Itô versus Stratonovich white-noise limits for systems with inertia and colored multiplicative noise

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(Received 28 October 2003; published 29 September 2004)

We consider the dynamics of systems in the presence of inertia and colored multiplicative noise. We study the limit where the particle relaxation time and the correlation time of the noise both tend to zero. We show that the limiting equation for the particle position depends on the magnitude of the particle relaxation time relative to the noise correlation time. In particular, the limiting equation should be interpreted either in the Itô or Stratonovich sense, with a crossover occurring when the two fast-time scales are of comparable magnitude. At the crossover the limiting stochastic differential equation is neither of Itô nor of Stratonovich type. This means that, after adiabatic elimination, the governing equations have different drift fields, leading to different physical behavior depending on the relative magnitude of the two fast-time scales. Our findings are supported by numerical simulations.

DOI: 10.1103/PhysRevE.70.036120 PACS number(s): 02.50.Fz, 05.10.Gg, 05.40.Jc

I. INTRODUCTION

Many problems of physical interest are described in terms of variables with widely separated characteristic time scales. Often one is interested in obtaining a coarse-grained, macroscopic description for the slow variables alone. The fast variables are eliminated through a process of adiabatic elimination. A simple example is the derivation of the Smoluchowski equation from the full phase space dynamics (i.e., Kramers' equation) through elimination of the momentum variables [1]. The formalism is very well developed for the case of additive noise. However, in the presence of multiplicative noise it is not *a priori* clear whether the limiting equation should be interpreted in the Itô or Stratonovich sense [2–4]. In particular, if the noise is colored, with short correlation time, the double limit of eliminating momentum variables and noise correlation requires careful analysis. This is the so-called Itô to Stratonovich problem [5]. Despite its importance, it has not yet been fully analyzed, although, in [6], a one-dimensional case is treated through an asymptotic study of the corresponding Fokker-Planck equation. We provide a systematic analysis of the problem, using both strong [stochastic-differential-equation- (SDE-) based] and weak (Fokker-Planck-based) convergence techniques, and incorporating the work in [6] as a special case. A time discrete problem in which two time scales resonate and lead to a non-Stratonovich correction to the stochastic integral in the continuous limit has been recently analyzed in [7].

Dynamical equations subject to multiplicative noise have been studied extensively over the last 20 years [8,9], in particular in connection to noise-induced phase transitions and to the dynamics of fronts. The starting point for investigations along these lines is a first-order-in-time (ordinary or partial) stochastic differential equation in position space subject to multiplicative white noise. This model leads to very rich and interesting dynamics, yet it seems that from a physical and modeling point of view it is often more natural to consider systems with inertia and subject to colored multiplicative noise. To our knowledge, so far there have been a limited number of investigations in this direction. There is some work, however, studying the effect of inertia or colored noise separately. As examples we mention the study of the effect of non-zero noise correlation time on noise-induced phase transitions in [10] and the study of the effect of inertia on the dynamics of fronts which was undertaken in [11]; there it was shown that inertial effects of any magnitude suppress the external white-noise influence on the velocity of fronts, essentially because no Stratonovich correction appears when inertia is present.

In this article we undertake a systematic study of the problem of adiabatic elimination for systems where inertia as well as multiplicative noise with finite correlation time is taken into account. The presence of inertia induces another characteristic time scale in the system, that of the particle relaxation time. We show that the limiting equation describing the dynamics in position space, when both particle relaxation time and noise correlation time tend to zero, depends on the relative magnitude of the two fast-time scales of the system. In particular, when the particle relaxation time is large compared to the noise correlation time, then the multiplicative noise in the limiting stochastic differential equation should be interpreted in the sense of Itô. This includes the case studied in [11] where the noise has zero correlation time. On the contrary, when the particle relaxation time is small compared to the correlation time of the noise, then the limiting SDE should be interpreted in the Stratonovich sense. This regime includes the noninertial case for which it is well

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known that the limit as the noise correlation time tends to zero leads to the Stratonovich interpretation of SDE's—e.g., [[12,13], Sec. 10.3]. The transition between Itô and Stratonovich limits occurs when the particle relaxation time and the noise correlation time are comparable in magnitude. In this case the limiting equation cannot be interpreted in either the Itô or the Stratonovich sense and the correction to the Itô stochastic integral depends on the specific details of the colored noise [see Eq. (12) below]. The important message here is that the limiting system, after adiabatic elimination, will be different depending on the ratio of the noise correlation time to the particle relaxation time. The different limits will have different physical behavior. For example, the invariant distribution, stability thresholds, and velocities of coherent structures will depend critically on the ratio of time scales.

We remark that the model that we study in this paper, Eqs. (4) and (3) below, does not satisfy a fluctuation-dissipation relation, as this model is not meant to describe systems in thermal equilibrium. As an example of such a model which fits within the framework of this work we mention the Maxey-Riley model [14] for the motion of inertial particles in a Gaussian velocity field as considered in [15,16]. Systems with inertia and delta correlated multiplicative noise which do satisfy the fluctuation-dissipation theorem have been considered by various authors, e.g. [2,4,17]. Our analysis covers the adiabatic elimination results reported in [2,4] in the regime where the noise correlation time goes to 0 faster than the particle relaxation time, the limiting equation being Eq. (8) below.

We emphasize that the results reported in this paper are provable and hold for a much wider class of systems than the ones considered here. For brevity of exposition we will present only formal calculations in the simplest possible setting. The details of the rigorous strong convergence theorems are presented in [18].

This paper is organized as follows: in Sec. II we present the model equations that we will consider, together with the appropriate rescaling. We derive the limiting equations that hold in the various parameter regimes and we present simple heuristics which justify the limits and which are made rigorous in [18]; we also present some numerical experiments that exemplify our analytical findings. In Sec. III we present some extensions of the results of Sec. II. In Sec. IV we present an alternative derivation of the limiting equations based on asymptotic analysis of the Chapman-Kolmogorov equation. Finally, Sec. V is devoted to discussion and conclusions.

