GEOMETRIC ANALYSIS SECTIONS 7,8

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Section 7. Introduction to Perelman's work on Ricci flow

[Pages 1-11 in G. Perelman *The entropy formula for the Ricci flow and its geometric applications* arxiv:math/0211159]

BACKGROUND I

Bounds on the curvature and injectivity radius give good control of Riemannian metrics.

For example, suppose that (M_i, g_i) is a sequence of complete Riemannian n-manifolds with base points $p_i \in M_i$.

Suppose that that there are ρ, C such that for each manifold $|\mathrm{Riem}| \leq C\rho^{-2}$ and $\mathrm{inj} \geq \rho$.

Then there is a subsequence which converges to a $C^{1,\alpha}$ limit $(M_\infty, g_\infty, p_\infty)$, with curvature tensor in L^q for all q.

Without the lower bound on the injectivity radius, more complicated *collapsing* phenomena can occur.

For example, there is a sequence of metrics on S^3 with bounded curvature which "converge" to the round S^2 by collapsing the fibres of the Hopf fibration $S^3 \to S^2$.

Control of the volume of balls gives control of the injectivity radius.

If $|\mathrm{Riem}| \leq \rho^{-2}$ and the injectivity radius is much less than ρ there is a ball B_r of radius $r = O(\rho)$ with very small volume ratio

$$\frac{\mathrm{Vol}(B_r)}{r^n}.$$

This is related to Sobolev inequalities. In the above situation there is a function f supported in B_r such that $\|\nabla f\|_{L^1}$ is very small compared to $\|f\|_{L^{n/n-1}}$.

For example, let S^1_ϵ be the circle of length $\epsilon << 1$ and $M = \mathbf{R} \times S^1_\epsilon$.

Let f be a smoothing of the characteristic function of a unit ball.

Then

$$\|\nabla f\|_{L^1} \sim \epsilon \quad \|f\|_{L^2} \sim \epsilon^{1/2}.$$

BACKGROUND II

Suppose that we have a functional $\mathcal{F}$ on the space $\mathcal{M}$ of Riemannian metrics on a compact manifold M.

The derivative of $\mathcal{F}$ at g is a tensor $S(g) = S_{ij}$.

$$\delta\mathcal{F} = \int_M \langle \delta g, S\rangle d\mathrm{vol}_g.$$

That is, we are assuming the standard L^2 metric on the infinite dimensional space $\mathcal{M}$ to regard the derivative of $\mathcal{F}$ as a vector field on $\mathcal{M}$.

If $\mathcal{F}$ is invariant under the diffeomorphism group Diff the tensor S satisfies the identity $\nabla^* S = 0$ where $\nabla^* S = \sum \nabla_j S_{ij}$.

This is because for any vector field v on M the Lie derivative of g along v is the symmetric part of $\nabla\alpha$ where α is the 1-form corresponding to v, and Diff-invariance implies that

$$\int_M \langle \nabla\alpha, S\rangle = \int_M \langle \alpha, \nabla^* S\rangle$$

vanishes for all α.

Formally, we have a gradient flow $\frac{\partial g}{\partial t} = S(g_t)$. (Formal in the sense that solutions may not exist, even for a short time.)

This induces (formally) a flow on $\mathcal{M}/\mathrm{Diff}$.

More generally, we could have some other assignment $g \mapsto S(g)$—*i.e.* a vector field on the infinite dimensional space $\mathcal{M}$.

Formally this defines a flow on $\mathcal{M}$, but if S does not satisfy the identity $\nabla^* S = 0$ it is not the gradient flow of a Diff-invariant functional.

The Einstein-Hilbert function I has derivative

$$S_{ij} = R_{ij} - \frac{R}{2} g_{ij}.$$

The identity $\nabla^* S = 0$ is the contracted second Bianchi identity.

We cannot hope to find Einstein metrics by minimising or maximising I (subject to the volume 1 constraint) because I is not bounded above or below.

The formal flow associated to I is

$$\frac{\partial g}{\partial t} = -2R_{ij} + Rg_{ij}.$$

This does not have short time solutions, for generic initial conditions.

To understand this, consider small variations $g_{ij} = \delta_{ij} + h_{ij}$ about the flat metric on a torus. We can impose the "gauge fixing" condition $\nabla_j h_{ij} = 0$. Then one finds that to first order $R_{ij} = -\Delta h_{ij}$ and the linearised version of the flow is

$$\frac{\partial h_{ij}}{\partial t} = 2\Delta h_{ij} - \Delta \mathrm{Tr}(h)\delta_{ij}.$$

This is the heat equation for the trace-free part of h and a backward heat equation for the trace.

