}\,\nabla T^{1,\delta} \overset{2}{\rightharpoonup} \nabla_y T_2(\mathbf{x}, \mathbf{y}, t) \tag{5.16b}$$

2. $\alpha = 0$:

$$T^{1,\delta} \overset{2}{\rightharpoonup} T_1(\mathbf{x}, \mathbf{y}, t) \tag{5.17a}$$

$$\delta\,\nabla T^{1,\delta} \overset{2}{\rightharpoonup} \nabla_y T_1(\mathbf{x}, \mathbf{y}, t) \tag{5.17b}$$

3. $\alpha \in (0, 1)$:

$$T^{1,\delta} \overset{2}{\rightharpoonup} T_1(\mathbf{x}, \mathbf{y}, t) \tag{5.18a}$$

$$\delta^{1+\alpha}\,\nabla T^{1,\delta} \overset{2}{\rightharpoonup} \nabla_y T_2(\mathbf{x}, \mathbf{y}, t) \tag{5.18b}$$

The weak formulation of (5.14) is:

$$\begin{aligned} &- \int_{\mathbb{R}^+\times\mathbb{R}^d} T^{1,\delta}\,\phi_t^\delta\,dx\,dt \;-\; \int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\nabla\phi^\delta T^{1,\delta}\,dx\,dt \\ &- \delta^{1+\alpha}\int_{\mathbb{R}^+\times\mathbb{R}^d} \nabla\phi^\delta\cdot\mathcal{K}^\delta\nabla T^{1,\delta}\,dx\,dt = -\int_{\mathbb{R}^+\times\mathbb{R}^d}\nabla\phi^\delta\cdot\mathcal{K}^\delta\nabla T_0\,dx\,dt \end{aligned} \tag{5.19}$$

We start with the case $\alpha = 0$. We write the two scale limit $T_1(\mathbf{x},\mathbf{y},t)$ in the form:

$$\begin{aligned} T_1(\mathbf{x},\mathbf{y},t) &= \langle T_1(\mathbf{x},\mathbf{y},t)\rangle + \tilde{T}_1(\mathbf{x},\mathbf{y},t) \\ &:= \overline{T}_1(\mathbf{x},t) + \tilde{T}_1(\mathbf{x},\mathbf{y},t) \end{aligned} \tag{5.20}$$

We now choose a test function $\phi(x,t) \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$ to pass to the two-scale limit:

$$\begin{aligned} &- \int_{\mathbb{R}^+\times\mathbb{R}^d} \overline{T}_1\,\phi_t\,dx\,dt \;-\; \int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\nabla\phi\overline{T}_1\,dx\,dt \\ &- \int_{\mathbb{R}^+\times\mathbb{R}^d} \nabla\phi\cdot\int_Y \mathcal{K}\nabla_y\tilde{T}_1\,dx\,dt\,dy = -\int_{\mathbb{R}^+\times\mathbb{R}^d}\nabla\phi\nabla T_0\,dx\,dt \end{aligned} \tag{5.21}$$

Now we wish to obtain an equation for $\tilde{T}_1$. To do this, we choose a test function of the form $\phi^\delta = \delta\,\phi_1(\mathbf{x},\frac{\mathbf{x}}{\delta},t)$ with $\phi_1(\mathbf{x},\mathbf{y},t)\in J_p$ and pass to the two scale limit in (5.19) to obtain:

$$\begin{aligned} -\int_{\mathbb{R}^+\times\mathbb{R}^d}\mathbf{V}\cdot\int_Y\nabla_y\phi_1\tilde{T}_1\,dx\,dt\,dy \;&-\; \int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y\nabla_y\phi_1\cdot\mathcal{K}\nabla_y\tilde{T}_1\,dx\,dt\,dy \\ &= -\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y\phi_1\mathbf{v}\cdot\nabla T_0\,dx\,dt\,dy \end{aligned} \tag{5.22}$$

We could content ourselves with the system of equations (5.21), (5.22) as being the weak formulation of the homogenized system of equations for T_1. However, we can decouple the above equations by using the standard separation of variables trick:

$$\tilde{T}_1(\mathbf{x},\mathbf{y},t) = \chi(\mathbf{x},\mathbf{y})\cdot\nabla_x T_0(\mathbf{x},t) \tag{5.23}$$

Inserting (5.23) into (5.22) we obtain:

$$
\begin{aligned}
- & \int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\left(\int_Y \nabla_y\phi_1\,\chi\right)\cdot\nabla_x T_0\,dx\,dt\,dy \\
- & \int_{\mathbb{R}^+\times\mathbb{R}^d} \left(\int_Y \nabla_y\phi_1\cdot\mathcal{K}\nabla_y\chi\right)\cdot\nabla_x T_0\,dx\,dt\,dy = -\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \phi_1\mathbf{v}\cdot\nabla T_0\,dx\,dt\,dy
\end{aligned}
$$

Equation (5.24) is the weak formulation of the elliptic cell problem

$$
(\mathbf{V}+\mathbf{v})\cdot\nabla_y\chi-\Delta_y\chi=-\mathbf{v} \tag{5.24}
$$

in Y with periodic boundary conditions. Using now (5.23) in (5.21) we obtain:

$$
\begin{aligned}
- & \int_{\mathbb{R}^+\times\mathbb{R}^d} \overline{T}_1\,\phi_t\,dx\,dt \;-\; \int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\nabla\phi\overline{T}_1\,dx\,dt \\
+ & \int_{\mathbb{R}^+\times\mathbb{R}^d} \nabla\phi\cdot\left(\int_Y \mathbf{v}\,\chi\right)\nabla_x\tilde{T}_0\,dx\,dt\,dy = -\int_{\mathbb{R}^+\times\mathbb{R}^d}\nabla\phi\nabla T_0\,dx\,dt
\end{aligned} \tag{5.25}
$$

where the definition of $\mathcal{K}$ and the fact that $\langle\chi\rangle=0$ were used. Equation (5.25) is the weak formulation of the inhomogeneous transport equation for $\overline{T}_1$:

$$
\frac{\partial\overline{T}_1}{\partial t}+\mathbf{V}(\mathbf{x})\cdot\nabla\overline{T}_1=\nabla(\mathcal{K}^*\nabla T_0) \tag{5.26a}
$$

$$
\overline{T}_1(\mathbf{x},t=0)=0 \tag{5.26b}
$$

with $\mathcal{K}^*=\mathcal{I}-\langle\mathbf{v}\otimes\chi\rangle$.

Now we proceed with the case $\alpha\in(-1,0)$. As before, we choose a test function $\phi(x,t)\in\mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$ to pass to the two-scale limit in (5.19):

$$
\begin{aligned}
- & \int_{\mathbb{R}^+\times\mathbb{R}^d} T_1\,\phi_t\,dx\,dt \;-\; \int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\nabla\phi T_1\,dx\,dt \\
- & \int_{\mathbb{R}^+\times\mathbb{R}^d} \nabla\phi\cdot\int_Y \mathcal{K}\nabla_y T_2\,dx\,dt\,dy = \int_{\mathbb{R}^+\times\mathbb{R}^d}\nabla\phi\nabla T_0\,dx\,dt
\end{aligned} \tag{5.27}
$$

The next step, as before, is to obtain an equation for $T_2(\mathbf{x},t,\mathbf{y})$. With the choice of test function $\phi^\delta=\delta\,\phi_1(\mathbf{x},\frac{\mathbf{x}}{\delta},t)$ with $\phi_1(\mathbf{x},\mathbf{y},t)\in J_p$ the two scale limit of the second term in

(5.19) gives:

$$
\begin{aligned}
-\int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\nabla\phi^\delta T^{1,\delta}\,dx\,dt &= -\int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\nabla_y\phi_1(\mathbf{x},\mathbf{y}=\frac{\mathbf{x}}{\delta},t)T^{1,\delta}\,dx\,dt \\
&\to -\int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{V}\cdot\left(\int_Y \nabla_y\phi_1(\mathbf{x},y,t)\,dy\right)T_1(\mathbf{x},t)\,dx\,dt \\
&= 0 \qquad (5.28)
\end{aligned}
$$

We compute the two scale limits of the other terms in (5.19) to get:

$$
-\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \nabla_y\phi_1\cdot\mathcal{K}\nabla_y T_2\,dx\,dt\,dy = -\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \phi_1\mathbf{v}\cdot\nabla T_0\,dx\,dt\,dy \qquad (5.29)
$$

We can now decouple equations (5.27) and (5.29) using separation of variables:

$$
T_2(\mathbf{x},\mathbf{y},t) = \chi(\mathbf{y})\cdot\nabla_x T_0(\mathbf{x},t) \qquad (5.30)
$$

We obtain

$$
-\int_{\mathbb{R}^+\times\mathbb{R}^d}\left(\int_Y \nabla_y\phi\cdot\mathcal{K}\nabla_y\chi\right)\cdot\nabla_x T_0\,dx\,dt\,dy = -\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \phi_1\mathbf{v}\cdot\nabla T_0\,dx\,dt\,dy
$$

Which is the weak formulation of the cell problem:

$$
\mathbf{v}\cdot\nabla_y\chi - \Delta_y\chi = -\mathbf{v} \qquad (5.31)
$$

The method of two scale convergence explains in a transparent fashion why the mean flow does not enter into the cell problem for $\alpha\in(-1,0)$: The better control over the $L^2_{loc}(\mathbb{R}^+;L^2(\mathbb{R}^d))$ norm of the gradient of $T^{1,\delta}$ enables us to conclude that the two scale limit T_1 is independent of the variable $\mathbf{y}$ and consequently the limit of the second term in (5.19) vanishes, (5.28).

