CODING EXERCISES - GEOMETRY

LUIGI LUNARDON

Introduction

The coding exercises in this problem sheet are meant to be an integration to the exercises assigned during the Topics in Geometry course. They can be solved in Mathematica, Python, C++, or whatever language you prefer. There is no particular order to follow; usually the code required is not too advanced, and the math is already introduced in one of the topics lectures. The exercises marked with $*$ are a bit more difficult than the others, either because the code is more involved, or because they require some more advanced background.

Acknowledgement. We would like to thanks all the LSGNT students who proposed some exercises: Benjamin Aslan, Dougal Davis, Brad Doyle, Matthew Habermann, Ben Heuer, Jordan Hofmann, Tim King, Thomas Sharpe, David Sheard, Bruno Ricieri Souza Roso, Samuel Stark, Zak Turcinovic.

1. Geometry

Exercise 1 (Singular polynomial)**.** Given a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ write a program which determines whether or not the associated variety $\{f(x_1, \ldots, x_n) = 0\} \subset \mathbb{C}^n$ is smooth.

Exercise 2 (Resolution of singularities for plane curves $*$)**.** The geometry of curves is often simpler than the one of higher dimensional varieties. A good example is given by the problem of resolution of singularities: we know that in characteristic 0 given a variety with any type of singularity, then there exists an algorithmic embedded resolution of the singularities, obtained by blow ups at smooth centers (this is a remarkable result by Hironaka); the situation is simpler if the variety is a curve, and the algorithm becomes easier to implement (see [Ful89, Chapter 7]). Write a code that given a plane curve C:

(1) verifies if C is smooth, and if not finds the singular points of C;
(2) blows-up $\mathbb{A}^2$ in each singular points of C and compute the proper transform of C after each blow-up;
(3) repeats the two previous points until we construct a resolution of C. Remember that after each blow-up, you have to verify that the proper transform is smooth in each of the coordinates charts.

Exercise 3 (Map associated to a line bundle)**.** Given a rational curve of degree 5 in $\mathbb{P}^3$ it is possible to show that it is always contained in a cubic surface. However, not all degree 5 rational curves in $\mathbb{P}^3$ are contained in a quadric surface (see, for instance [Har77, Exercise 5.6.2]).

Consider a non-constant map $\phi\colon \mathbb{P}^1 \to \mathbb{P}^3$ of degree 5, its image is a rational curve of degree 5; write a program which determines whether or not this curve is contained in a conic.

Exercise 4 (Geodesics)**.** Let X be a torus, find a parameterization of X in $\mathbb{R}^3$ and plot it. Choose the equation of some geodesics in these coordinates and solve them numerically (you can use the command in Mathematica NDSolve). Hence plot these geodesics on the torus.

Exercise 5 (Simplicial Homology $*$)**.** To compute the homology with integer coefficients of a simplicial complex, it is enough to reduce the boundary maps to their Smith normal form. The Smith normal form of an integer matrix $A \in \mathrm{Mat}_{n,m}$ is an integer matrix $B \in \mathrm{Mat}_{n,m}$ such that:

(1) there exist two invertible square matrices S and T such that $B = SAT$;
(2) if $i \neq j$ then $B_{i,j} = 0$;
(3) the diagonal elements satisfy $B_{i,i} | B_{i+1,i+1}$.

It can be proven that the diagonal elements are essentially unique. Implement an algorithm to reduce an integer matrix to its Smith normal form, some suggestions can be found in [DHSW03]. Test your code by computing the simplicial homology of the following complexes: circle, 3-simplex, $\mathbb{RP}^2$.

Exercise 6 (Differential forms)**.** Fix an n-dimensional vector space V and a basis $\mathcal{B}_V$. Write a code that:

(1) associates to two forms $\omega_1 \in \Lambda^h V$ and $\omega_2 \in \Lambda^m V$ the interior product $\omega_1 \wedge \omega_2 \in \Lambda^{h+k} V$;
(2) determines whether or not there exist elements $v_1, \dots, v_{h+k} \in V$ such that

$$\omega_1 \wedge \omega_2 = v_1 \wedge \cdots \wedge v_{h+k}.$$

Exercise 7 (Chern classes)**.** It is well known that the number or lines on a smooth cubic surface is 27. It is possible to prove a little weaker result (*i.e.* that there are 27 lines on the generic cubic surface) using Chern classes, see for instance [EH16, 6.2.1].

(1) Write a program that computes the expected number of lines on a general hypersurface of degree $2n-3$ in $\mathbb{P}^n$.
(2) Write a program that computes the expected number of 2-planes on a general quartic hypersurface in $\mathbb{P}^7$ ([EH16, Exercise 6.47]).