The good PDE is the Ricci flow

$$\frac{\partial g}{\partial t} = -2R_{ij}$$

with linearisation

$$\frac{\partial h_{ij}}{\partial t} = 2\Delta h_{ij}.$$

But this is not the gradient flow of a Diff-invariant functional, since $\nabla^* \mathrm{Ricci} = \sum \nabla_j R_{ij}$ does not vanish.

END OF BACKGROUND

A plausible variational approach to finding Einstein metrics is to *maximise* μ_g over conformal classes, where $\mu(g)$ is the conformal invariant consider in the previous section, obtained by *minimising* within a conformal class.

In dimension n one parametrises a conformal class by $\tilde{g} = u^{4/n-2}g$ and μ_g is the minimum of

$$Q_g(u) = \int_M |\nabla u|^2 + \alpha R,$$

subject to the constraint $\|u\|_{L^p} = 1$ with the constants

$$\alpha = \frac{n-2}{4(n-1)} \quad p = \frac{2n}{n-2}.$$

But *let's do the calculations for general* α, p.

Calculus

Suppose we have two families of functions $A_s(x), B_s(x)$ and we define $\mathcal{F}(s)$ to be the minimum of A_s subject to the constraint $B_s = 1$.

For fixed s, at the minimum $x(s)$ we have $\nabla A = \lambda \nabla B$.

One finds that

$$\frac{\partial}{\partial s}\mathcal{F} = \frac{\partial A}{\partial s} - \lambda \frac{\partial B}{\partial s}, \qquad (***)$$

evaluated at $x(s)$.

In our situation, s runs over the space of metrics and we have two functions on the space of positive functions $uC^\infty(M)$. We have

$$\mathcal{F}(g) = \mathrm{min} Q_g(u)$$

subject to $\int u^p d\mathrm{vol}_g = 1$.

(We are ignoring the question of the existence of the minimiser for the moment.)

The minimising function u satisfies

$$-\Delta u + \alpha R u = p\lambda u^{p-1}, \qquad (****)$$

for some Lagrange multipier λ.

Writing $\delta g = h$ with $\mathrm{trace} h = H$, the equations (***), (****) and the formula for the variation in the scalar curvature give: $\delta\mathcal{F}$=

$$\int_M -h_{ij}\nabla_i u \nabla_j u + \frac{H}{2}(|\nabla u|^2 + \alpha R u^2) + \alpha(-\Delta H + \nabla_i\nabla_j h_{ij} - R_{ij}h_{ij})u^2 + \dots$$

$$\dots + p^{-1} H(\Delta u - \alpha R u)u.$$

Integrating by parts, we get the sum of the integrals of

$$h_{ij}(-\nabla_i u \nabla_j u - \alpha R_{ij} u^2 + \alpha \nabla_i \nabla_j (u^2),$$

and

$$H\left(\frac{1}{2}|\nabla u|^2 + (\alpha/2 - \alpha/p)Ru^2 + p^{-1}u\Delta u - \alpha\Delta(u^2)\right).$$

We can write this as

$$\delta \mathcal{F} = \int_M \sum S_{ij} h_{ij} d\mathrm{vol}$$

where S is the tensor

$$\left(-\alpha R_{ij} + \alpha \nabla_i \nabla_j (u^2) - \nabla_i u \nabla_j u \right) + \dots$$

$$\dots + \left(\frac{1}{2} |\nabla u|^2 + (\alpha/2 - \alpha/p) R u^2 + p^{-1} u \Delta u - \alpha \Delta (u^2) \right) g_{ij}$$

In the Yamabe case the we get

$$S = u^2 \, \widetilde{\mathrm{Ricci}}_0$$

where $\widetilde{\mathrm{Ricci}}$ is the Ricci curvature of the conformal metric $u^{4/n-2}g$ and $\widetilde{\mathrm{Ricci}}_0$ is its trace-free part.

We already knew that!

Perelman found another special case, when $p = 2$ and $\alpha = 1/4$. (These are the values of the limits of the Yamabe parameters as $n \to \infty$.)

In this case the Euler-Lagrange equation is linear and $\mathcal{F}(g)$ is just the first eigenvalue of the operator $-\Delta + R/4$.