Now we study the case $\alpha\in(0,1)$. First we remark that the control over the gradient of $T^{1,\delta}$ that we have is too weak and the two scale limit T_1 depends on $\mathbf{y}$. Moreover, we do not have any information concerning $\nabla_y T_1$. Now the solution still looks like $T^\delta\sim T_0+\delta^{1+\alpha}T_1+\delta T_2$, however the term T_1 is weaker than the correction T_2. Since, though, $<T_2>=0$ this term is will only contribute to the cell problem through the two-scale limit $\nabla_y T_2$ and not to the effective equation.

The first step is the same as in the $\alpha = 0$ case: We use the decomposition (5.20) and a test function $\phi(x,t) \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ to conclude:

$$\begin{aligned} &- \int_{\mathbb{R}^+ \times \mathbb{R}^d} \overline{T}_1 \, \phi_t \, dx \, dt - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \nabla \phi \overline{T}_1 \, dx \, dt \\ &- \int_{\mathbb{R}^+ \times \mathbb{R}^d} \nabla \phi \cdot \int_Y \mathcal{K} \nabla_y T_2 \, dx \, dt \, dy = - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \nabla \phi \cdot \nabla T_0 \, dx \, dt \end{aligned} \tag{5.32}$$

Now we choose the test function $\phi^\delta = \delta \, \phi_1(\mathbf{x}, \frac{\mathbf{x}}{\delta}, t)$ with $\phi_1(\mathbf{x}, \mathbf{y}, t) \in J_p$. For the two scale limit of the second term in (5.19) we have:

$$\begin{aligned} - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \nabla \phi^\delta T^{1,\delta} \, dx \, dt &= - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \nabla_y \phi_1(\mathbf{x}, \mathbf{y} = \frac{\mathbf{x}}{\delta}, t) T^{1,\delta} \, dx \, dt \\ &\to - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \left(\int_Y \nabla_y \phi_1(\mathbf{x}, y, t) \, T_1(\mathbf{x}, \mathbf{y}, t) \, dy \right) dx \, dt \\ &= - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \left(\int_Y \nabla_y \phi_1(\mathbf{x}, y, t) \, \tilde{T}_1(\mathbf{x}, \mathbf{y}, t) \, dy \right) dx \, dt \end{aligned}$$

The limits of the other terms in (5.19) are the same as before. We finally get:

$$\begin{aligned} - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \int_Y \nabla_y \phi_1 \tilde{T}_1 \, dx \, dt \, dy &- \int_{\mathbb{R}^+ \times \mathbb{R}^d} \int_Y \nabla_y \phi_1 \cdot \mathcal{K} \nabla_y T_2 \, dx \, dt \, dy \\ &= - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \int_Y \phi_1 \mathbf{v} \cdot \nabla T_0 \, dx \, dt \, dy \end{aligned} \tag{5.33}$$

So far we have obtained two equations for the three unknowns $\overline{T}_1$, $\tilde{T}_1$, T_2. We need another equation to complete the system. To derive this equation we choose a test function $\phi^\delta = \delta^{1+\alpha} \, \phi_2(\mathbf{x}, \frac{\mathbf{x}}{\delta}, t)$, $\phi_2(\mathbf{x}, \mathbf{y}, t) \in J_p$. All terms in (5.19) disappear in the limit $\delta \to 0$ apart from the second term:

$$\begin{aligned} - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \nabla \phi^\delta T^{1,\delta} \, dx \, dt &= \delta^{1+\alpha} \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \phi_2 \cdot \nabla T^{1,\delta} \, dx \, dt \\ &\to \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \left(\int_Y \phi_2(\mathbf{x}, y, t) \, \nabla_y T_2(\mathbf{x}, \mathbf{y}, t) \, dy \right) dx \, dt \end{aligned}$$

Consequently, we obtain the equation:

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \left(\int_Y \phi_2(\mathbf{x}, y, t) \, \nabla_y T_2(\mathbf{x}, \mathbf{y}, t) \, dy \right) dx \, dt = 0 \tag{5.34}$$

Example

The system of equations (5.32), (5.33) and (5.34) is the system of homogenized equations. We now see that in general we will not be able to decouple these three equations, as we did before for $\alpha \in (-1, 0]$. In particular, equation (5.34) does not imply in general that $T_2 = 0$. Let us present an example when this is possible. We consider the problem in two dimensions with a constant mean flow $\mathbf{V} = (V_1, V_2)$. Let us first assume that $\frac{V_1}{V_2} \notin \mathbb{Q}$, which of course implies that at least one of the two components of the mean flow is irrational. Then equation (5.34) has only the trivial solution and we get $T_2 = T_2(\mathbf{x}, t)$. Then, equation (5.33) reduces to:

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \int_Y \nabla_y \phi \tilde{T}_1 \, dx\, dt\, dy \; = - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \int_Y \phi_1 \mathbf{v} \cdot \nabla T_0 \, dx\, dt\, dy \tag{5.35}$$

which is a well posed equation. In this case we have no enhancement in the diffusivity and the homogenized equation for $\overline{T}_1$ is:

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} \overline{T}_1 \, \phi_t \, dx\, dt \; + \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{V} \cdot \nabla \phi \overline{T}_1 \, dx\, dt \; = - \int_{\mathbb{R}^+ \times \mathbb{R}^d} \nabla \phi \cdot \nabla T_0 \, dx\, dt \tag{5.36}$$

which is the weak formulation of the inhomogeneous transport equation for $\overline{T}_1$:

$$\frac{\partial \overline{T}_1}{\partial t} + \mathbf{V}(\mathbf{x}) \cdot \nabla \overline{T}_1 = \Delta T_0 \tag{5.37a}$$

$$\overline{T}_1(\mathbf{x}, t = 0) = 0 \tag{5.37b}$$

Remarks Concerning the Well Posedness of the Homogenized Equations

The arguments presented so far show that the functions $(\overline{T}_1(\mathbf{x}, t), \tilde{T}_1(\mathbf{x}, t, \mathbf{y}))$, $(T_1(\mathbf{x}, t), T_2(\mathbf{x}, t, \mathbf{y}))$ and $(\overline{T}_1(\mathbf{x}, t), \tilde{T}_1(\mathbf{x}, t, \mathbf{y}), T_2(\mathbf{x}, t, \mathbf{y}))$ are solutions of the systems of equations (5.21, 5.22), (5.27, 5.29) and (5.32, 5.33, 5.34) for $\alpha = 0$, $\alpha \in (-1, 0)$, and $\alpha \in (0, 1)$, respectively. Let us now discuss the uniqueness of solutions to these equations (well-posedness of the transport equation for T_0 is clear).

We start with the equations for the parameter range $\alpha \in (-1, 0)$. We assume that there exists a second set of solutions $(T_1^1(\mathbf{x}, t), T_2^1(\mathbf{x}, t, \mathbf{y}))$ that satisfy equations (5.27, 5.29). The difference $\widehat{T}_2 = T_2 - T_2^1$ satisfies the homogeneous equation:

$$\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \nabla_y\phi_1\cdot\mathcal{K}\nabla_y\widehat{T}_2\,dx\,dt\,dy = 0 \tag{5.38}$$

We choose $\phi_1 = \widehat{T}_2$ and use the antisymmetry of $\mathcal{H}$ to obtain:

$$\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y |\nabla_y\widehat{T}_2|^2\,dx\,dt\,dy = 0 \tag{5.39}$$

Since now $\widehat{T}_2 \in L^2_{loc}(\mathbb{R}^+\times\mathbb{R}^d; H^1(Y)/\mathbb{R})$ we conclude that $\widehat{T}_2 \equiv 0$ and consequently the solution of (5.29) is unique. Now use ((5.27) to obtain a homogeneous equation for the difference $\widehat{T}_1 = T_1 - T_1^1$:

$$\int_{\mathbb{R}^+\times\mathbb{R}^d}\widehat{T}_1\,\phi_t\,dx\,dt\; + \int_{\mathbb{R}^+\times\mathbb{R}^d}\mathbf{V}\cdot\nabla\phi\widehat{T}_1\,dx\,dt = 0 \tag{5.40}$$

The choice $\phi = \widehat{T}_1$, together with the incompressibility of the mean flow, immediately give $\widehat{T}_1 \equiv 0$ and well-posedness is proved. The case $\alpha = 0$ can be treated similarly: the incompressibility of the mean flow implies that (5.39) still holds, with $\widehat{T}_2$ replaced by $\widehat{\tilde{T}}_1$.