Exercise 8 (Toric Geometry)**.** To any fan of strongly convex rational cones $\Delta \subset \mathbb{R}^n$ it is possible to associate a toric variety X, and a necessary preliminary step is the construction of the dual fan Δ^*. The fan Δ contains many important information about the toric variety. For instance, X is smooth if and only if any cone $\sigma \in \Delta$ is generated by a basis of the lattice $\mathrm{Span}_{\mathbb{R}}(\sigma) \cap \mathbb{Z}^n$. See [Ful93] for more details on these topics.

Let $\{v_1, \dots, v_m\} \subset \mathbb{Z}^2$ be a primitive set of generators of a complete fan $\Delta \subset \mathbb{R}^2$. Write a program that:

(1) computes the dual fan Δ^*;
(2) deduces if the associated toric variety is smooth.

Exercise 9 (Degeneration of toric variety $*$)**.** Let M be the lattice

$$M = \{(x, y_0, y_1, \dots, y_n) \in \mathbb{Z}^{n+2} \mid y_0 + \cdots + y_n = 0\}$$

and $\Sigma_0 \subset \mathbb{Z}^{n+3}$ be defined as

$$\Sigma_0 = \{(x, y_0, y_1, \dots, y_n, z) \in \mathbb{Z}^{n+3} \mid y_0 + \cdots + y_n = 0, z > 0, xz = y_0^2 + y_1^2 + \cdots + y_n^2\}..$$

Denote by Σ_1 the convex hull of Σ_0 in $M \otimes \mathbb{R} \oplus \mathbb{R} \cong \mathbb{R}^{n+2}$. Define an infinite fan Σ in $M \otimes \mathbb{R}$ (supported on the half space $x > 0$) by projecting the faces of the polytope Σ_1, and let X be the associated non-finite type toric variety (obtained by gluing together infinitely many toric affine charts). The group $M_0 = \{(t_0, t_1, \dots, t_n) \mid t_0 + \cdots + t_n = 0\}$ acts on M preserving Σ by the formula

$$(t_0, t_1, \dots, t_n) \cdot (x, y_0, y_1, \dots, y_n) = (x, y_0 + xt_0, \dots, y_n + xt_n).$$

Projection to the first coordinate $M \to \mathbb{Z}$ defines a morphism $X \to \mathbb{A}^1$ such that the generic fibre is a torus. After taking the quotient by M_0, the generic fibre is an abelian variety (isomorphic to a product of copies of the Tate curve), and the special fibre (corresponding to $x = 1$ in the fan) is a projective (toroidal) degeneration of this abelian variety. The following computation is the first step in working out the natural cell decomposition of this degeneration into tori.

Describe the slice $x = 1$ for a set of representatives for the orbits of the top-dimensional cones of Σ under the action of M_0 for $n = 1, 2, 3, 4$. For $n = 1$ and 2 this is easy by hand, for $n = 3, 4$ it is better to use a computer.

Exercise 10 (Tropical Geometry)**.** Let $p \in \mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ be a Laurent polynomial

$$p(x_1, \dots, x_r) = \sum_{i \in I} c_i x_1^{n_{1,i}} \dots x_r^{n_{r,i}}$$

the tropicalization of $p(x_1, \dots, x_r)$ is the function $\mathrm{Trop}(p) \colon \mathbb{R}^r \to \mathbb{R}$

$$\mathrm{Trop}(p)(x_1, \dots, x_r) = \bigoplus_{i \in I} |c_i| \otimes x_1^{n_{1,i}} \otimes \cdots \otimes x_n^{n_{r,i}} = \min_{i \in I} \left(|c_i| + n_{1,i} x_1 + \cdots + n_{r,i} x^r\right).$$

The associated tropical hypersurface $\mathcal{H}_p \subset \mathbb{R}^r$ is defined to be the non differentiability locus of $\mathrm{Trop}(p)$; this set coincides with the sets of points such that there exist two distinct elements $a, b \in I$ such that:

$$\min_{i \in I} \left(|c_i| + n_{1,i} x_1 + \cdots + n_{r,i} x^r\right) = |c_a| + n_{1,a} x_1 + \cdots + n_{r,a} x^r = |c_b| + n_{1,b} x_1 + \cdots + n_{r,b} x^r;$$

for a more detailed explanation see [MS15].

Given a polynomial $p(x, y)$, write a program that draws the associated tropical curve.

Exercise 11 (Hodge Theory)**.** To a smooth projective variety $X \subset \mathbb{P}^n$ it is possible to associate its Hodge-Deligne polynomial

$$\mathcal{H}(x,y) = \sum_{p,q\geq 0} (-1)^{p+q} h^{p,q}(X) x^p y^q, \qquad h^{p,q}(X) = \dim\left(H^q(X,\Omega^p)\right).$$

If $X = X_{d_1,\dots,d_r} \subset \mathbb{P}^{n+r}$ is a smooth complete intersection, then it is possible to compute the Hodge-Deligne polynomial knowing only the degrees $d_1,\dots,d_r$. The strategy to follow is explained in [Hir66, Appendix one, Theorem 22.1.1 and Theorem 22.1.2].