We have

$$(1/2)|\nabla u|^2 + (1/2)u\Delta u - (1/4)\Delta(u^2) = 0.$$

Also:

$$-\nabla_i u \nabla_j u + (1/4)\nabla_i \nabla_j (u^2) = \nabla_i \nabla_j (f)$$

where $f = -\frac{1}{2}\log u$.
So in this case

$$S_{ij} = (-1/4)[R_{ij} + \nabla_i \nabla_j f]u^2.$$

Under the flow

$$\frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f), (*****)$$

we have

$$\frac{d\mathcal{F}}{dt} = -\int |\mathrm{Ricci} + \nabla^2 f|^2 \, u^2 \, d\mathrm{vol}.$$

But the flow (*****) is equivalent to Ricci flow modulo diffeomorphisms so we conclude that the same formula holds under Ricci flow.
The point is that $\nabla^2 f$ is the Lie derivative of g along the vector field $\mathrm{grad} f$.

Another way of saying this is that since $\nabla^*(S) = 0$ we have

$$\int \langle S, \nabla^2 f \rangle = 0.$$

So

$$\int \langle S, \mathrm{Ricci} \rangle = \int \langle S, \mathrm{Ricci} + \nabla^2 f \rangle$$

On $\mathcal{M}/\mathrm{Diff}$ the Ricci flow is the gradient flow of the functional $\mathcal{F}$ with respect to the metric defined using the modified volume form $u^2 d\mathrm{vol}_g$.

Now let Φ be a function on $\mathbf{R}$ and consider for each metric g a functional

$$Q_g(u) = \int_M |\nabla u|^2 + R/4u^2 + \Phi(u),$$

subject to the constraint that $\int u^2 = 1$. Assume that there is a unique minimiser and define a functional on $\mathcal{M}$ as before. The Euler-Lagrange equation for the minimum is

$$-\Delta u + (R/4)u + \frac{1}{2}\Phi'(u) = \lambda u.$$

We get the integral of

$$\left(u\Phi'(u) - 2\Phi(u)\right) H$$

as an additional term in the formula for the variation, so we get an additional term $(u\Phi'(u) - 2\Phi(u))\, g_{ij}$ in S.

Choose $\Phi(u) = -u^2 \log u$, so $u\Phi' - \Phi = u^2$. This defines a functional which we call $\mathcal{W}$. The associated tensor is

$$S_{ij} = (-1/4)[R_{ij} + \nabla_i \nabla_j f + g_{ij}]u^2.$$

So the functional $\mathcal{W}$ is increasing under the flow

$$\frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f + g_{ij}). \qquad (*******)$$

But this flow is also essentially equivalent to Ricci flow. If $g(t)$ is a Ricci flow defined for $t < T_0$ and if we fix any T, the path of metrics

$$g^*(t) = (T - t)^{-1} g(t)$$

defined for $t < \min T, T_0$ satisfies

$$\frac{\partial g^*}{\partial t} = -2(T - t)^{-2}(\mathrm{Ricci}(g^*) + g^*),$$

which is equivalent to (*******) up to diffeomorphisms and a reparametrisation of the "time" variable.

We conclude that:

for a Ricci flow $g(t)$ the quantity

$$\mathcal{W}\left((T-t)^{-1}g(t)\right)$$

is an increasing function of t.

On any compact Riemannian manifold (N, G) a *Logarithmic Sobolev inequality* holds.

There is a constant $C = C(G)$ such that for all positive functions u with L^2 norm 1:

$$\int |\nabla u|^2 - u^2 \log u \geq C.$$

Suppose that the manifold (N, G) has small curvature, $\mathrm{Riem} \leq 1/100$ say, and there is metric ball of radius 1 with very small volume ν.

Let v be a suitable smoothing of the characteristic function of this unit ball and u be the multiple of v chosen so that the integral of u^2 is 1. So u is approximately $\nu^{-1/2} v$ and

$$\int |\nabla u|^2 \sim \nu(\nu^{-1/2})^2 = 1$$

while

$$\int u^2 \log u \sim \nu(\nu^{-1/2})^2 \log \nu^{-1/2} = -(1/2) \log \nu$$

So the constant C for this metric is very negative: at most about $(1/2) \log \nu << 0$.

That is: *the best constant in the logarithmic Sobolev inequality detects volume collapsing* (just as for the ordinary Sobolev inequalities).

It is clear, given the above, that our functional

$$Q_g(u) = \int |\nabla u|^2 + (R/4)u^2 - u^2 \log u$$

is bounded below on the functions with L^2 norm 1 so the infimum $\mathcal{W}(g)$ is well-defined.

It can be shown that there is a minimiser (as we assumed in our preceding discussion).

Application to singularity formation in Ricci flow.
Suppose that $g(t)$ is a Ricci flow defined on a time interval $[0, T_0)$.

Without real loss of generality suppose that $T_0 = 1$.

We have some number C so that $\mathcal{W}(\lambda\, g(0)) \geq C$ for all $\lambda \in [1/2, 2]$.