We now turn to the case $\alpha \in (0,1)$. We combine equations (5.33, 5.34) to obtain:

$$\begin{aligned} & - \int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \Big(\mathbf{V}\cdot\Big(\nabla_y\phi_1\tilde{T}_1 + \nabla_y\phi_2 T_2\Big) \;+\; \nabla_y\phi_1\cdot\mathcal{K}\nabla_y T_2\Big)\;dx\,dt\,dy \\ & = \;-\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \phi_1\mathbf{v}\cdot\nabla T_0\,dx\,dt\,dy \end{aligned} \tag{5.41}$$

We assume again that there exists another set of solutions $(\tilde{T}_1^1(\mathbf{x},t,\mathbf{y}), T_2^1(\mathbf{x},t,\mathbf{y}))$ and obtain a homogeneous equation for the differences $\widehat{\tilde{T}}_1 = \tilde{T}_1 - \tilde{T}_1^1$ and $\widehat{T}_2 = T_2 - T_2^1$:

$$\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \Big(\mathbf{V}\cdot\Big(\nabla_y\phi_1\widehat{\tilde{T}}_1 + \nabla_y\phi_2\widehat{T}_2\Big) \;+\; \nabla_y\phi_1\cdot\mathcal{K}\nabla_y\widehat{T}_2\Big)\;dx\,dt\,dy = 0 \tag{5.42}$$

We choose $\phi_1 = \widehat{T}_2$ and $\phi_2 = \widehat{\tilde{T}}_1$ to obtain:

$$\int_{\mathbb{R}^+\times\mathbb{R}^d}\int_Y \nabla_y\widehat{T}_2\cdot\mathcal{K}\nabla_y\widehat{T}_2\,dx\,dt\,dy = 0 \tag{5.43}$$

Proceeding now as before we conclude that $\widehat{T}_2 \equiv 0$ and consequently T_2 is determined uniquely. Using this information we can immediately show that equation (5.33) uniquely determines $\overline{T}_1$.

On the other hand, this system of equations does not enable us to uniquely determine $\tilde{T}_1$. Indeed, from (5.42) and since $\widehat{T}_2 \equiv 0$ we obtain:

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} \int_Y \mathbf{V} \cdot \nabla_y \phi_1 \widehat{\tilde{T}}_1 \, dx \, dt \, dy = 0 \tag{5.44}$$

However, from this equation we cannot conclude in general that $\widehat{\tilde{T}}_1$, apart from the specific case described in the example above. For the general case where $\mathbf{V}(\mathbf{x})$ is a smooth, incompressible vector field, a more detailed analysis is required.

5.4 Remarks on the Homogenization Theorem

Let us finish this chapter with a few remarks on the homogenized equations derived in the previous section, together with a discussion of various extensions of the homogenization theorem. First, let us remark that the results of the previous section, for $\alpha \in (-1, 0]$ can be rephrased as follows: the solution to the original advection diffusion equation is of the form

$$T^\delta = T_0 + \delta^{1+\alpha} T_1 + o(\delta^{1+\alpha}) \tag{5.45}$$

verifying a posteriori the multiple scale techniques that were used in chapters 2 and 3. The technique that we used can be continued in order to obtain equations for the higher order terms in the expansion. Moreover, the rescaling 5.12 that we used the analogous rescalings for the higher order terms in the expansion can become a part of the unknowns: only one rescaling will lead us to a nontrivial and non singular two-scale limit. We mention that a similar technique, the method of *development in* Γ*-convergence*, has been introduced to treat singularly perturbed variational problems.

We have argued that the method of two-scale convergence provides us with a systematic way of obtaining equations for the terms of various orders in the multiple scales expansion. From this point of view, it seems that two-scale convergence is the method of choice since it enables us to obtain the effective equations and to verify them in one step. However, as we have already seen in chapter 3 and is also discussed elsewhere, for example [18, 103], the creation of initial and boundary layers might affect the higher order terms in the expansion with respect to δ. In chapter 3 we had to take care of the initial layer in order to be able to obtain sharp estimates for the error term in the expansion. On the other hand, there does not seem to exist an a priori way of treating the initial and boundary layers within the framework of two-scale convergence and this will affect the error estimates when trying to obtain a single equation for the first two terms in the expansion.

In the previous section we studied the problem for steady velocity fields. However this restriction is not necessary and the results can be easily extended to cover spatiotemporally dependent velocity fields. The definition and basic compactness theorems of two-scale convergence where extended to cover the case where there are oscillations in both space and time with different speeds of oscillation in the time variable by Holmbom in [53]. For example, the analogue of theorem (5.1) is:

THEOREM 5.4 *Let T^δ be a uniformly bounded sequence in $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)$. Then there exists a function $T_0(\mathbf{x}, t, \mathbf{y}, \tau)$ in $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d \times Y \times [0,1])$ and a subsequence, still denoted by T^δ, such that for any test function $\psi(\mathbf{x}, t, \mathbf{y}, \tau)$ and for all $\gamma > 0$*

$$\lim_{\delta \to 0} \int_{\mathbb{R}^+ \times \mathbb{R}^d} T^\delta(\mathbf{x}, t) \psi\left(\mathbf{x}, t, \frac{\mathbf{x}}{\delta}, \frac{t}{\delta^\gamma}\right) dxdt = \int_{\mathbb{R}^+ \times \mathbb{R}^d} \int_0^1 \int_Y T_0(\mathbf{x}, t, \mathbf{y}, s) \psi(\mathbf{x}, t, \mathbf{y}, s)\, dydxdtds \tag{5.46}$$

The class of test functions is taken to be that of functions $\psi(\mathbf{x}, t, \mathbf{y}, \tau)$ which are smooth with compact support and periodic in both $\mathbf{y}$ and τ with period 1. Holmbom then used his results to study the problem of homogenization for linear parabolic equations with coefficients oscillating in both space and time (this corresponds to the case where the mean flow is $O(\delta)$ weak compared to the fluctuations). Using the similar techniques to those used in the previous section we can derive the effective equations for the first two terms in the expansion as well as the cell problems. Using test functions of the form $\phi^\delta = \delta^{1-\alpha} \phi_1\left(\mathbf{x}, t, \frac{\mathbf{x}}{\delta}, \frac{t}{\delta^\gamma}\right)$ will enable us to show that the structure of the cell problem depends upon the value of γ. For brevity we shall omit the details.

In the previous section we assumed that the initial conditions are smooth, independent of δ. However, this is not necessary. We could consider, for example, the initial conditions to be $T^\delta_{in}(\mathbf{x}) \in L^2(\mathbb{R}^2)$ such that $T^\delta_i n \to T_{in}$ strongly in $L^2(\mathbb{R}^2)$. The homogenization result would be exactly the same in this case.

In this work we have been concerned with the effect of a spatiotemporally dependent mean flow on the homogenization for advection-diffusion equations. Certain of the results presented so far can be extended to the case of advection-reaction-diffusion equations, provided that the nonlinearity is not too strong. For example, we can use two-scale convergence to study the problem of homogenization for the following system of an advection-diffusion equation coupled with an advection-reaction-diffusion equation through a KPP nonlinearity, that was studied by Majda and Souganidis in [72]:

$$\frac{\partial T^\delta(\mathbf{x},t)}{\partial t} + \left(\mathbf{V}(\mathbf{x}) + \frac{1}{\delta}\mathbf{v}(\frac{\mathbf{x}}{\delta})\right) \cdot \nabla T^\delta(\mathbf{x},t) = \mathcal{K}\Delta T^\delta(\mathbf{x},t) \;\; in\, \Omega \times (0,T) \tag{5.47a}$$

$$\frac{\partial Z^\delta(\mathbf{x},t)}{\partial t} + \left(\mathbf{V}(\mathbf{x},t) + \frac{1}{\delta}\mathbf{v}(\frac{\mathbf{x}}{\delta})\right) \cdot \nabla Z^\delta(\mathbf{x},t) = \mathcal{K}\Delta Z^\delta(\mathbf{x},t) + Z^\delta \left(T^\delta - Z^\delta\right) \;\; in\, \Omega \times (0,T) \tag{5.47b}$$

$$T^\delta(\mathbf{x},t) = 0 \;\; on\, \partial\Omega \times (0,T), \tag{5.47c}$$

$$Z^\delta(\mathbf{x},t) = 0 \;\; on\, \partial\Omega \times \{t=0\}, \tag{5.47d}$$

$$T^\delta(\mathbf{x},t) = T_0(\mathbf{x}) \;\; on\, \Omega \times \{t=0\}, \tag{5.47e}$$

$$Z^\delta(\mathbf{x},t) = Z_0(\mathbf{x}) \;\; on\, \Omega \times \{t=0\}, \tag{5.47f}$$