Given a smooth complete intersection $X_{d_1,\dots,d_r} \subset \mathbb{P}^{n+r}$, write a program that compute its Hodge diamond and Hodge-Deligne polynomial.

Exercise 12 (Lie group and algebra $*$)**.** Let G be a Lie Group and $\mathfrak{g}$ the associated lie algebra. Differential forms on G can be restricted to left-invariant differential forms, which are preserved under the exterior differential. Thus it follows that the de Rham complex restricts to a complex of left-invariant differential forms which gives rise to 'left-invariant' cohomology groups. It is a theorem by Cartan that these are isomorphic to the de Rham cohomology groups if G is compact. All the data in the left-invariant complex can be translated into linear maps between exterior powers of $\mathfrak{g}$.

In this exercise we do these computations using $G = SL(n)$, choose your favourite field.

(1) Compute the structure constants, roots and Weyl group of $\mathfrak{sl}_n$, see [Hum78].
(2) Compute the de Rham cohomology of $SL(n)$. Can anything be said about the ring structure?

Exercise 13 (Morse Theory)**.** Fix two integers $p,q \geq 2$ and consider the polynomial

$$f(x,y) = x^p y + y^q x.$$

By considering a linear projection from the smooth fiber to $\mathbb{C}$, we can think of the smooth fibre of the Morsified function as the total space of an auxilliary fibration. Write a program in Mathematica that:

(1) calculates the critical points of the Morsified function, and plots the critical values;
(2) calculates the critical points of the auxiliary fibration above a given point and plot the critical values;
(3) using the Mathematica function 'animate', track what happens to the critical values of the auxiliary fibration as you move along a vanishing path. Show that for any vanishing path, precisely two of the critical values of the auxiliary fibration come together.

2. Number Theory

Exercise 14 (Power division function)**.** Write a program that for given $k,n \in \mathbb{N}$ compute the power divisor function

$$\sigma_k(n) = \sum_{0<d|n} d^k.$$

Exercise 15 (Elliptic curves $*$)**.** Let $y^2z - (x^3 + azx^2 + bxz^2 + cz^3) \subset K[x,y,z]$ be a non-singular polynomial. This polynomial defines a smooth elliptic curve $\mathcal{C} \subset \mathbb{P}^2$.

Elliptic curves admit a structure of abelian group, we choose as identity the point $E = [0,1,0]$, and the sum is defined as following. Given two points $P,Q \in \mathcal{C}$, the point $-(P+Q)$ is the third intersection point between the line through P and Q and $\mathcal{C}$ (intersection is counted with multiplicity). To find the point $P+Q$ we have just to repeat the previous construction, with the points $-(P+Q)$ and E. Choose $a,b,c \in \mathbb{Z}$ and write a program that:

(1) given a couple of distinct points $P,Q \in \mathcal{C}$ produces as output the coordinates of the point $P+Q$,
(2) given a point $P \in \mathcal{C}$ and an integer n, computes the value of nP; to compute $2P$, consider the tangent line to $\mathcal{C}$ passing through P.
(3) Modify your code to make it works over $\mathbb{F}_p$, remember to verify that the elliptic curve is still smooth after reduction modulo p.
(4) Use the previous points to calculate the canonical height of a point up to a desired precision.
(5) Consider the curve given by the equation $y^2 = x^3 - 7x + 10$. Show that this curve has at least 27 points with integer coordinates; consider the subgroup generated by $P = [1,2,1]$ and $Q = [2,2,1]$. ([Har77, 4.4.18])

Exercise 16 (Generalized theta series $*$)**.** Chose a positive-defined quadratic form Q and, using the theory of generalized theta series, write a program which computes the representation numbers

$$r_Q(n) = \#\{v \in \mathbb{Z}^4 : v^t Q v = n\}, \qquad n \in \mathbb{N}.$$

Exercise 17 (Modular forms)**.** For this exercise, we recommend to use either Sage or Magma, with one of the packages for modular forms (without them, the first part is a starred exercise).

(1) This part is just to get familiar with some of the commands in the package: define some character, pick a weight, and compute a basis for some space of modular forms. Compute the Hecke operators, compute some eigenforms and see that their q-expansions are as desired.
(2) ($*$) Write a code that, given a modular form f_1 of some tame level N coprime to p, weight k and some character χ_1 that's ramified at p, verifies if it is true that for any character χ_2 satisfying $\chi_2^p \equiv \chi_1 \mod p$, there is a modular form f_2 such that $f_2^p \equiv f_1$ in q-expansion.

References

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