II. ONE-DIMENSIONAL DYNAMICS

Consider the following Langevin equation with multiplicative colored noise:

$$\tau\ddot{x} = f(x)\,\eta_0(\nu t) - \dot{x}. \tag{1}$$

The parameter τ is the nondimensional relaxation time of the particle velocity, and $f(x)$ is a sufficiently smooth function which is bounded, together with its first two derivatives. Overdots denote differentiation with respect to time. The colored noise $\eta_0(t)$ is an Ornstein-Uhlenbeck (OU) process which—when the initial data is stationary—is a Gaussian process with $\langle\eta_0(t)\rangle=0$, $\langle\eta_0(t)\eta_0(s)\rangle=(\lambda/2\alpha)e^{-\alpha|t-s|}$. It satisfies the equation [19]

$$\dot{\eta}_0 = -\alpha\eta_0 + \sqrt{\lambda}\xi,$$

where $\xi(t)$ is the standard white-noise process in one dimension with $\langle\xi(t)\rangle=0$, $\langle\xi(t)\xi(s)\rangle=\delta(t-s)$.

We are interested in studying the long-time behavior of solutions to Eq. (1) in the limit when τ as well as the parameter $1/\nu$, which controls the relaxation time of the colored process, tends to zero. To this end, we rescale the nondimensional parameters of the problem as $\tau=\epsilon^c\tau_0$, $\nu=\nu_0/\epsilon^a$, where τ_0, ν_0 are $\mathcal{O}(1)$ numbers and $\epsilon\ll 1$. We also rescale time by $t=T/\epsilon^b$. The equation of motion becomes, after multiplying through by ϵ^{-b},

$$\epsilon^{c+b}\tau_0 x'' = f(x)\frac{\eta_0\left(\frac{\nu_0}{\epsilon^{a+b}}T\right)}{\epsilon^b} - x',$$

where primes denote differentiation with respect to T. We set $\nu_0=1$ for notational simplicity, chose $b=1$, $a=1$, $c>-1$ [20], and use the original notation t for time to obtain

$$\tau_0\epsilon^\gamma\ddot{x} = \frac{f(x)\,\eta_0(t/\epsilon^2)}{\epsilon} - \dot{x}, \tag{2}$$

with $\gamma=c+1>0$. Using the scaling properties of Brownian motion [[21], p. 104], we can rewrite $\eta_0(t/\epsilon^2)=\eta(t)$ where the rescaled OU process $\eta(t)$ satisfies the equation

$$\dot{\eta} = -\frac{\alpha}{\epsilon^2}\eta + \frac{\sqrt{\lambda}}{\epsilon}\xi. \tag{3}$$

In view of Eq. (3) the equation of motion (2) can be written as

$$\tau_0\epsilon^\gamma\ddot{x} = \frac{f(x)\,\eta(t)}{\epsilon} - \dot{x}. \tag{4}$$

From the exact solution of Eq. (3) it is easy to conclude that $\eta(t)=O(1)$ [22]. From Eqs. (3) and (4) it becomes evident that the particle velocity relaxation time is $O(\epsilon^\gamma)$, while the noise correlation time is $O(\epsilon^2)$. It is therefore expected that resonance phenomena will appear when $\gamma=2$.

We are interested in obtaining the limiting equation for the particle position as $\epsilon\to 0$. We use the variation of constants formula to solve for the particle velocity. Letting $\dot{x}(t)=y(t)$, $x_0=x(0)$, and $y_0=y(0)$, we obtain

$$\dot{x}(t) = y_0\exp\left(-\frac{t}{\tau_0\epsilon^\gamma}\right) + \frac{1}{\tau_0\epsilon^\gamma}\int_0^t \exp\left(-\frac{t-s}{\tau_0\epsilon^\gamma}\right)\frac{f(x(s))\,\eta(s)}{\epsilon}ds. \tag{5}$$

From this equation, after an integration by parts, we obtain an integral equation for the particle position:

$$x(t) = x_0 + \tau_0\epsilon^\gamma y_0\left[1 - \exp\left(-\frac{t}{\tau_0\epsilon^\gamma}\right)\right] + \int_0^t \frac{f(x(s))\,\eta(s)}{\epsilon}ds - \int_0^t \exp\left(-\frac{t-s}{\tau_0\epsilon^\gamma}\right)\frac{f(x(s))\,\eta(s)}{\epsilon}ds. \qquad (6)$$

Clearly, we have $\tau_0\epsilon^\gamma y_0(1-e^{-t/\epsilon^\gamma})=O(\epsilon^\gamma)$ as $\epsilon\to 0$. Next, upon using Eq. (5) we can obtain sharp estimates of the moments of the particle velocity $y(t)=\dot{x}(t)$ [18]. These estimates enable us to conclude that, roughly speaking, the particle velocity is of order $O(\epsilon^{-\min(1,\gamma/2)})$. Using these estimates one can show that

$$\int_0^t \exp\left(-\frac{t-s}{\tau_0\epsilon^\gamma}\right)\frac{f(x(s))\,\eta(s)}{\epsilon}ds = O(\epsilon^{\max(\gamma/2,\gamma-1)}).$$

Thus the contribution to the limiting equations [Eqs. (8), (11), and (12) below] comes only from the first integral on the right-hand side of Eq. (6). In order to analyze this term we integrate by parts once using Itô formula to obtain

$$\int_0^t \frac{f(x(s))\,\eta(s)}{\epsilon}ds = \frac{\sqrt{\lambda}}{\alpha}\int_0^t f(x(s))\xi(s)ds + \frac{\epsilon}{\alpha}\int_0^t f'(x(s))y(s)\,\eta(s)ds + O(\epsilon). \qquad (7)$$

We will use the notation $J(t)$ for the second term on the right-hand side of Eq. (7).