(for simplicity, consider the problem in a bounded domain with smooth boundary and with Dirichlet boundary conditions but this is not necessary. Moreover, more general types of nonlinearities can be considered). Standard energy estimates enable us to obtain the uniform bounds $||T^\delta(x,t)||_{L^2(0,T;H_0^1(\Omega))} \leq C$, $||Z^\delta(x,t)||_{L^2(0,T;H_0^1(\Omega))} \leq C$ (we have to make certain assumptions concerning the initial conditions of Z^δ). Using these estimates we can pass to the two scale limit to obtain the following system of homogenized equations:

$$\frac{\partial T(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x}) \cdot \nabla T(\mathbf{x},t) = \mathcal{K}\Delta T(\mathbf{x},t) + \nabla \cdot (\mathcal{K}_T \cdot \nabla T) \quad in\, \Omega \times (0,T) \tag{5.48a}$$

$$\frac{\partial Z(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x}) \cdot \nabla Z(\mathbf{x},t) = \mathcal{K}\Delta Z(\mathbf{x},t) + \nabla \cdot (\mathcal{K}_T \cdot \nabla Z) + Z\,(T - Z) \quad in\, \Omega \times (0,T) \tag{5.48b}$$

$$T(\mathbf{x},t) = 0 \quad on\, \partial\Omega \times (0,T), \tag{5.48c}$$

$$Z(\mathbf{x},t) = 0 \quad on\, \Omega \times \{t = 0\}, \tag{5.48d}$$

$$T(\mathbf{x},t) = T_0(\mathbf{x}) \quad on\, \Omega \times \{t = 0\}, \tag{5.48e}$$

$$Z(\mathbf{x},t) = Z_0(\mathbf{x}) \quad on\, \Omega \times \{t = 0\}, \tag{5.48f}$$

The effective diffusivity tensor $\mathcal{K}_T$ is given by the standard formula $\mathcal{K}_T^{ij} = \langle \nabla\chi_i \cdot \nabla\chi_j \rangle$ and χ satisfies the elliptic cell problem $\mathbf{v} \cdot \nabla\chi - \mathcal{K}\Delta\chi = -\mathbf{v}$.

We emphasize the fact that in order to be able to use the method of two-scale convergence we need to have good uniform estimates of the solution to the PDE under investigation in the appropriate Sobolev spaces. However, for stronger nonlinearities (for example, assuming that the reaction is $O(\delta^{-1})$ compared to the diffusion in the example above or for Hamilton-Jacobi equations) these estimates are no longer available. For this kind of problems an "L^∞ variant" of the two-scale convergence method, the *perturbed test function method* that was developed by Evans, [38, 39], within the framework of viscosity solutions is the appropriate tool. This method was used in a series of papers by Majda and Souganidis, for example [71, 72], to study the problem of periodic homogenization for advection-reaction-diffusion equations.

CHAPTER 6
DISCUSSION AND CONCLUSIONS

In this work we studied the problem of periodic homogenization for advection–diffusion equations with mean flow. We showed that the effective transport of the passive tracers is greatly influenced by the presence and properties of the mean flow. In our work we identified three regimes, depending upon the relative strength of the mean flow relative to the fluctuations. We showed that for weak mean flows the effective equation is an advection–diffusion equation with a constant effective diffusion tensor whose value depends only upon the fluctuations. In the second regime where the mean flow is equal in strength with the fluctuations we showed that the effective diffusion tensor is a function of space and time with values depending upon both the mean flow and the slow modulations of the fluctuations. The spatial dependence of the effective diffusion tensor leads to an additional drift term which contributes to the large scale velocity with which the passive tracers are being transported. In the third regime where the mean flow is stronger than the fluctuations we rigrously showed that the homogenized transport is governed by a complicated system of equations where both the fast and the slow variables are present. One cannot in general decouple this system of equations. We showed through asymptotic analysis and numerical experiments that resonant enhanced diffusion phenomena might appear in this regime.

Further, we studied the symmetry properties of the effective diffusion tensor. We studied the dependence of the antisymmetric part of the effective diffusion tensor on the Peclet number for steady flows and we derived necessary and sufficient conditions that ensure the symmetry of $\mathcal{K}^*$.

The results of this work can be extended in various directions. First, it would be very interesting to study the same problem in the case where the fluctuations in the velocity are modeled as a short range correlated, sufficiently mixing random velocity field. This is a more natural setting in view of applications of the theory to the study of the transport of passive tracers in the atmosphere and ocean.

Another very interesting problem is that of periodic homogenization for compressible velocity fields. There have been very few works on this problem even in the case where the mean flow is absent [79, 114] and none, to our knowledge, to the case of compressible velocity fields with mean flow. The compressibility of the velocity field leads to new interesting physical phenomena such as the trapping of tracer particles and we think that this is a very exciting area of research.

From the point of view of physical oceanography it would also be very interesting to study the effects of stratification on the effective transport of passive tracers using ideas from homogenization theory. Since the physical properties of passive tracers are the same along lines of constant density [50], the natural coordinate system for this problem is that of *isopycnal* coordinates in which the role of the vertical coordinate and density are interchanged. In these new coordinates the velocity field is no longer incompressible and the study of homogenization for compressible velocity fields becomes relevant. The final goal of such a research project would be the systematic derivation of the so called Gent-McWilliams parametrization [46, 49] for some simple classes of fluid flows.

APPENDIX A
TIME INDEPENDENT VELOCITY FIELD WITH WEAK MEAN FLOW

A.1 Introduction

In chapter 5 we derived rigorously the effective equations for the first two terms in the multiple scales expansion using the method of two scale convergence. We restricted ourselves to the case $\alpha \in (-1, 1)$, α being the exponent that controls the strength of the fluctuations relative to the mean flow. The case where $\alpha = -1$, i.e. the case where the mean flow is $O(\delta)$-weak compared to the fluctuations can be treated using the same method: the fact that diffusion is $O(1)$ enables us to obtain uniform estimates in $L^2_{loc}(\mathbb{R}^+; H^1_{loc}(\mathbb{R}^d))$ and we can pass to the two scale limit to obtain an advection-diffusion equation for the first term in the expansion where the diffusion is always enhanced. Thus, in this case we don't have to derive equations for higher order terms in the expansion, since at the length and time scales where the homogenization theorem is valid advection due to the mean flow and enhancement in the diffusion due to the fluctuations are balanced.

In this appendix we shall present an alternative proof of the homogenization theorem for weak mean flows using the Trotter-Kato theorem, [89, pp. 87-88], [99, p. 35] [115, pp.269-272], from the theory of analytic semigroups. In the next section we present some properties of the fluctuating part of the velocity field and, in particular, we construct the stream matrix. Then we prove the homogenization theorem. In the last section we present some further remarks and comments.

A.2 Construction of the Stream Matrix

Consider the following initial-boundary value problem for the advection–diffusion equation with homogeneous Dirichlet boundary conditions:

$$\frac{\partial T^\delta(\mathbf{x},t)}{\delta t} + \left(\mathbf{V}(\mathbf{x}) + \frac{1}{\delta}\mathbf{v}(\frac{\mathbf{x}}{\delta})\right) \cdot \nabla T^\delta(\mathbf{x},t) = \Delta T^\delta(\mathbf{x},t), \quad \text{in } \Omega \times \{t > 0\} \tag{A.1a}$$

$$T^{\delta}(\mathbf{x},t) = T_{in}(\mathbf{x}) \quad \text{on } \Omega \times \{t=0\} \tag{A.1b}$$

$$T^{\delta}(\mathbf{x},t) = 0, \quad \text{on } \partial\Omega \times \{t>0\} \tag{A.1c}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with smooth boundary. To ease the notation we have set $Pe_l = 1$. Both components of the velocity field are incompressible. The fluctuations are periodic with unit period:

$$\nabla \cdot \mathbf{V}(\mathbf{x}) = 0 \tag{A.2a}$$

$$\nabla_y \cdot \mathbf{v}(\mathbf{y}) = 0 \tag{A.2b}$$

$$\mathbf{v}(\mathbf{y}+\mathbf{e}_j) = \mathbf{v}(\mathbf{y}) \;\; j = 1,..,d \tag{A.2c}$$

$$\langle \mathbf{v}(\mathbf{y}) \rangle := \int_Y \mathbf{v}(\mathbf{y})\, \mathrm{d}y = \mathbf{0}, \tag{A.2d}$$

where $\mathbf{y} \equiv \frac{\mathbf{x}}{\delta}$, ∇_y denotes the gradient with respect to $\mathbf{y}$, $\{\mathbf{e}_j\}_{j=1}^d$ denotes the unit vector in the j^{th} coordinate direction. As in the previous chapter we shall use the notation $Y = [0,1]^d$. We assume that the velocity field is smooth enough to have a Fourier series expansion:

$$\mathbf{v}(\mathbf{y}) = \sum_{\mathbf{k}\neq\mathbf{0}} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} \mathbf{v}_{\mathbf{k}} \tag{A.3}$$