At any given time $\tau \in (1/2, 1)$ write $\max_M |\mathrm{Riem} g(\tau)| + 1 = \rho^{-2}$.

Take $T = \tau + \epsilon\rho^2$ for a suitable small number ϵ, say $\epsilon = 1/100$. So T lies in the interval $[1/2, 2]$.

Let $g^*(t) = (T - t)^{-1}\, g(t)$ for $0 \leq t \leq \tau$.

By the monotonicity result

$$\mathcal{W}(g^*(\tau) \geq \mathcal{W}(T^{-1} g(0) \geq C. \quad (*******)$$

Write $G = g^*(\tau)$. By construction the curvature of G is bounded by ϵ. For a function supported in a unit ball in (M, G) the scalar curvature term in the formula for Q_G is very small.
It follows from (*******) and the preceding discussion that a unit ball in (M, G) cannot have a very small volume.

Hence the injectivity radius of (M, G) is at least δ for some computable number δ.

Suppose we have a sequence of times $t_i \to 1$.
Let $p_i \in M$ be a point where the maximum value of $|\mathrm{Riem}(g(t_i)|$ is attained. Define ρ_i as above and $G_i = \epsilon^{-1}\rho_i^{-1} g(t_i)$.

The based Riemannian manifolds (M, G_i, p_i) have bounded curvature and a lower bound on the injectivity radius.

So, possibly passing to a subsequence, there is a $C^{1,\alpha}$ limit.

If in fact the size of the curvature of $g(t_i)$ at p_i becomes large then the size of the curvature of G_i at p_i is at least 1 so one can hope to show that the limit is non-trivial (*i.e.* not $\mathbf{R}^n$).

Results of Hamilton shows that one also gets “blow up limits” of the Ricci flow, which is an “ancient solution”, defined for all $t < 0$.

Analysis of such limits leads to an understanding of possible ways in which singularities can form in the Ricci flow, particularly in dimension 3.

Section 8. Singular perturbation problems

One approach that can be used to construct solutions to a variety of equations involves constructing an approximate solution and then seeking to deform this to an exact solution. Suppose we set up our equation as $\mathcal{F}(A) = 0$ for some nonlinear map $\mathcal{F}$ on an open set in a Banach space H_1 taking values in a Banach space H_2. Let A_0 be an approximate solution in that $\|\mathcal{F}(A_0)\|$ is small in a suitable sense. Let L be the derivative of $\mathcal{F}$ at A_0 so

$$\mathcal{F}(A_0 + a) = \mathcal{F}(A_0) + La + Q(a).$$

Suppose that L has a right inverse S and write $a = S\sigma$. The equation to be solved is

$$\sigma = T(\sigma)$$

where

$$T(\sigma) = -Q(S(\sigma)) - \mathcal{F}(A_0).$$

We seek a solution using the contraction mapping principle. This will work if there is some r such that

$$\|Q(S\sigma_1) - Q(S\sigma_2)\| <\leq (1/2)\|\sigma_1 - \sigma_2\|,$$

(say) for all $\|\sigma_1\|, \|\sigma_2\| < r$ and

$$\|\mathcal{F}(A_0)\| << r.$$

For example, suppose that Q is a quadratic function with

$$\|Q(A_1) - Q(A_2)\| \leq C\|A_1 - A_2\| \; \|A_1 + A_2.\|$$

Suppose that $\|S\| = \epsilon^{-1}$.

Then

$$\|Q(S\sigma_1) - Q(S\sigma_2)\| \leq C\epsilon^{-2}\|\sigma_1 - \sigma_2\| \; \|\sigma_1 + \sigma_2\|,$$

and we will find our solution provided that

$$\|\mathcal{F}(A_0)\| \leq c\epsilon^2,$$

where $c = (10C)^{-1}$, say.

One type of application of this idea involves “gluing constructions”.

Example Let $T = \mathbf{C}^2/\Lambda$ be a 2-dimensional complex torus. The involution of T induced by $z \mapsto -z$ on $\mathbf{C}^2$ has $2^4 = 16$ fixed points. Taking the quotient and blowing up these points gives a Kummer surface X. This admits Kahler metrics and has zero first Chern class so by Yau’s Theorem it has Kähler metrics with zero Ricci curvature: one in each Kähler class.

We can produce some of these metrics, independent of Yau's theorem by gluing.

X contains 16 holomorphic 2-spheres. A neighbourhood of one of these is holomorphically equivalent to a neighbourhood of the zero section in the total space Y of the cotangent bundle of S^2. There is an explicit Kähler metric of zero Ricci curvature on Y. The complement of the union of the spheres in X has a flat metric induced from T.