We consider first the case $\gamma\in(0,2)$. We use the aforementioned estimates on the moments of the particle velocity, together with the fact that $\eta(t)/\epsilon=(\sqrt{\lambda}/\alpha)\xi(t)-(\epsilon/\alpha)\dot{\eta}(t)$ to show that $J(t)=O(\epsilon^{1-\gamma/2})$. Thus, for $\gamma\in(0,2)$ and ϵ sufficiently small the particle position $x(t)$ satisfies the following equation:

$$\dot{x} = \frac{\sqrt{\lambda}}{\alpha}f(x)\xi + O(\epsilon^{\min(1-\gamma/2,\gamma/2)}).$$

Consequently, as $\epsilon\to 0$, $x(t)$ converges to $X(t)$ which satisfies the following Itô SDE:

$$\dot{X} = \frac{\sqrt{\lambda}}{\alpha}f(X)\xi. \qquad (8)$$

The fact that for $\gamma<2$ we obtain the limiting equation (8) can be explained intuitively as follows: in this parameter regime the particle relaxation time—which is of the order of ϵ^γ—is large compared to the relaxation time of the noise and consequently the particle experiences a rough noise with practically zero correlation time. This means that for $\gamma<2$ the OU process is not viewed from the point of view of the particle as a smooth approximation to white noise and this results in the limiting equation being an Itô SDE.

Now we proceed with the case $\gamma\geq 2$. We perform an integration by parts on $J(t)$ in Eq. (7) and use the estimates on the moments of the particle position, together with standard tools from stochastic calculus, to obtain

$$\frac{\epsilon}{\alpha}\int_0^t f'(x(s))y(s)\,\eta(s)ds = \frac{1}{\alpha}\int_0^t f'(x(s))f(x(s))\,\eta^2(s)ds - \tau_0\epsilon^{\gamma-1}\int_0^t f'(x(s))y(s)\,\eta(s)ds + O(\epsilon^{\gamma-1}). \qquad (9)$$

Another integration by parts [23] yields

$$\int_0^t f'(x(s))f(x(s))\,\eta^2(s)ds = \frac{\lambda}{2\alpha}\int_0^t f'(x(s))f(x(s))ds + O(\epsilon). \qquad (10)$$

Furthermore, using the fact that, for $\gamma\geq 2$, $y(t)\eta(t)=O(\epsilon^{-1})$ we conclude that

$$\int_0^t f'(x(s))y(s)\,\eta(s)ds = O(\epsilon^{-1}),$$

since $f'(x)$ is assumed to be bounded. From this we conclude that the last integral on the right-hand side of Eq. (9) is of order $O(\epsilon^{\gamma-2})$. Now it is evident that, for $\gamma>2$ and for ϵ sufficiently small, the particle position satisfies the equation

$$\dot{x} = \frac{\lambda}{2\alpha^2}f'(x)f(x) + \frac{\sqrt{\lambda}}{\alpha}f(x)\xi + O(\epsilon^{\min(\gamma-2,1)}).$$

We take the limit $\epsilon\to 0$ to obtain the limiting Itô SDE for $\gamma>2$:

$$\dot{X} = \frac{\lambda}{2\alpha^2}f'(X)f(X) + \frac{\sqrt{\lambda}}{\alpha}f(X)\xi. \qquad (11)$$

In this parameter regime the particle relaxation time is small compared to that of the noise. Consequently, for $\gamma>2$, the rescaled OU process is indeed a smooth Gaussian approximation to white noise giving a Stratonovich correction to the drift and leading to the Itô SDE (11), in agreement with standard theorems [[12,13], Sec. 10.3, [24]]: the case $\gamma=\infty$ leads to the case of tracer particles whose relaxation time is zero and covered precisely by these standard theorems.

Now we consider the case $\gamma=2$. We combine Eqs. (9) and (10) with $\gamma=2$ to obtain

$$\frac{\epsilon}{\alpha}\int_0^t f'(x(s))y(s)\,\eta(s)ds = -\tau_0\epsilon\int_0^t f'(x(s))y(s)\,\eta(s)ds + \frac{\lambda}{2\alpha^2}\int_0^t f'(x(s))f(x(s))ds + O(\epsilon).$$

We solve the above equation for the integral on the left-hand side:

$$\epsilon\int_0^t f'(x(s))y(s)\,\eta(s)ds = \frac{\lambda}{2\alpha(1+\alpha\tau_0)}\int_0^t f'(x(s))f(x(s))ds + O(\epsilon).$$

Substituting this expression into the last integral in Eq. (7) we conclude that the particle position, when $\gamma=2$ and ϵ is sufficiently small, is given by the Itô equation

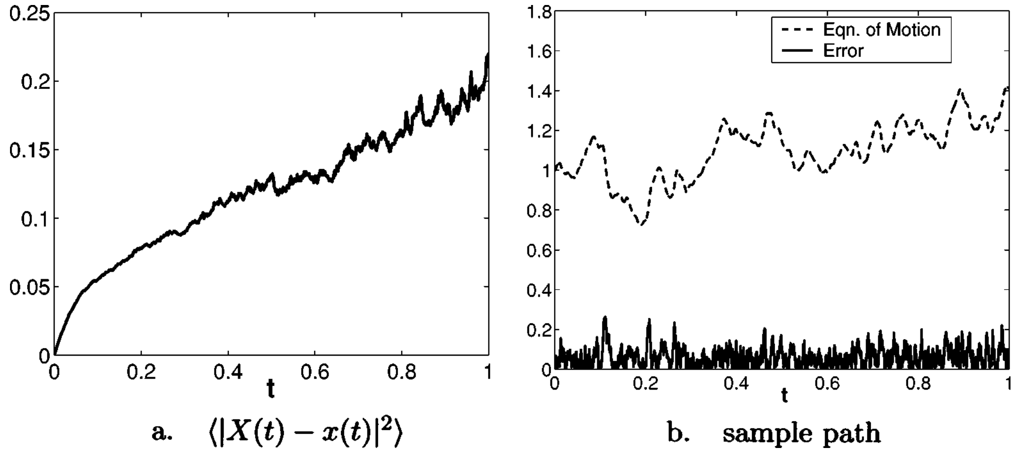

FIG. 1. Difference between the solution of the limiting equation and the equation of motion for $\gamma=1$, for $\epsilon=0.1$.

$$\dot{x} = \frac{\lambda}{2\alpha^2(1+\tau_0\alpha)} f'(x)f(x) + \frac{\sqrt{\lambda}}{\alpha} f(x)\xi + O(\epsilon).$$

We pass to the limit to obtain the Itô SDE for $\gamma=2$:

$$\dot{X} = \frac{\lambda}{2\alpha^2(1+\tau_0\alpha)} f'(X)f(X) + \frac{\sqrt{\lambda}}{\alpha} f(X)\xi. \qquad (12)$$

For the case $\gamma=2$ the particle relaxation time is comparable in magnitude to the noise correlation time and a resonance mechanism prevails which results in the limiting stochastic differential equation containing a correction to the drift which is not the standard Stratonovich one. The drift correction depends on the friction coefficient of the OU process α as well as the particle relaxation time τ_0. We also remark that we can formally derive the limiting equations (8) and (11) from Eq. (12) through varying τ_0: taking the limit $\tau_0 \to \infty$ in Eq. (12)—which corresponds to the regime $\gamma < 2$—we obtain the Itô equation (8); on the other hand, the limit $\tau_0 \to 0$—which corresponds to the case $\gamma > 2$—leads to the Stratonovich equation (11).