Since we have assumed that $\langle \mathbf{v}(\mathbf{y}) \rangle = 0$ the $\mathbf{k} = \mathbf{0}$ term in the Fourier series disappears. We have the following lemma (see, for example, [67, ch. 2]):

LEMMA A.1 *The velocity field satisfying (A.2) has the following property:*
$\mathbf{v}_{\mathbf{k}} \cdot \mathbf{k} = 0 \;\; \forall\, \mathbf{k} \in \mathbb{Z}^d$

Proof:

From the incompressibility condition we get:

$$\nabla_y \cdot \mathbf{v} = 2\pi i \sum_{\mathbf{k}\neq\mathbf{0}} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} \mathbf{v_k} \cdot \mathbf{k} = \mathbf{0} \Rightarrow \mathbf{v_k} \cdot \mathbf{k} = \mathbf{0} \;\; \forall\, \mathbf{k} \, \in \, \mathbb{Z}^d \tag{A.4}$$

Now we wish to construct the stream matrix. We write (A.3) componentwise:

$$v_p(\mathbf{y}) = \sum_{\mathbf{k}\neq\mathbf{0}} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} v_{\mathbf{k}}^p \;\; p = 1, .., d \tag{A.5}$$

We have:

LEMMA A.2 *There exist an antisymmetric matrix $\mathcal{H} = \mathcal{H}(\mathbf{y})$ such that*

$$v_q(\mathbf{y}) = -\sum_{p=1}^{d} \frac{\partial \mathcal{H}_{pq}}{\partial y_p}, \;\; q = 1..d \tag{A.6}$$

Moreover, $\langle \mathcal{H}_{pq} \rangle = 0$, $p, q = 1, .., d$.

Proof:

We define $\mathcal{H}$ as:

$$\mathcal{H}_{pq} := \frac{1}{2\pi i} \sum_{\mathbf{k}\neq 0} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} \frac{k_q v_{\mathbf{k}}^p - k_p v_{\mathbf{k}}^q}{\mathbf{k}^2} \tag{A.7}$$

We have:

$$\begin{aligned}
-\sum_{p=1}^{d} \frac{\partial \mathcal{H}_{pq}}{\partial y_p} &= -\sum_{p=1}^{d} \sum_{\mathbf{k}\neq 0} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} k_p \frac{k_q v_{\mathbf{k}}^p - k_p v_{\mathbf{k}}^q}{\mathbf{k}^2} \\
&= -\sum_{\mathbf{k}\neq 0} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} \sum_{p=1}^{d} k_p \frac{k_q v_{\mathbf{k}}^p - k_p v_{\mathbf{k}}^q}{\mathbf{k}^2} \\
&= -\sum_{\mathbf{k}\neq 0} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} \frac{k_q}{\mathbf{k}^2} \mathbf{k} \cdot \mathbf{v_k} + \sum_{\mathbf{k}\neq 0} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} v_{\mathbf{k}}^q \\
&= \sum_{\mathbf{k}\neq 0} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} v_{\mathbf{k}}^q \\
&= v_q(\mathbf{y}), \;\; q = 1, \ldots, d,
\end{aligned} \tag{A.8}$$

where we have used lemma A.1. The average of each component of the stream matrix is:

$$\begin{aligned} \langle \mathcal{H}_{pq} \rangle &= \int_Y \mathcal{H}_{pq}(\mathbf{y})\, \mathrm{d}y \\ &= \int_Y \frac{1}{2\pi i} \sum_{\mathbf{k}\neq 0} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} \frac{k_q v_{\mathbf{k}}^p - k_p v_{\mathbf{k}}^q}{\mathbf{k}^2}\, \mathrm{d}y \\ &= 0, \quad p,q = 1,..,d \end{aligned} \tag{A.9}$$

Now we observe that:

$$\mathbf{v}\left(\mathbf{y} = \frac{\mathbf{x}}{\delta}\right) = -\nabla_y \cdot \mathcal{H}\left(\mathbf{y} = \frac{\mathbf{x}}{\delta}\right) = -\delta\, \nabla \cdot \mathcal{H}\left(\frac{\mathbf{x}}{\delta}\right), \tag{A.10}$$

Hence, we can write the advection term due to the velocity fluctuations as:

$$\begin{aligned} \frac{1}{\delta}\mathbf{v}\left(\frac{\mathbf{x}}{\delta}\right) \cdot \nabla T^\delta(\mathbf{x},t) &= -\nabla \cdot \mathcal{H} \cdot \nabla T^\delta \\ &= -\nabla \cdot (\mathcal{H} \cdot \nabla T^\delta), \end{aligned} \tag{A.11}$$

since $\mathcal{H}$ is antisymmetric and the Hessian of T^δ is symmetric. Using the stream matrix we can rewrite the advection-diffusion equation in the following form:

$$\frac{\partial T^\delta(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x}) \cdot \nabla T^\delta(\mathbf{x},t) = \nabla \cdot (\mathcal{K}^\delta \nabla T^\delta(\mathbf{x},t)) \tag{A.12}$$

where:

$$\mathcal{K}^\delta := I + \mathcal{H}(\frac{\mathbf{x}}{\delta}) \tag{A.13}$$

Clearly, $\mathcal{K}^\delta$ is $\mathbf{y}-$periodic, positive definite but not symmetric. The positive definiteness of $\mathcal{K}^\delta$ implies that the differential operator $-\nabla(\mathcal{K}^\delta\nabla) + \mathbf{V}(\mathbf{x})\cdot\nabla$ is uniformly elliptic, [40, p. 294].

We are interested in studying the behavior of the solution of (A.12) in the limit $\delta \to 0$. In particular, we want to prove that T^δ converges, in the appropriate norm, to a function $\overline{T}$ which satisfies the following advection-diffusion equation:

$$\frac{\partial \overline{T}(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x}) \cdot \nabla \overline{T}(\mathbf{x},t) = \nabla \cdot (\mathcal{K}^* \cdot \nabla \overline{T}(\mathbf{x},t)) \tag{A.14}$$

with the same initial and boundary conditions. The effective diffusivity tensor $\mathcal{K}^*$ is given by the standard formula[14]:

$$\mathcal{K}^*_{ij} := \delta_{ij} + \langle \nabla \chi^i \cdot \nabla \chi^j \rangle \tag{A.15}$$

and $\boldsymbol{\chi}(\mathbf{y})$ is solution of the cell problem:

$$\mathbf{v}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \boldsymbol{\chi} - \Delta_{\mathbf{y}} \boldsymbol{\chi} = -\mathbf{v}(\mathbf{y}) \tag{A.16}$$

Alternative formulations of $\mathcal{K}^*$ and the cell problem are [14, pp. 15-17]:

$$\mathcal{K}^*_{ij} = \delta_{ij} - \langle \mathcal{K}^*_{ik} \frac{\partial \chi^j}{\partial y_k} \rangle \tag{A.17a}$$

$$\mathcal{K}_1 \, \chi^j = - \frac{\partial \mathcal{K}_{ij}}{\partial y_i}, \tag{A.17b}$$

where $\mathcal{K}_{ij}(\mathbf{y}) := \delta_{ij} + \mathcal{H}_{ij}(\mathbf{y})$, $\mathcal{K}_1 := -\frac{\partial}{\partial y_i}(\mathcal{K}_{ij}(\mathbf{y})\frac{\partial}{\partial y_j})$. We have adopted the summation convention. Now we are ready to prove the homogenization theorem.