We can write down an approximate solution by "gluing" 16 copies of Y, scaled by factors $\epsilon_1, \dots \epsilon_{16}$, to the flat metric.

The parameters ϵ_i and the choice of flat metric fix the Kähler class.

When the ϵ_i are sufficiently small the strategy outlined above can be used to deform the approximate solution to an exact one.

This does not replace Yau's proof because we cannot treat all Kähler classes.

But it gives more in that one knows almost exactly what the metrics are.

Note that there is some connection between these gluing constructions and our discussion of the Yamabe problem.

In this course we will not say more about gluing but we discuss *adiabatic problems*.

By this we mean geometry on a manifold which is a product with one factor much larger than the other, or more generally a fibration with the fibre much smaller than the base.

We will discuss one model problem, but the same techniques apply to many other problems.

Let X, T, N be Riemannian manifolds with X and T compact. For $\epsilon > 0$ we consider the metric $g_\epsilon = g_X + \epsilon^{-1} g_T$ on $T \times X$. We want to consider harmonic maps from $(T \times X, g_\epsilon)$ to N.

Recall that any map $f : T \to N$ has a tension field $\tau_T(f)$ and the harmonic map condition is $\tau_T = 0$. At a solution f there is a linearised operator J_f ("Jacobi operator"), which is a self-adjoint operator on sections of $f^*(TN)$.

For example if $T = S^1$, so we have a closed geodesic, this is the classical Jacobi operator

$$J(v) = \nabla_t^2 v + R(v, \underline{t})\underline{t},$$

where R is the curvature tensor of N and $\underline{t}$ is the tangent vector to the geodesic.

The quadratic form $\langle J_f v, v \rangle$ defines the second variation of the harmonic maps energy.

Suppose that we have some connected moduli space M of harmonic maps from X to N.

We assume that M is Morse-Bott, which is to say that it is a manifold with tangent spaces equal to the kernels of the Jacobi operators.

The L^2 metric on sections of $f^*(TN)$ defines a Riemannian metric on M.

Suppose that $\phi : T \rightarrow M$ is a harmonic map. We say that ϕ is rigid if the Jacobi operator of ϕ has zero kernel.

A map $\phi : T \to M$ defines a map $F_\phi : T \times X \to N$ in an obvious way.
That is, we are using

$$\mathrm{Maps}\,(T \times X, N) = \mathrm{Maps}\,(T, \mathrm{Maps}\,(X, N)).$$

Theorem

Suppose that M is Morse-Bott and a local minimum of the energy functional. and suppose that $\phi : T \rightarrow M$ is rigid. Then for small ϵ the map F_ϕ can be deformed slightly to give a harmonic map from $T \times X, g_\epsilon)$ to N.

For any map $F : T \times X \to N$ the tension field with respect to g_ϵ can be written as $\tau_X + \epsilon \tau_T$ where τ_X, τ_T are the tension fields of the restrictions of F to the slices $\{t\} \times X, T \times \{x\}$.

Our map F_ϕ is characterised by two properties:

- $\tau_X(F_\phi = 0$
- $\pi \tau_T(F_\phi) = 0$

where for each fixed t the map π acts as the projection from sections of the pull-back of TN over $\{t\} \times X$ to the kernel of the Jacobi operator of the map $\phi(t) : X \to N$.

BACKGROUND

Finite-dimensional analogue

Let $V \to Z$ be a rank r real vector bundle over an r-dimensional manifold.

Suppose that σ is a section of V which vanishes on a submanifold $Y \subset Z$ and that at each point of Y the kernel of the derivative of σ is the tangent space of Y.

Let $Q \to Y$ be the vector bundle formed by the cokernels of the derivative of σ, so we have a quotient map $\pi : V|_Y \to Q$.

Suppose that τ is another section of V. Then $\pi(\tau)$ is a section of Q over Y. Suppose that $\pi(\tau)$ has a transverse zero at a point $y_0 \in Y$.

Then, for small ϵ the section $\sigma + \epsilon\tau$ of V over Z has a zero near to y_0.

To see this, consider the case $r = 2$ with $\mathrm{dim Y} = 1$.

We can suppose that $Z = \mathbf{R}^2$ and that V is the trivial bundle with $\sigma(x, y) = (x, 0)$, so Y is the y-axis. We can suppose that $y_0 = (0, 0)$.

We have $\tau(x, y) = (f_1(x, y), f_2(x, y))$ and the hypothesis is that f_2 vanishes at the origin but $\frac{\partial f_2}{\partial y}$ does not vanish there.