From the above discussion it is clear that the rate at which the particle position $x(t)$ converges to the solution $X(t)$ of the limiting equation depends crucially on γ. In particular, the convergence rate deteriorates as γ tends to 2^- and 2^+. This is to be expected, since the limiting equation depends discontinuously on γ and has a jump at $\gamma=2$. For $\gamma=2$ the convergence rate is quadratic, when measured in mean square.

We exemplify the above theoretical findings with some simple numerical experiments for the specific choice $f(x) = x$. For this function the limiting equations can be solved explicitly. We solve numerically the equation of motion (4) for various values of γ. In Figs. 1(a), 2(a), and 3(a) we present the difference between the solutions of the limiting equations and the equations of motion measured in mean square for $\gamma=1$, 2, and 3, respectively. In Figs. 1(b), 2(b), and 3(b) we present sample paths of the solution of the equation of motion, as well as the pathwise error of the limiting equation, for the same values of γ. From these graphs we see that $x(t)$ and $X(t)$ are very close—in particular for $\gamma \geq 2$—even pathwise and not only in the mean-square sense. The proof of this fact is based on the calculations presented in this section together with some nontrivial estimates. The resulting theorem holds for the case where the colored noise is infinite dimensional and is presented in [18]. We also observe that the solution of the equation of motion is smoother than the solution of the limiting equation. This is to be expected since $X(t)$ is a Markovian approximation to the non

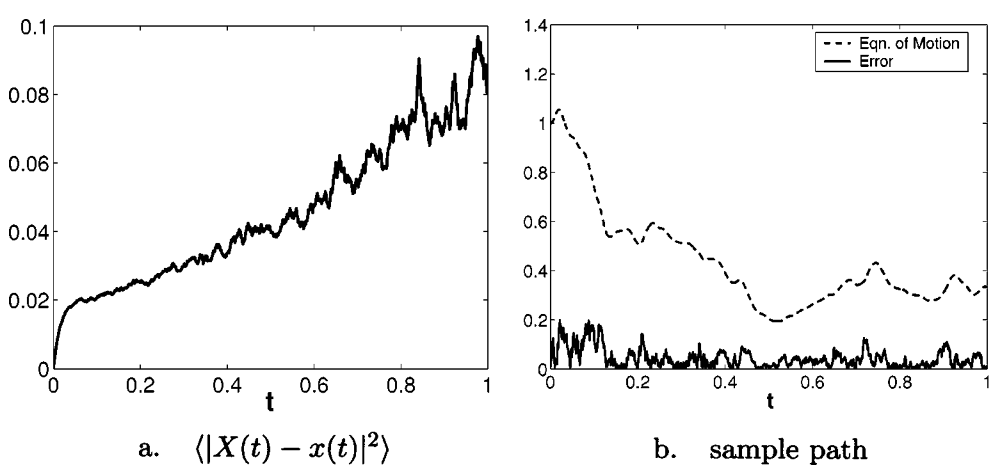

FIG. 2. Difference between the solution of the limiting equation and the equation of motion for $\gamma=2$, for $\epsilon=0.1$.

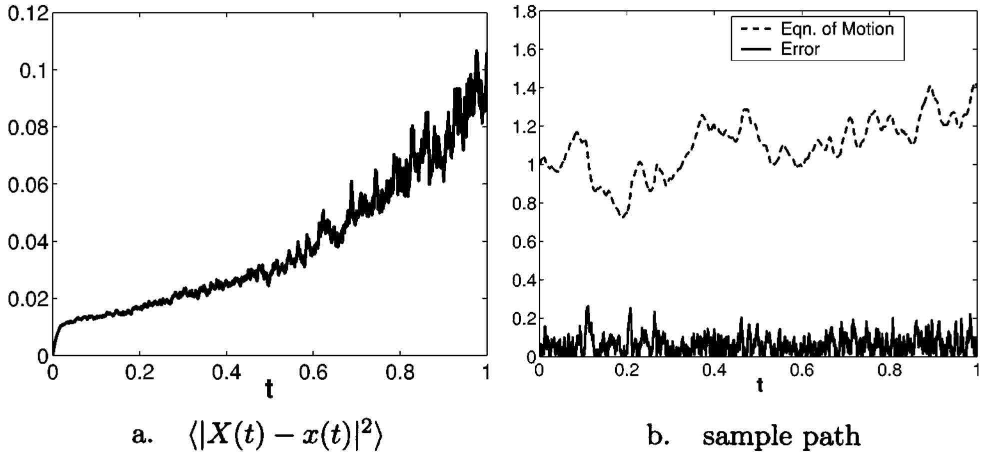

FIG. 3. Difference between the solution of the limiting equation and the equation of motion for $\gamma=3$, for $\epsilon=0.1$.

Markovian process $x(t)$ and, hence, less smooth [[25], Chap. 10].

In Fig. 4 we present the numerically calculated convergence rate showing excellent agreement between the theoretical prediction and numerical observations. The error, when measured in mean square, is of the form $C_1\epsilon^{\gamma}+C_2\epsilon^{2-\gamma}$ when $\gamma<2$, $C_3\epsilon^2$ for $\gamma=2$, and $C_4\epsilon^2+C_5\epsilon^{2(\gamma-2)}$ when $\gamma>2$. Consequently, except for $\gamma=2$, nonlinear regression analysis is needed in order to calculate the convergence rate which is $2-\gamma$ and $2(\gamma-2)$ for $1\leqslant\gamma\leqslant 2$ and $2<\gamma\leqslant 3$, respectively.