A.3 The Homogenization Theorem

In this section we shall prove the homogenization theorem. Our method will be to use the Trotter-Kato theorem in order to reduce the problem to that of the homogenizing the corresponding elliptic equation. More precisely, we shall show that it is enough to prove strong convergence of the resolvents of the corresponding generators in $L^2(\Omega)$. Then we shall use Tartar's method of oscillating test functions to study the elliptic problem. We have the following:

THEOREM A.1 *Consider the initial boundary value problem for the advection-diffusion equation:*

$$\frac{\partial T^\delta(\mathbf{x},t)}{\partial t} + \mathbf{V}(\mathbf{x}) \cdot \nabla T^\delta(\mathbf{x},t) = \nabla \cdot \left(\mathcal{K}^\delta \nabla \cdot T^\delta(\mathbf{x},t) \right), \quad in\ \Omega \times \{t > 0\} \tag{A.18a}$$

[14] In fact, this is the symmetric part of the effective diffusivity. However, since $\mathcal{K}^*$ is constant only the symmetric part is important for the homogenized equation

$$T^{\delta}(\mathbf{x},t) = T_{in}(\mathbf{x}), \quad on\; \Omega \times \{t=0\} \tag{A.18b}$$

$$T^{\delta}(\mathbf{x},t) = 0, \quad in\; \partial\Omega \times \{t>0\}, \tag{A.18c}$$

where Ω is an open bounded domain of R^d with smooth boundary. Assume that $T_{in}(\mathbf{x}) \in L^2(\Omega)$, $\mathbf{V}(\mathbf{x}) \in (L^{\infty}(\Omega, R^d))^d, \mathcal{H}(\mathbf{y}) \in (L^{\infty}(Y))^d$. Then the solution $T^{\delta}(\mathbf{x},t)$ of (A.18) converges to the solution $\overline{T}(\mathbf{x},t)$ of (A.14) with the same initial and boundary conditions. The effective diffusion tensor is given by formulas (A.15), (A.16). The convergence is strong in $L^2(\Omega)$ uniformly in every finite interval of $t \geq 0$: $T^{\delta}(\mathbf{x},t) \to \overline{T}(\mathbf{x},t)$ strongly in $L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$.

Proof:

First we prove that the operator

$$A^{\delta} := \frac{\partial}{\partial x_i}\left(\mathcal{K}^{\delta}_{ij}\frac{\partial}{\partial x_j}\right) - V^j\frac{\partial}{\partial x_j} \tag{A.19}$$

generates a contraction semigroup in $L^2(\Omega)$. The domain of definition of A^{δ} is taken to be:

$$D(A^{\delta}) := H^1_0(\Omega) \cap H^2(\Omega) \tag{A.20}$$

$D(A^{\delta})$ is dense in $L^2(\Omega)$. Moreover, the operator A^{δ} is dissipative:

$$\begin{aligned}
(\mathcal{K}^{\delta}u, u)_{L^2(\Omega)} &= \int_{\Omega}\frac{\partial}{\partial x_i}\mathcal{K}^{\delta}_{ij}\frac{\partial u}{\partial x_j}u\,\mathrm{d}x - \int_{\Omega}V^j\frac{\partial u}{\partial x_j}u\,\mathrm{d}x \\
&= -\int_{\Omega}\mathcal{K}^{\delta}_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\,\mathrm{d}x \\
&\leq -||\nabla u||^2_{L^2(\Omega)} \\
&\leq 0 \quad \forall\, u \in\; H^1_0(\Omega) \cap H^2(\Omega)
\end{aligned} \tag{A.21}$$

The second integral vanishes on account of the incompressibility of the mean flow:

$$\begin{aligned}
\int_{\Omega}V^j\frac{\partial u}{\partial x_j}u\,\mathrm{d}x &= \int_{\Omega}\frac{\partial(V^j u)}{\partial x_j}u\,\mathrm{d}x \\
&= -\int_{\Omega}V^j u\frac{\partial u}{\partial x_j}\,\mathrm{d}x
\end{aligned} \tag{A.22}$$

Moreover, we have that $R(I - A^\delta) = L^2(\Omega)$. To see this, we have to prove that the elliptic problem

$$(I - A^\delta)u^\delta = f, \quad x \in \Omega \tag{A.23a}$$

$$u^\delta = 0, \quad x \in \partial\Omega \tag{A.23b}$$

has a unique weak solution for every $f \in L^2(\Omega)$. We have to verify that the bilinear form $B^\delta[u, v] : H_0^1(\Omega) \times H_0^1(\Omega) \mapsto \mathbb{R}$

$$B^\delta[u, v] := \int_\Omega \mathcal{K}_{ij}^\delta \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x + \int_\Omega V^j \frac{\partial u}{\partial x_j} v \, \mathrm{d}x + \int_\Omega uv \, \mathrm{d}x \tag{A.24}$$

satisfies the conditions of the Lax-Milgram lemma. We readily check that $B^\delta[u, v]$ is continuous and coercive :

$$\begin{aligned} |B^\delta[u, v]| &= \Big| \int_\Omega \mathcal{K}_{ij}^\delta \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x + \int_\Omega V^j \frac{\partial u}{\partial x_j} v \, \mathrm{d}x + \int_\Omega uv \, \mathrm{d}x \Big| \\ &\leq ||\mathcal{K}_{ij}^\delta||_{L^\infty} \int_\Omega |\nabla u||\nabla v| \, \mathrm{d}x + ||V^j||_{L^\infty} \int_\Omega |\nabla u||v| \, \mathrm{d}x + \int_\Omega |u| \, |v| \, \mathrm{d}x \\ &\leq C||u||_{H_0^1(\Omega)}||v||_{H_0^1(\Omega)} \end{aligned} \tag{A.25}$$

$$\begin{aligned} B^\delta[u, u] &= \int_\Omega \mathcal{K}_{ij}^\delta \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, \mathrm{d}x + \int_\Omega V^j \frac{\partial u}{\partial x_j} u \, \mathrm{d}x + \int_\Omega u^2 \, \mathrm{d}x \\ &= \int_\Omega \delta_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \int_\Omega u^2 \, \mathrm{d}x \\ &\geq \left[\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_\Omega u^2 \, \mathrm{d}x \right] \\ &:= C||u||_{H_0^1(\Omega)}^2 \end{aligned} \tag{A.26}$$

Hence, from the Lumer-Phillips theorem ([115, p. 250], [89, p. 14]) we deduce that A^δ generates a contraction semigroup in $L^2(\Omega)$.

Let now A denote the differential operator associated with the homogenized equation:

$$A := \frac{\partial}{\partial x_i} \left(\mathcal{K}_{ij}^* \frac{\partial}{\partial x_j} \right) - V^j \frac{\partial}{\partial x_j} \tag{A.27}$$

According to the Trotter-Kato theorem (, [89, pp. 87-88], [99, p. 35], [115, pp.269-272]), in order to prove that the solution of (A.12) converges to that of (A.14) as $\delta \to 0$, it is enough to have strong convergence in $L^2(\Omega)$ of the resolvent of A^δ to that of A:

$$(I - A^\delta)^{-1} f \to (I - A)^{-1} f \qquad \text{strongly in } L^2(\Omega) \qquad \forall\, f \in L^2(\Omega) \tag{A.28}$$

In other words, we want strong convergence in $L^2(\Omega)$ of the weak solution of (A.23) to the weak solution of:

$$(I - A)u = f, \ \ x \in \Omega \tag{A.29a}$$

$$u = 0, \ \ x \in \partial\Omega \tag{A.29b}$$

for every $f \in L^2(\Omega)$. This result is proven in theorem 2. Hence, we finally get:

$$e^{A^\delta t} T_0(\mathbf{x}) \to e^{A\,t} T_0(\mathbf{x}) \ \ \forall\, T_0(\mathbf{x}) \in L^2(\Omega) \tag{A.30}$$

uniformly in each finite interval of $t \geq 0$.

Now we want to prove strong convergence of the resolvents in $L^2(\Omega)$. We shall accomplish this using Tartar's method of oscillating test functions, see for example [28, ch. 8].

THEOREM A.2 *The weak solution u^δ of (A.23) converges strongly in $L^2(\Omega)$ to the weak solution u of (A.29).*

Proof:

The weak formulation of (A.23) is:

$$B^\delta[u^\delta, v] = (f, v) \ \ \forall\, v \ \in \ H_0^1(\Omega) \tag{A.31}$$

The weak formulation of (A.29) is:

$$B[u, v] = (f, v) \ \ \forall\, v \ \in \ H_0^1(\Omega) \tag{A.32}$$

where $B[u,v]$ is defined like $B^\delta[u,v]$ with $\mathcal{K}_{ij}^\delta$ replaced by $\mathcal{K}_{ij}^*$ and $(\cdot,\cdot)$ stands for the $L^2(\Omega)$ inner product. First we prove that u^δ is bounded in $H_0^1(\Omega)$. We have:

$$\begin{aligned} B^\delta[u^\delta,u^\delta] &= (f,u^\delta) \\ &\leq ||f||_{L^2(\Omega)}||u^\delta||_{L^2(\Omega)} \\ &\leq ||f||_{L^2(\Omega)}||u^\delta||_{H_0^1(\Omega)} \end{aligned} \tag{A.33}$$