So f_2 vanishes on a curve γ through the origin transverse to the y-axis.

It is clear that for small ϵ the function $x + \epsilon f_1$ has a zero on γ. This gives a zero of $\sigma + \epsilon\tau$.

There is another analogue of “intermediate infinite-dimensionality”.
Let B be a Riemannian manifold, V a real-valued function on P and λ a real parameter.
We can deform the harmonic maps energy for maps $f : T \to B$ using the potential function V:

$$E_\lambda(f) = \int_T |\nabla f|^2 + \lambda \int_T V \circ f.$$

If $T = S^1$ this is the Lagrangian for the motion of a particle on B in the potential $-V$.

Suppose that V is a Morse-Bott function with a critical submanifold $M \subset B$ which is a local minimum and that $\phi : T \to M$ is a rigid harmonic map.

Then for large λ there is a solution of the Euler-Lagrange equations for E_λ which is close to ϕ.
Our problem fits into this setting with B the infinite dimensional space of maps from X to N, V the energy functional and $\lambda = \epsilon^{-1}$

There are analogues of our Theorem in gauge theory. If M is a Morse-Bott moduli space of minimising Yang-Mills connections over a manifold X and $\phi : T \to M$ is a rigid harmonic map then we get a Yang-Mills connection over $(T \times X, g_\epsilon)$ for small ϵ. (Theorem of Y. Hong)

In suitable situations where the moduli space M is a complex manifold there are analogues for *holomorphic* maps $\phi : T \to M$ and solutions of first-order equations over $T \times X$, such as the Seiberg-Witten and instanton equations (Theorem of G. Dostoglou and D. Salamon.)

This is what underlies the correspondence between Ozsvath-Szabo's “Heegard Floer Theory” and Seiberg-Witten Theory. The Heegard theory is built using holomorphic maps to the vortex moduli spaces (symmetric products).

It is also related to Taubes' correspondence between holomorphic curves and solutions of the deformed Seiberg-Witten equation.

The whole discussion can be extended to replace the product $X \times T$ with a fibre bundle $\mathcal{X} \to T$.

END OF BACKGROUND

START OF OUTLINE PROOF OF THE THEOREM.

Linear Theory

Begin by considering a strictly positive self-adjoint operator J over X equal to $-\Delta_X$ plus lower order terms (for example $-\Delta_X + 1$).

Let Δ_m be the Laplace operator on $\mathbf{R}^m$. Then $L = J - \Delta_m$ is an operator on functions on $X \times \mathbf{R}^m$ equal to the Laplacian on $\mathbf{R}^m \times X$ plus lower order terms.

Proposition 1
There is a bounded right inverse

$$L^{-1} : C^{,\alpha} \rightarrow C^{2,\alpha}.$$

It suffices to get a bound on $L^{-1} : C^0 \rightarrow C^0$. For if $\rho \in C^{,\alpha}$ and $f = L^{-1}\rho$ so that $Lf = \rho$ then by applying the standard elliptic estimates in balls of fixed size we get

$$\|f\|_{C^{2,\alpha}} \leq C \left(\|f\|_{C^0} + \|\rho\|_{C^{,\alpha}} \right)$$

First there is an easy L^2 theory.
Define a norm on compactly supported functions on $T \times X$

$$\|f\|_1^2 = \langle Jf, f\rangle + \langle \nabla_T f, \nabla_T f\rangle.$$

Then $\|f\|_1 \geq c\|f\|_{L^2}$ where c^{-1} is the first eigenvalue of J on X so the completion H under $\|\|_1$ includes L^2.
For $\rho \in L^2$ the map $g \mapsto \langle g, \rho\rangle$ is bounded on H and so by the Riesz representation theorem there is an f such that

$$\langle g, \rho\rangle = \langle g, f\rangle_1$$

for all g, which says that f is weak solution of $Lf = \rho$.

We can study this using separation of variables. Let ψ_λ be an orthonormal basis of eigenfunctions for J over X. Write

$$\rho = \sum_\lambda \rho_\lambda \psi_\lambda$$

so ρ_λ are functions on $\mathbf{R}^m$.
A function $f = \sum f_\lambda \psi_\lambda$ satisfies $Lf = \rho$ if

$$(-\Delta_m + \lambda) f_\lambda = \rho_\lambda. \quad (****)$$

The equation (****) on $\mathbf{R}^m$ can be solved using Fourier transform methods, or otherwise.
The Green's function for $-\Delta_m + 1$ is a function $K(r)$ with a standard singularity at the origin and exponential decay: $K(r) \leq Ce^{-\alpha r}$, for $r > 1$ say.