III. EXTENSION TO HIGHER DIMENSIONS AND GENERALIZATIONS

The systematic adiabatic elimination procedure described in the previous section can be extended in a straightforward way to cover the case of a particle moving in $\mathbb{R}^d$ under the influence of a sufficiently smooth and uniformly bounded potential and for infinite-dimensional colored multiplicative noise. Let us consider the following rescaled equations of motion:

$$\epsilon^{\gamma}\ddot{x}_i = -\frac{\partial V(x)}{\partial x_i} + \frac{f_{ij}(x)\eta_j(t)}{\epsilon} - \dot{x}_i,\ i=1,\ldots,d.$$

Here and below double or triple appearance of an index denotes summation. The processes $\{\eta_j(t)\}_{j=1}^{\infty}$ are a set of independent OU processes:

$$\dot{\eta}_j = -\frac{\alpha_j}{\epsilon^2}\eta_j + \frac{\sqrt{\lambda_j}}{\epsilon}\xi_j,\ j=1,\ldots,\infty,$$

where $\langle\xi_j(t)\rangle=0$, $\langle\xi_i(t)\xi_j(s)\rangle=\delta_{ij}\delta(t-s)$. Then, under various technical conditions on the functions $\{f_{ij}(x)\}_{i,j=1}^{d,\infty}$ and the spectrum of the noise $\{\lambda_j\}_{j=1}^{\infty}$, one can prove that the governing equations make sense and that, as $\epsilon\to 0$, the particle position $x(t)$ converges to $X(t)$ which satisfies the following SDE:

$$\dot{X}_i = \begin{cases} -\dfrac{\partial V(x)}{\partial x_i} + \dfrac{\sqrt{\lambda_j}}{\alpha_j} f_{ij}(X)\xi_j, & \gamma<2 \text{ Itô},\\[2ex] -\dfrac{\partial V(x)}{\partial x_i} + \dfrac{\lambda_j}{2\alpha_j^2(1+\tau_0\alpha_j)}\dfrac{\partial f_{ij}(X)}{\partial X_k} f_{kj}(X) + \dfrac{\sqrt{\lambda_j}}{\alpha_j} f_{ij}(X)\xi_j, & \gamma=2,\\[2ex] -\dfrac{\partial V(x)}{\partial x_i} + \dfrac{\lambda_j}{2\alpha_j^2}\dfrac{\partial f_{ij}(X)}{\partial X_k} f_{kj}(X) + \dfrac{\sqrt{\lambda_j}}{\alpha_j} f_{ij}(X)\xi_j, & \gamma>2 \text{ Stratonovich}. \end{cases}$$

The above limiting equations are derived without any specific assumptions on the functions $\{f_{ij}(x)\}_{i,j=1}^{d,\infty}$, other than them being sufficiently smooth and bounded. However, there are instances where the Stratonovich correction vanishes identically and the limiting equations are the same independently of γ. To see this, we rewrite the drift correction to the Itô stochastic differential in the form

$$\frac{\partial f_{ij}(X)}{\partial X_k} f_{kj}(X) = \frac{\partial(f_{ij}(X)f_{kj}(X))}{\partial X_k} - f_{ij}(X)\frac{\partial f_{kj}(X)}{\partial X_k}. \quad (13)$$

Now, when the noise is isotropic we have that $f_{ij}(X)f_{kj}(X) \propto \delta_{ik}$ and consequently the first term on the right-hand side of Eq. (13) vanishes. Furthermore, the second term on the right-hand side of the above equation will also vanish identically

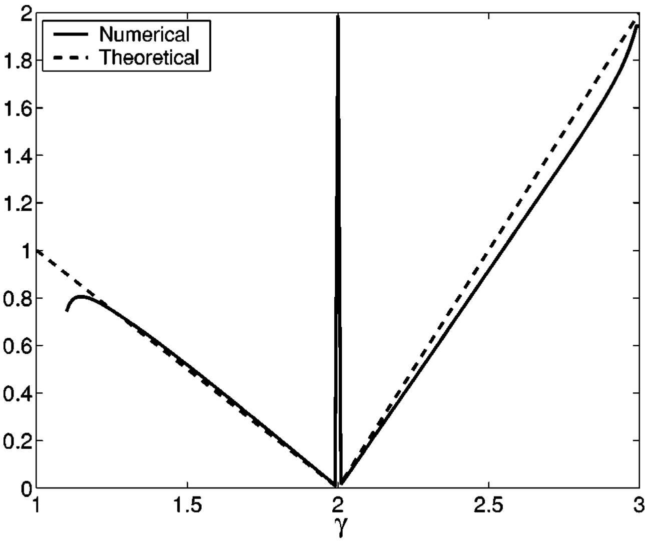

FIG. 4. Convergence rate.

when the noise is divergence free—i.e., when $\partial f_{kj}(X)/\partial X_k = 0$. As an example where this is indeed the case we mention a model for the motion of an inertial particle moving in a random incompressible velocity field on the two-dimensional unit torus: such a model was introduced in [16] and analyzed in [15,26]. This model comprises Stokes' law, Eq. (1), for the motion of the particles and the velocity field is given by an infinite-dimensional OU process [15], $v(x,t) = \Sigma_k k^{\perp}\phi_k(x)\eta_k(t)$, where $k^{\perp}=(k_2,-k_1)^T$ and $\phi_k(x)$, $k_1,k_2 = 1,\ldots,\infty$ are the eigenfunctions of the Laplacian in two dimensions with periodic boundary conditions. In this setting, the limit that we study in this paper corresponds to that of a rapidly decorrelating fluid when the inertia of the particles also tends to zero. For this problem, which was our initial motivation for the undertaking of this work, the isotropy and incompressibility of the velocity field result in the Stratonovich correction vanishing identically and the limiting equation being that of Itô for all $\gamma > 0$ [16,18]. In these analyses the assumptions that one has to impose on the spectrum of the noise are more severe for $\gamma \geqslant 2$, since more integrations by parts, using Itô formula, are required in the proofs.