On the other hand, from the continuity of $B^\delta[u^\delta,u^\delta]$ we can get a lower bound for $B^\delta[u^\delta,u^\delta]$:

$$B^\delta[u^\delta,u^\delta] \leq C||u^\delta||^2_{H_0^1(\Omega)} \tag{A.34}$$

Combining (A.33) and (A.34) we obtain:

$$||u^\delta||_{H_0^1(\Omega)} \leq C_1 \tag{A.35}$$

Let us now define:

$$\xi_i^\delta := \mathcal{K}_{ij}^\delta \frac{\partial u^\delta}{\partial x_j} \tag{A.36}$$

Now $\mathcal{K}_{ij}^\delta \in L^\infty(R^n)$, which together with (A.35) readily gives:

$$||\xi_i^\delta||_{L^2(\Omega)} \leq C_2 \tag{A.37}$$

The above inequalities imply that we can extract subsequences, still denoted by u^δ, ξ_i^δ such that:

$$u^\delta \rightharpoonup u \qquad \text{weakly in } H_0^1(\Omega) \tag{A.38a}$$

$$\xi_i^\delta \rightharpoonup \xi_i \qquad \text{weakly in } L^2(\Omega) \tag{A.38b}$$

Using the definition of $B^\delta[u^\delta, v]$ and ξ_i^δ we can rewrite (A.31) in the form:

$$(\xi_i^\delta, \frac{\partial v}{\partial x_j}) + (V^j \frac{\partial u^\delta}{\partial x_j}, v) + (u^\delta, v) = (f, v) \;\; \forall v \in H_0^1(\Omega) \tag{A.39}$$

Using (A.38) we can pass to the limit $\delta \to 0$ and obtain:

$$(\xi_i, \frac{\partial v}{\partial x_j}) + (V^j \frac{\partial u}{\partial x_j}, v) + (u, v) = (f, v) \;\; \forall v \in H_0^1(\Omega) \tag{A.40}$$

Our aim now is to prove that u, the limit of u^δ, satisfies (A.29). We shall accomplish this by obtaining an expression for ξ_i using the adjoint problem. The adjoint operator of $\mathcal{K}_1$ is:

$$\mathcal{K}_1^* := -\frac{\partial}{\partial y_i}\left(\mathcal{K}_{ji}\frac{\partial}{\partial y_j}\right) \tag{A.41}$$

Let now w be the weak solution of:

$$\mathcal{K}_1^* w = 0, \tag{A.42}$$

such that $w - P$ is $\mathbf{y}-$periodic, $P(\mathbf{y})$ being a homogeneous polynomial of degree 1. Further, define

$$w - P = -\hat{\chi} \tag{A.43}$$

Consequently, $\hat{\chi}$ satisfies the equation

$$\mathcal{K}_1^* \hat{\chi} = \mathcal{K}_1^* P \tag{A.44}$$

We require that $\hat{\chi}$ is $y-$periodic. Further, we define w^δ as follows:

$$\begin{aligned} w^\delta &= \delta w(\frac{x}{\delta}) \\ &= P(x) - \delta\hat{\chi}(\frac{x}{\delta}) \end{aligned} \tag{A.45}$$

Now w^δ satisfies the following problem:

$$(\mathcal{K}^\delta)^* w^\delta = 0 \tag{A.46}$$

where:

$$\mathcal{K}^\delta := -\frac{\partial}{\partial x_i}\left(\mathcal{K}^\delta_{ij}\frac{\partial}{\partial x_j}\right) \tag{A.47}$$

and $(K^\delta)^*$ is its adjoint. We choose a test function of the form $v = w^\delta\phi$ with $\phi \in C_0^\infty(\Omega)$. We use this test function in (A.31). We also take the $L^2(\Omega)$ inner product of (A.46) with ϕu^δ. Subtracting the resulting expressions we obtain:

$$B^\delta[u^\delta, \phi\, w^\delta] - (u^\delta\phi, (\mathcal{K}^\delta)^* w^\delta) = (f, \phi\, w^\delta) \tag{A.48}$$

The above equation implies:

$$(\xi_i^\delta, \frac{\partial \phi}{\partial x_i} w^\delta) - \int_\Omega \mathcal{K}^\delta_{ij} u^\delta \frac{\partial \phi}{\partial x_j}\frac{\partial w^\delta}{\partial x_i}\,\mathrm{d}x + \int_\Omega V^j \frac{\partial u^\delta}{\partial x_j}(\phi\, w^\delta)\,\mathrm{d}x + \int_\Omega u^\delta\,\phi\, w^\delta\,\mathrm{d}x = (f, \phi\, w^\delta) \tag{A.49}$$

Now we want to take the limit $\delta \to 0$. The first term on the left hand side of (A.49) converges to $(\xi_i, \frac{\partial \phi}{\partial x_i} P)$ as the product of a weakly convergent and a strongly convergent sequence. The third and fourth terms as well as the right hand side converge since both u^δ and w^δ are strongly convergent sequences in $L^2(\Omega)$. For the second term on the left hand side of (A.49) we proceed as follows. First we define the following operator:

$$(f(x))^\delta := f(\frac{x}{\delta}) \tag{A.50}$$

We have:

$$\int_\Omega \mathcal{K}^\delta_{ij} u^\delta \frac{\partial \phi}{\partial x_j}\frac{\partial w^\delta}{\partial x_i}\,\mathrm{d}x = \int_\Omega (\mathcal{K}_{ij}\frac{\partial w}{\partial y_i})^\delta u^\delta \frac{\partial \phi}{\partial x_j}\,\mathrm{d}x \tag{A.51}$$

Notice that $u^\delta \frac{\partial \phi}{\partial x_j} \in L^1(\Omega)$ and consequently the integral in (A.51) is well defined. Moreover, we have:

$$\int_\Omega u^\delta \frac{\partial \phi}{\partial x_j}\,\mathrm{d}x \to \int_\Omega u \frac{\partial \phi}{\partial x_j}\,\mathrm{d}x \tag{A.52a}$$

$$(\mathcal{K}_{ij}\frac{\partial w}{\partial x_i})^\delta \rightharpoonup \langle \mathcal{K}_{ij}\frac{\partial w}{\partial y_i}\rangle \qquad \text{weak-* in } L^\infty(\Omega) \tag{A.52b}$$

For the proof of (A.52b) we refer to [33, ch.2, theorem 1.5]. We can easily see that $u^\delta \frac{\partial \phi}{\partial x_j}$ converges strongly in $L^1(\Omega)$. Thus, since the integrand in (A.51) consists of the product of a strongly convergent sequence in $L^1(\Omega)$ and a weakly-* convergent sequence in $L^\infty(\Omega)$, (A.52) implies that:

$$\int \mathcal{K}_{ij}^\delta u^\delta \frac{\partial \phi}{\partial x_j} \frac{\partial w^\delta}{\partial x_i} \,\mathrm{d}x \rightarrow \langle \mathcal{K}_{ij} \frac{\partial w}{\partial y_i} \rangle \int_\Omega u \frac{\partial \phi}{\partial x_j} \,\mathrm{d}x \tag{A.53}$$

In appendix B we present a detailed derivation of the above equation. From the above considerations we finally deduce that the limit of (A.49) as $\delta \rightarrow 0$ is:

$$(\xi_i, \frac{\partial \phi}{\partial x_i} P) - \langle \mathcal{K}_{ij} \frac{\partial w}{\partial y_i} \rangle \int_\Omega u \frac{\partial \phi}{\partial x_j} \,\mathrm{d}x + \int_\Omega V^j \frac{\partial u}{\partial x_j} \phi \, P \,\mathrm{d}x + \int_\Omega u \, \phi \, P \,\mathrm{d}x = (f, \phi P) \tag{A.54}$$

or,

$$(\xi_i, \frac{\partial \phi}{\partial x_i} P) - \langle \mathcal{K}_{ij} \frac{\partial w}{\partial y_i} \rangle (u, \frac{\partial \phi}{\partial x_j}) + (V^j \frac{\partial u}{\partial x_j}, \phi \, P) + (u, \phi \, P) = (f, \phi \, P) \tag{A.55}$$

From (A.40), for $v = \phi \, P$, we get:

$$(\xi_i, \frac{\partial (\phi \, P)}{\partial x_i}) + (V^j \frac{\partial u}{\partial x_j}, \phi \, P) + (u, \phi \, P) = (f, \phi \, P) \tag{A.56}$$

Combining the above two equations we obtain:

$$(\xi_i \frac{\partial P}{\partial x_i}, \phi) = \langle \mathcal{K}_{ij} \frac{\partial w}{\partial y_i} \rangle (\frac{\partial u}{\partial x_j}, \phi) \;\; \forall \phi \in C_0^\infty(\Omega) \tag{A.57}$$

Thus:

$$\xi_i \frac{\partial P}{\partial x_i} = \langle \mathcal{K}_{ij} \frac{\partial w}{\partial y_i} \rangle \frac{\partial u}{\partial x_j} \tag{A.58}$$