For example in dimension $m = 1$ one has $K(r) = e^{-r}$.

It is then straightforward to show that the Green's function for L has exponential decay and is integrable in the sense that

$$\int |G((t,x),(t',x')|dtdx \leq C,$$

which immediately gives $\|L^{-1}\rho\|_{C^0} \leq C\|\rho\|_{C^0}$.

Now let J_t be a smooth family of such operators on X parametrised by T which we can regard as an operator J on $T \times X$. Let $\Lambda = J - \epsilon\Delta_T$, an operator over $T \times X$.
The leading term is minus the Laplacian of $(T \times X, g_\epsilon)$

Proposition 2

For small ϵ the operator Λ has a right inverse with $C^{,\alpha} \mapsto C^{2,\alpha}$ operator norm bounded, independent of ϵ.

As before, it suffices to get $C^0 \mapsto C^0$ bounds.

Let B be a small ball of radius δ in T, in the fixed metric g_T and σB be a larger concentric ball of radius $\sigma\delta$.
In the metric $\epsilon^{-1} g_T$ the ball σB is almost isometric to a large ball of radius $\sigma\delta\epsilon^{-1/2}$ in $\mathbf{R}^m$.
Over $\sigma B \times X$ we can regard Λ as a small perturbation of an operator of the kind we considered in Proposition 1, and thus construct an inverse P, so that for a function ρ supported in $B \times X$ we have $\Lambda P\rho = \rho$ over $\sigma B \times X$.

Take a cover of T by such small balls B_i and let χ_i be a subordinate partition of unity.

For each i, let $\tilde{\chi}_i$ be a function supported on σB_i equal to 1 on B_i.
Then we have local inverses P_i and we define

$$P(\rho) = \sum_i \tilde{\chi}_i P_i(\chi_i \rho).$$

Then

$$\Lambda P(\rho) - \rho = \nabla^2 \tilde{\chi}_i * P_i(\chi\rho) + \nabla\tilde{\chi} * \nabla P_i(\chi_i\rho). \quad (*****)$$

The support of $\nabla\tilde{\chi}_i$ is separated from the support of χ_ρ. Adjusting parameters and using the exponential decay we can make the right hand side of (*****) very small compared with $\|\rho\|$ and thus prove Proposition 2.

Now consider a family of semi-positive operators J_t with kernels H_t forming a vector bundle $H \to T$.

Let π_0 be the projection from functions on $T \times X$ to sections of H. We have linear operator

$$\lambda = -\pi_0 \Delta_T : \Gamma(H) \to \Gamma(H).$$

This is second order self-adjoint differential operator of Laplace type.

In the product case, when J_t is constant and H is a trivial vector bundle, λ is just $-\Delta_T$.

Define Λ as before.

Let H_t^+ be the L^2-orthogonal complement of H_t, the span of the eigenfunctions with positive eigenvalues. These form an infinite-dimensional vector bundle over T and we have

$$C^\infty(T \times X) = \Gamma(H^+) \oplus \Gamma(H).$$

With respect to this the operator Λ has a “matrix” description:

$$\Lambda = \begin{pmatrix} \Lambda_{++} & \Lambda_{+0} \\ \Lambda_{0+} & \Lambda_{00} \end{pmatrix}$$

and $\Lambda_{00} = \epsilon\lambda$.

Proposition 3

- The operator Λ_{++} is invertible and the inverse has operator norm bounded independent of ϵ.
- If λ is invertible then, for small ϵ, Λ is invertible and the inverse has operator norm bounded by $C\epsilon^{-1}$.

We skip the proof. The statement is clearly true in the product case. The main point can be seen for 2×2 matrices

$$\begin{pmatrix} 1 & B\epsilon \\ C\epsilon & D\epsilon \end{pmatrix}$$

which are invertible for small ϵ if $D \neq 0$.

This completes our discussion of the linear theory.
We go back to our harmonic map problem for $(T \times X, g_\epsilon) \to N$.
We suppose that N is an open subset in $\mathbf{R}^q$ with some Riemannian metric having Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ so a map is a vector valued function u.

(Using the same device as in Section 3 we can treat the general N this way.)

We want to solve the equation $\mathcal{F}(u) = 0$ where

$$\mathcal{F}(u) = \tau_X(u) + \epsilon\tau_T(u).$$

We have $\tau_X(u) = \Delta_X u + \Gamma^{\lambda}_{\mu\nu}\nabla_X u^{\mu}.\nabla_X u^{\nu}$ and similarly for τ_T.

Here Γ is evaluated at the point $u(t, x)$.