IV. ALTERNATIVE DERIVATION OF THE LIMITING EQUATIONS

In this section we present an alternative derivation of the limiting equations for the one-dimensional case, based on a singular perturbation expansion of the associated Chapman-Kolmogorov equation. The singular perturbation approach is rigorously justified by general theorems of Kurtz [27]; see also [28]. A similar derivation, which uses the Fokker-Planck picture can be found in [6].

We start with Eqs. (3) and (4), which we write as a first-order system

$$\dot{x} = \frac{1}{\sqrt{\tau_0 \epsilon^{\gamma}}} v,$$

$$\dot{v} = \frac{f(x)\eta}{\epsilon\sqrt{\tau_0\epsilon^{\gamma}}} - \frac{v}{\tau_0\epsilon^{\gamma}},$$

$$\dot{\eta} = -\frac{\alpha}{\epsilon^2}\eta + \frac{\sqrt{\lambda}}{\epsilon}\xi. \qquad (14)$$

Note that $v(t)$ differs from $y(t)=\dot{x}(t)$ by an ϵ-dependent scaling factor. As has already been mentioned, the OU process $\eta(t)$ satisfies $\eta(t)=O(1)$. Considering then the equation for $v(t)$, we observe that $\gamma=2$ is a threshold value: for $\gamma>2$, the fastest component is $v(t)$ which remains exponentially close to the "slow manifold" $v(t)\approx \epsilon^{\gamma/2-1}\sqrt{\tau_0} f(x(t))\eta(t)$. Thus, as in Sec. II, we separate the analysis into the cases $\gamma=2$, $\gamma<2$, and $\gamma>2$.

A. Case $\gamma=2$

For $\gamma=2$, the Chapman-Kolmogorov equation for $u(t,x,v,\eta)$ takes the form

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon^2}L_0 u + \frac{1}{\epsilon}L_1 u, \qquad (15)$$

where

$$L_0 = -\alpha\eta\frac{\partial}{\partial\eta} + \frac{\lambda}{2}\frac{\partial^2}{\partial\eta^2} + \left(\frac{f(x)\eta}{\sqrt{\tau_0}} - \frac{v}{\tau_0}\right)\frac{\partial}{\partial v},$$

$$L_1 = \frac{v}{\sqrt{\tau_0}}\frac{\partial}{\partial x}.$$

Here the dynamics in v and η have comparable rates, whereas the dynamics in x are an order of magnitude slower.

In order to analyze Eq. (15) we expand its solution in a power series, $u=u_0+\epsilon u_1+\cdots$. Substituting this expansion in the equation gives a hierarchy of equations:

$$L_0 u_0 = 0,$$

$$L_0 u_1 = -L_1 u_0,$$

$$L_0 u_2 = -L_1 u_1 + \frac{\partial u_0}{\partial t}.$$

The leading-order equation $L_0 u_0 = 0$ implies that $u_0 = u_0(t,x)$. The next equation in the hierarchy, $L_0 u_1 = -L_1 u_0$, can be solved explicitly:

$$u_1(t,x,v,\eta) = \left(\sqrt{\tau_0}v + \frac{f(x)\eta}{\alpha}\right)\frac{\partial u_0}{\partial x} + c(t,x).$$

The third equation $L_0 u_2 = -L_1 u_1 + \partial u_0/\partial t$ is solvable only if the right-hand side integrates to zero (with respect to both v and η) against densities ρ invariant under the Fokker-Planck operator L_0^*, which implies

$$\frac{\partial u_0}{\partial t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L_1 u_1 \rho\, dv\, d\eta. \qquad (16)$$

The only density invariant under L_0^* is the Gaussian density $\rho(v,\eta)\sim\mathcal{N}(0,\Sigma)$ with covariance matrix

$$\Sigma = \frac{\lambda}{2\alpha}\begin{pmatrix} \frac{\tau_0 f^2(x)}{1+\tau_0\alpha} & \frac{\sqrt{\tau_0}f(x)}{1+\tau_0\alpha} \\ \frac{\sqrt{\tau_0}f(x)}{1+\tau_0\alpha} & 1 \end{pmatrix}.$$

Substituting ρ, u_1, and L_1 into the solvability condition (16) yields

$$\frac{\partial u_0}{\partial t} = \frac{\lambda\tau_0 f^2(x)}{2\alpha(1+\tau_0\alpha)}\frac{\partial^2 u_0}{\partial x^2} + \frac{\lambda f(x)}{2\alpha^2(1+\tau_0\alpha)}\frac{\partial}{\partial x}\left[f(x)\frac{\partial u_0}{\partial x}\right]$$
$$= \frac{\lambda}{2\alpha^2}f^2(x)\frac{\partial^2 u_0}{\partial x^2} + \frac{\lambda}{2\alpha^2(1+\tau_0\alpha)}f(x)f'(x)\frac{\partial u_0}{\partial x}.$$

We identify the latter as the Chapman-Kolmogorov equation for the reduced Itô SDE

$$\dot{X} = \frac{\lambda}{2\alpha^2(1+\tau_0\alpha)}f(X)f'(X) + \frac{\sqrt{\lambda}}{\alpha}f(X)\xi,$$

which coincides with Eq. (12).

B. Case $\gamma<2$

For $\gamma<2$, we have fast dynamics in both v and η, but the dynamics in η are faster. This calls for a perturbative expansion in two *independent* small parameters ϵ and $\epsilon^{\gamma/2}$.

The Chapman-Kolmogorov equation takes then the form

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon^2}L_0 u + \frac{1}{\epsilon\epsilon^{\gamma/2}}L_1 u + \frac{1}{\epsilon^{\gamma}}L_2 u + \frac{1}{\epsilon^{\gamma/2}}L_3 u, \qquad (17)$$

where

$$L_0 = -\alpha\eta\frac{\partial}{\partial\eta} + \frac{\lambda}{2}\frac{\partial^2}{\partial\eta^2},\quad L_1 = \frac{f(x)\eta}{\sqrt{\tau_0}}\frac{\partial}{\partial v},$$

$$L_2 = -\frac{v}{\tau_0}\frac{\partial}{\partial v},\quad L_3 = \frac{v}{\sqrt{\tau_0}}\frac{\partial}{\partial x}.$$