Now we choose $P(y) = y_i$. We define $w - y_i = -\hat{\chi}_i$ and consequently $\hat{\chi}_i$ satisfies the equation $\mathcal{K}_1^* \hat{\chi} = \mathcal{K}_1^* y_i$. Thus, (A.58) gives:

$$\xi_i = \langle \mathcal{K}_{kj}(y) \frac{\partial (y_i - \hat{\chi}^i)}{\partial y_k} \rangle \, \frac{\partial u}{\partial x_j} \tag{A.59}$$

Inserting now (A.59) into (A.40) we obtain:

$$(\langle \mathcal{K}_{ij} - \mathcal{K}_{kj}\frac{\partial \hat{\chi}^i}{\partial y_k}\rangle \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}) + (V^j \frac{\partial u}{\partial x_j}, v) + (u, v) = (f, v) \;\; \forall v \in H_0^1(\Omega) \tag{A.60}$$

Now we have to prove that u satisfies the homogenized equation, i.e. that (A.60) coincides with (A.32). Comparing (A.32) with (A.60) and using (A.17a) we conclude that we have to prove that:

$$\langle \mathcal{K}_{ik}\frac{\partial \chi^j}{\partial y_k}\rangle = \langle \mathcal{K}_{kj}\frac{\partial \hat{\chi}^i}{\partial y_k}\rangle \tag{A.61}$$

We multiply the adjoint cell problem $\mathcal{K}^*\hat{\chi}^i = \mathcal{K}^* y_i$ by χ^j and integrate over the unit box Y (this is just the $L^2(Y)$ inner product). We have:

$$\begin{aligned}(\chi^j, \mathcal{K}_1^*\hat{\chi}^i)_Y &= (\chi^j, \mathcal{K}_1^* y_i)_Y \\ &= -(\chi^j, \frac{\partial \mathcal{K}_{ik}}{\partial y_k})_Y \\ &= (\frac{\partial \chi^j}{\partial y_k}, \mathcal{K}_{ik})_Y \\ &= \langle \mathcal{K}_{ik}\frac{\partial \chi^j}{\partial y_k}\rangle \end{aligned} \tag{A.62}$$

Now we multiply (A.17b) by $\hat{\chi}_i$ and integrate over Y to obtain:

$$\begin{aligned}(\hat{\chi}^i, \mathcal{K}_1\chi^j)_{T^d} &= -(\hat{\chi}^i, \frac{\partial \mathcal{K}_{kj}}{\partial y_k})_Y \\ &= (\frac{\partial \hat{\chi}^j}{\partial y_k}, \mathcal{K}_{kj})_Y \\ &= \langle \mathcal{K}_{kj}\frac{\partial \hat{\chi}^i}{\partial y_k}\rangle \end{aligned} \tag{A.63}$$

Equations (A.62) and (A.63) now imply (A.61).

So far we have proven that $u^\delta \rightharpoonup u$ weakly in $H_0^1(\Omega)$, u being the solution of the homogenized equation. By the Rellich compactness theorem we know that embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact, which implies that $u^\delta \to u$ strongly in $L^2(\Omega)$. Now the proof is complete.

A.4 Comments

Let us first review the method that we used in order to prove the homogenization theorem in the previous section. First we used the Trotter-Kato theorem to reduce the problem to the proof of the strong convergence of the resolvents in $L^2(\Omega)$. Then we used the adjoint to the cell problem to build appropriate test functions that enabled us to pass to the limit and obtain the strong convergence.

This method seems to be more difficult to use than the two-scale convergence for various reasons: First, we had to work hard in order to construct the appropriate test functions, in contrast to the method of two-scale convergence where it was enough to consider test functions of the form $\phi^\delta \sim \phi_0(\mathbf{x}, t) + \delta\, \phi_1(\mathbf{x}, t, \frac{\mathbf{x}}{\delta})$. Moreover, it is more difficult to extend this method to the case of time dependent velocity fields. Finally, the above method cannot be immediately used to study homogenization for advection-reaction-diffusion equations, since the Trotter-Kato theorem is no longer valid.

On the other hand, this approach has certain advantages. First, we can use it to prove homogenization for non-periodic oscillations. In this case we will not have an explicit expression of the effective diffusion tensor in terms of the solution of the cell problem. However, we can still show that the solution of the corresponding elliptic problem converges strongly in $L^2(\Omega)$ to the solutions of an elliptic equation and that the diffusivity is always enhanced. We refer to [28, ch. 13] for results of this form. This in turn implies convergence of solutions for the parabolic problem.

More importantly, we can use this method to prove the homogenization theorem for random velocity fields with short range correlations which are sufficiently mixing. In fact, a variant of the techniques developed by Papanicolaou and Varadhan [87] can be used in order to define the appropriate test functions in the random setting. Then, the Trotter-Kato theorem gives the homogenization theorem for the evolution problem.

APPENDIX B
PROOF OF EQUATION (A.53)

We first prove the following lemma:

LEMMA B.1 *Let $g_\epsilon \rightharpoonup g$ weak-* in $L^\infty(\Omega)$ and let $f_\epsilon \to f$ strongly in $L^1(\Omega)$. Then :*

$$\int_\Omega f_\epsilon \, g_\epsilon \, \mathrm{d}x \to \int_\Omega f \, g \, \mathrm{d}x \tag{B.1}$$

Proof:

We have:

$$f_\epsilon \, g_\epsilon = -(f_\epsilon - f)\,(g_\epsilon - g) + f\,g - f\,g_\epsilon - f_\epsilon\,g \tag{B.2}$$

Thus:

$$\int_\Omega f_\epsilon \, g_\epsilon \, \mathrm{d}x = -\int_\Omega (f_\epsilon - f)\,(g_\epsilon - g)\,\mathrm{d}x + \int_\Omega f\,g\,\mathrm{d}x - \int_\Omega f\,g_\epsilon\,\mathrm{d}x - \int_\Omega f_\epsilon\,g\,\mathrm{d}x \tag{B.3}$$

We study each term in the right hand side of the (B.3) separately. We bound the first term as follows:

$$\begin{aligned} \int_\Omega (f_\epsilon - f)\,(g_\epsilon - g)\,\mathrm{d}x &\leq \int_\Omega |(f_\epsilon - f)|\,|(g_\epsilon - g)|\,\mathrm{d}x \\ &\leq ||f_\epsilon - f||_{L^1(\Omega)}\,||g_\epsilon - g||_{L^\infty(\Omega)} \\ &\to 0 \end{aligned} \tag{B.4}$$

Since g_ϵ is weakly-* convergent in $L^\infty(\Omega)$, the third term converges to $-\int_\Omega f\,g\,\mathrm{d}x$. f_ϵ is strongly convergent in $L^1(\Omega)$, hence the fourth term converges to $-\int_\Omega f\,g\,\mathrm{d}x$ as well. Consequently, we have:

$$-\int_\Omega (f_\epsilon - f)\,(g_\epsilon - g)\,\mathrm{d}x + \int_\Omega f\,g\,\mathrm{d}x - \int_\Omega f\,g_\epsilon\,\mathrm{d}x - \int_\Omega f_\epsilon\,g\,\mathrm{d}x \to \int_\Omega f\,g\,\mathrm{d}x, \tag{B.5}$$

and hence:

$$\int_\Omega f_\epsilon\, g_\epsilon \,\mathrm{d}x \to \int_\Omega f\, g \,\mathrm{d}x \tag{B.6}$$

This completes the proof of the lemma.

In our case we have that $u_\epsilon \to u$ strongly in $L^2(\Omega)$ and that $\phi \in C_0^\infty(\Omega)$, consequently $u_\epsilon \frac{\partial \phi}{\partial x_j} \in L^1(\Omega)$ and moreover $u_\epsilon\, \phi \to u\, \frac{\partial \phi}{\partial x_j}$ strongly in $L^1(\Omega)$. On the other hand, we know that $(\mathcal{K}_{ij} \frac{\partial w}{\partial x_i})^\epsilon \rightharpoonup < K_{ij} \frac{\partial w}{\partial y_i} >$ weak-* in $L^\infty(\Omega)$. Upon using the lemma, we get:

$$\begin{aligned} \int_\Omega \mathcal{K}_{ij}^\epsilon u^\epsilon \frac{\partial \phi}{\partial x_j} \frac{\partial w^\epsilon}{\partial x_i} \,\mathrm{d}x &\to \int_\Omega \langle \mathcal{K}_{ij} \frac{\partial w}{\partial y_i} \rangle u \frac{\partial \phi}{\partial x_j} \,\mathrm{d}x \\ &= \langle \mathcal{K}_{ij} \frac{\partial w}{\partial y_i} \rangle \int_\Omega u \frac{\partial \phi}{\partial x_j} \,\mathrm{d}x \end{aligned} \tag{B.7}$$

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