In this set-up, at any u we have a derivative D_u of $\mathcal{F}$. It is a linear map from $\mathbf{R}^q$-valued functions to $\mathbf{R}^q$-valued functions.

BUT we should keep in mind that if u is not a harmonic map this derivative does not have a completely geometric meaning: it depends on the "co-ordinates" i.e. the description of the maps from $T \times X$ to N as an open subset of a vector space.

We write $D_u = D_{u,X} + \epsilon D_{u,T}$ in the obvious way.

Let U be the function corresponding to $\phi : T \to M$.

This is our first attempt at an "approximate solution".

We have

$$\mathcal{F}(u) = \epsilon \tau_T(U)$$

and for each $t \in T$ we have a linearised operator J_t on X as considered above.

The kernel of J_t is the tangent space of M at $\phi(t)$. We have an operator J as before.

The condition that ϕ is a harmonic map says that $\pi_0(\tau_T(U)) = 0$.

So there is a function ξ such that $J_(\xi) = -\tau_T U$.

Let $u = U + \epsilon\xi$. This is an "improved approximate solution".

We have $D_{U,X} = J$ but $D_{u,X}$ will differ from this by $O(\epsilon)$. There is an operator μ such that

$$D_{u,X} = J + \epsilon\mu + O(\epsilon^2),$$

so

$$D_u = J + \epsilon(D_{U,T} + \mu) + O(\epsilon^2).$$

Let $\lambda : \Gamma(H) \to \Gamma(H)$ be defined by

$$\lambda = \pi_0 \circ (D_{U,T} + \mu).$$

Key Lemma λ is the Jacobi operator of the harmonic map $\phi : T \to M$.

To understand this consider a finite dimensional analogue with a function $f(x, y) = a(x^2 - y)^2$ on $\mathbf{R}^2$. This has the parabola $y = x^2$ as critical submanifold. Let g be another function on $\mathbf{R}^2$ whose restriction to the parabola has a nondegenerate critical point at the origin. Our problem is to find a critical point of $f + \epsilon g$ for small ϵ.

The hypothesis on g is *not* the same as saying that g_{xx} is non-vanishing at the origin.

This is a reflection of the fact that the Hessian does not have an intrinsic (coordinate independent) meaning at non-critical points.

If we parametrise the parabola by $(x = t, y = t^2)$ we have

$$g_{tt}(0) - g_{xx}(0) = 2g_y(0) \quad (*******)$$

Here g_{tt} is the analogue of the operator λ.

Let $\xi = (1/2a)g_y$ and consider the point $p = (0, \epsilon\xi)$. This point p is the analogue of u.

We have $\nabla(f + \epsilon g)(p) = O(\epsilon^2)$ and

$$f_{xx}(p) = f_{xx}(0) + \epsilon\xi f_{xxy} + O(\epsilon^2).$$

Since $f_{xxy} = 4a$ at the origin this is

$$f_{xx}(p) = f_{xx}(0) + \epsilon 2g_y + O(\epsilon^2)$$

The change in f_{xx} going from $(0, 0)$ to p matches up with the defect term (*******).

The operator $\Lambda = J + \epsilon(D_{U,T} + \mu)$ is not exactly of the form we considered before but the same arguments apply and we deduce that Λ^{-1} has operator norm bounded by $C\epsilon^{-1}$.

The derivative D_u differs from Λ by $O(\epsilon^2)$ so D_u^{-1} has the same bound.

We know that $\|\mathcal{F}(u)\|$ is $O(\epsilon^2)$ but this is not quite good enough to apply the contraction mapping argument.

That would work if $\|\mathcal{F}(u)\| \leq c\epsilon^2$ for some possibly small number c (computable from the various bounds we have).

So we look for an improved approximate solution of the form

$$u' = u + \epsilon\theta + \epsilon^2\eta$$

for a suitable $\theta \in \Gamma(H), \eta \in \Gamma(H^+)$.

We find that

$$\mathcal{F}(u') = \mathcal{F}(u) + \epsilon^2(J(\eta) + (D_{T,U} + \mu)(\theta)) + O(\epsilon^3).$$

So we can solve for θ, η to remove the $O(\epsilon^2)$ term in $\mathcal{F}(u)$. We get $\mathcal{F}(u') = O(\epsilon^3)$.

For small ϵ, this u' is a sufficiently good approximate solution to use the contraction mapping argument.

One can also go on to construct a formal power series solution

$$U + \sum_{i=1}^{\infty} \xi_i \epsilon^i + \theta_i \epsilon^i .$$

This means that one could get by with weaker bounds on the inverse operators.