The solution $u=u(t,x,v,\eta)$ is then expanded in a double power series

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots,$$

where, for every $k=0,1,\ldots,$

$$u_k = u_{k,0} + \epsilon^{\gamma/2}u_{k,1} + \epsilon^{\gamma}u_{k,2} + \cdots .$$

Substituting this double expansion into Eq. (17) we get a double hierarchy of equations:

$$0 = L_0 u_{0,k},\ k=0,1,\ldots,$$

$$0 = L_1 u_{0,0},$$

$$0 = L_0 u_{1,0} + L_1 u_{0,1},$$

$$0 = L_0 u_{1,1} + L_1 u_{0,2},$$

$$0 = L_2 u_{0,0},$$

$$0 = L_1 u_{1,0} + L_2 u_{0,1} + L_3 u_{0,0},$$

$$\frac{\partial u_{0,0}}{\partial t} = L_0 u_{2,0} + L_1 u_{1,1} + L_2 u_{0,2} + L_3 u_{0,1},$$

$$\vdots = \vdots .$$

The first line implies that for $k=0,1,\ldots$, $u_{0,k}=u_{0,k}(t,x,v)$, where as the second (or the fifth) line implies that $u_{0,0}=u_{0,0}(t,x)$. From the third and fourth lines we deduce that

$$u_{1,k}(t,x,v,\eta) = \frac{f(x)\eta}{\alpha\sqrt{\tau_0}}\frac{\partial u_{0,k+1}}{\partial v} + c_k(t,x,v).$$

The sixth line implies, equating powers of η, that $u_{1,0}$ does not depend on v, and hence $c_0(t,x,v)=0$. Moreover, it follows that

$$\frac{\partial u_{0,1}}{\partial v} = \sqrt{\tau_0}\frac{\partial u_{0,0}}{\partial x}$$

and

$$u_{0,1}(t,x,v) = \sqrt{\tau_0}v\frac{\partial u_{0,0}}{\partial x} + e(t,x).$$

Finally applying the solvability condition on the seventh line gives

$$\frac{\partial u_{0,0}}{\partial t} = \frac{\lambda f^2(x)}{2\alpha^2\tau_0}\frac{\partial^2 u_{0,2}}{\partial v^2} - \frac{v}{\tau_0}\frac{\partial u_{0,2}}{\partial v} + v^2\frac{\partial^2 u_{0,0}}{\partial x^2} + \frac{v}{\sqrt{\tau_0}}\frac{\partial e}{\partial x}.$$

The last step is to observe that this equation requires a solvability condition with respect to the v variables. Defining

$$G_0 u = \frac{\lambda f^2(x)}{2\alpha^2\tau_0}\frac{\partial^2 u}{\partial v^2} - \frac{v}{\tau_0}\frac{\partial u}{\partial v},$$

we have

$$G_0 u_{2,0} = \frac{\partial u_{0,0}}{\partial t} - v^2\frac{\partial^2 u_{0,0}}{\partial x^2} - \frac{v}{\sqrt{\tau_0}}\frac{\partial e}{\partial x},$$

which is solvable only if the right-hand side integrates to zero (with respect to v) against densities ρ that are invariant under G_0^*—i.e.,

$$G_0^*\rho = \frac{\lambda f^2(x)}{2\alpha^2\tau_0}\frac{\partial^2\rho}{\partial v^2} + \frac{1}{\tau_0}\frac{\partial(v\rho)}{\partial v} = 0.$$

The only invariant density being

$$\rho = \sqrt{\frac{\alpha^2}{\pi\lambda f^2(x)}}\exp\left(-\frac{\alpha^2\eta^2}{\lambda f^2(x)}\right),$$

the solvability condition reduces to

$$\frac{\partial u_{0,0}}{\partial t} = \frac{\lambda f^2(x)}{2\alpha^2}\frac{\partial^2 u_{0,0}}{\partial x^2},$$

which is the Chapman-Kolmogorov equation for the Itô SDE

$$\dot{X} = \frac{\sqrt{\lambda}}{\alpha}f(X)\xi.$$

C. Case $\gamma > 2$

For $\gamma > 2$ we have fast dynamics in both v and η with the dynamics in v being faster. We will proceed as in the previous subsection, by expanding the solution in two independent small parameters ϵ and $\epsilon^{\gamma/2}$. The case when $\gamma/2$ is an integer, when there is no need for an expansion in two independent parameters, can be treated similarly. The basic ingredient of the analysis—independently of whether $\gamma/2$ is an integer of not—is the invertibility of the operator L_0: the condition that the solutions of equation $L_0 f = g$ should be smooth, defined for all v in $(-\infty, \infty)$, ensures that this equation has a unique solution.

The Chapman-Kolmogorov equation takes the form

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon^{\gamma}} L_0 u + \frac{1}{\epsilon \epsilon^{\gamma/2}} L_1 u + \frac{1}{\epsilon^{\gamma/2}} L_2 u + \frac{1}{\epsilon^2} L_3 u, \tag{18}$$

where

$$L_0 = -\frac{v}{\tau_0}\frac{\partial}{\partial v}, \; L_1 = \frac{f(x)\eta}{\sqrt{\tau_0}}\frac{\partial}{\partial v},$$

$$L_2 = \frac{v}{\sqrt{\tau_0}}\frac{\partial}{\partial x}, \; L_3 = -\alpha\eta\frac{\partial}{\partial \eta} + \frac{\lambda}{2}\frac{\partial^2}{\partial \eta^2}.$$

Notice that the ordering of the two last terms on the right-hand side of Eq. (18) depends on γ: for $2 < \gamma < 4$ the term $(1/\epsilon^2)L_3$ comes before $(1/\epsilon^{\gamma/2})L_2 u$, whereas for $\gamma > 4$ the ordering is as shown in Eq. (18). However, as the subsequent analysis will show, the ordering of these two terms does not have any effect on the form of the limiting equation.

The solution $u = u(t, x, v, \eta)$ is then expanded in a double power series:

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots,$$

where, for every $k = 0, 1, \ldots,$

$$u_k = u_{k,0} + \epsilon^{\gamma/2} u_{k,1} + \epsilon u_{k,2} + \cdots .$$

Substituting this double expansion into Eq. (18) we get a double hierarchy of equations [29]:

$$\begin{aligned}
0 &= L_0 u_{k,0}, \; k = 0,1,2, \\
0 &= L_1 u_{0,0}, \\
0 &= L_0 u_{0,1} + L_2 u_{0,0} + L_1 u_{1,0}, \\
0 &= L_0 u_{1,1} + L_2 u_{1,0} + L_1 u_{2,0}, \\
0 &= L_3 u_{0,0}, \\
0 &= L_3 u_{1,0} + L_1 u_{0,1}, \\
\frac{\partial u_{0,0}}{\partial t} &= L_0 u_{0,2} + L_1 u_{1,1} + L_2 u_{0,1} + L_3 u_{2,0}, \\
&\vdots = \vdots .
\end{aligned} \tag{19}$$

The first line in Eqs. (19) implies that $u_{k,0} = u_{k,0}(x, \eta, t)$, for $k = 0$, 1, 2. Now the second line in Eqs. (19) is identically satisfied. Moreover, the fifth line implies that $u_{0,0} = u_{0,0}(x, t)$. The third line in the above equations, since $u_{1,0}$ is independent of v, becomes

$$L_0 u_{0,1} = -\frac{v}{\sqrt{\tau_0}}\frac{\partial u_{0,0}}{\partial x},$$

from which we conclude that

$$u_{0,1} = v\sqrt{\tau_0}\frac{\partial u_{0,0}}{\partial x} + c_{0,1}(x, \eta, t).$$

Similarly, from the fourth line in Eqs. (19) we obtain

$$u_{1,1} = v\sqrt{\tau_0}\frac{\partial u_{1,0}}{\partial x} + c_{1,1}(x, \eta, t).$$

The equation for $u_{1,0}$ becomes

$$L_3 u_{1,0} = -f(x)\eta\frac{\partial u_{0,0}}{\partial x}.$$

For this equation to be solvable, the right-hand side has to integrate (with respect to η) to zero against densities that are invariant under the Fokker-Planck operator L_3^*, which is the adjoint of L_3. The unique invariant density for this operator is

$$\rho(\eta) = \sqrt{\frac{\alpha}{\pi\lambda}}\exp\left(-\frac{\alpha\eta^2}{\lambda}\right).$$

Now, since $\int_{-\infty}^{\infty}\eta\rho(\eta)d\eta = 0$, the solvability condition for the equation for $u_{1,0}$ is satisfied and we obtain

$$u_{1,0} = f(x)\frac{\eta}{\alpha}\frac{\partial u_{0,0}}{\partial x}.$$

We combine this with the expression for $u_{1,1}$ that we obtained previously to deduce

$$u_{1,1} = v\sqrt{\tau_0}\frac{\eta}{\alpha}\frac{\partial}{\partial x}\left(f(x)\frac{\partial u_{0,0}}{\partial x}\right).$$

Let us now consider the last line in Eqs. (19). We use our findings so far to write this equation in the form

$$-\frac{v}{\tau_0}\frac{\partial u_{0,2}}{\partial v} = -\frac{\eta^2}{\alpha}f(x)\frac{\partial}{\partial x}\left(f(x)\frac{\partial u_{0,0}}{\partial x}\right) - v^2\frac{\partial^2 u_{0,1}}{\partial x^2} - \frac{v}{\sqrt{\tau_0}}\frac{\partial c_{0,1}}{\partial x} - L_3 u_{2,0} + \frac{\partial u_{0,0}}{\partial t}.$$

The only way for the above equation to have smooth solutions defined for all v in $(-\infty, \infty)$ is for the constant term in v to vanish [30]:

$$-\frac{\eta^2}{\alpha}f(x)\frac{\partial}{\partial x}\left(f(x)\frac{\partial u_{0,0}}{\partial x}\right) - L_3 u_{2,0} + \frac{\partial u_{0,0}}{\partial t} = 0.$$

This is an equation for $u_{2,0}$ in η. The solvability condition for this equation reads

$$\int_{-\infty}^{\infty}\left[-\frac{\eta^2}{\alpha}f(x)\frac{\partial}{\partial x}\left(f(x)\frac{\partial u_{0,0}}{\partial x}\right)+\frac{\partial u_{0,0}}{\partial t}\right]\rho(\eta)d\eta=0,$$

from which we deduce

$$\frac{\partial u_{0,0}}{\partial t}=\frac{\lambda}{2\alpha^2}f(x)\frac{\partial}{\partial x}\left(f(x)\frac{\partial u_{0,0}}{\partial x}\right).$$

This is the Chapman-Kolmogorov equation for the Stratonovich SDE:

$$\dot{X}=\frac{\lambda}{2\alpha^2}f(X)f'(X)+\frac{\lambda}{\alpha}f(X)\xi,$$

which is precisely Eq. (11).

V. CONCLUSIONS

We have shown that the interplay between inertial effects and colored multiplicative noise has a profound effect on the form of the Smoluchowski equation which describes the dynamics of the particle in the limit when particle relaxation time and noise correlation time tend to zero. In particular, the multiplicative noise in the limiting equation should either be interpreted in the Itô or the Stratonovich sense, depending on whether the noise correlation time tends to zero faster or slower than the particle relaxation time. Furthermore, when the two fast time scales of the problem are comparable in magnitude then a different limiting equation emerges which cannot be interpreted in either the Itô or the Stratonovich sense.

The solution of the limiting SDE can have very different properties depending on the interpretation of the multiplicative noise. For example, the noise can influence the velocity of kinks in stochastic reaction diffusion equations with multiplicative noise only when the equation is interpreted in the Stratonovich sense [11]. Hence, our findings suggest that great care has to be taken in any adiabatic elimination procedure for systems where more than one fast time scale is present to ensure that the limit which correctly captures the physics in identified.

ACKNOWLEDGMENTS

The authors are grateful to D. Cai and J.C. Mattingly for useful suggestions. They are also grateful to J.M. Sancho for useful suggestions and for providing them with Refs. [3,11] and to P.R. Kramer for a very careful reading of an earlier version of this paper. G.A.P. and A.M.S. are grateful to EPSRC for financial support. R.K. was supported in part by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities and in part by the Applied Mathematical Sciences subprogram of the Office of Energy Research of the U.S. Department of Energy under Contract DE-AC03-76-SF00098.

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