Differential Geometry, Part 3

Simon Donaldson

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Let V be a finite-dimensional Euclidean vector space. The *Clifford algebra* $Cl(V)$ is the algebra with unit generated by V subject to the relations

$$v_1 v_2 + v_2 v_1 = -2\langle v_1, v_2\rangle,$$

for $v_1, v_2 \in V$.

Let $e_1, \ldots .e_n$ be an orthonormal basis for V. It is clear that $Cl(V)$ has dimension 2^n and a basis $e_{i_1} \ldots e_{i_p}$ for $i_1 < i_2 \cdots < i_p$. There is a canonical vector space isomorphism $\sigma : C(V) \to \Lambda^* V$. So the Clifford algebra can be regarded as $\Lambda^* V$ with a modified product. The Clifford product maps $\Lambda^p \otimes \Lambda^q$ to

$$\Lambda^{p+q} \oplus \Lambda^{p+q-2} \oplus \Lambda^{p+q-4} \ldots,$$

and the highest degree term is the same as the wedge product.

If $\xi_1, \xi_2 \in \Lambda^2 \subset C(V)$ then, using the Clifford product, $\xi_1\xi_2 - \xi_2\xi_1 \in \Lambda^2$. So the commutator in $Cl(V)$ defines a bracket $\Lambda^2 \otimes \Lambda^2 \to \Lambda^2$. On the other hand $\Lambda^2 = \mathfrak{so}(V)$, the Lie algebra of $SO(V)$ and one checks that this gives the same bracket.
It follows that any representation of $Cl(V)$ on a vector space Σ (i.e. an algebra homomorphism $Cl(V) \to \mathrm{End}\,(\Sigma)$) defines a representation of $\mathfrak{so}(V)$ on Σ.

Defining a representation of $C(V)$ on Σ is the same as defining a map $\gamma : V \to \mathrm{End}\,\Sigma$ such that, if e_i is an orthonormal basis of V and $\gamma_i = \gamma(e_i)$, we have

$$\gamma_i^2 = -1 \quad , \gamma_i\gamma_j = -\gamma_j\gamma_i \qquad i \neq j$$

We will construct a representation for $\mathbf{R}^n$ by induction. Suppose $n = 2m$ is even and we have defined $S_{2m} = S_{2m}^+ \oplus S_{2m}^-$ such that $\gamma(v)$ maps $S^\pm$ to $S^\mp$ for $v \in \mathbf{R}^{2m}$.
Then we define $S_{2m+1} = S_{2m}^+ \oplus S_{2m}^-$ and

$$\gamma_{2m+1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Now given S_{2m+1} we define $S_{2m+2}^+ = S_{2m+2}^- = S_{2m+1}$ with

$$\gamma_{2m+2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus for each n we have a representation of $\mathfrak{so}_n$ on a complex vector space $S = S_n$. This defines a representation of the double cover $\mathrm{Spin}(n)$ of $SO(n)$. If we write $c \in \mathrm{Spin}(n)$ for the non-trivial element in the kernel of $\mathrm{Spin}(n) \to SO(n)$ then c acts as -1 on S.

Corollary

Suppose we have two orthonormal bases e_i, e'_i for V defining the same orientation. Then the inductive procedure above defines vector spaces S, S' using these bases. Up to sign there is a canonical isomorphism from S to S'.

Elements of S are called spinors.

For n even, the action of $Cl(V)$ on S induces an isomorphism

$$\gamma : Cl(V) \otimes \mathbf{C} \to \mathrm{End}\, S.$$

We have $\mathrm{Spin}(V) \mapsto \mathrm{End}(S)$ and this is the composite of γ an embedding $\mathrm{Spin}(V) \subset Cl(V)$. This is image of the exponential map

$$\exp_{Cl} : \Lambda^2 \to Cl(V)$$

defined by Clifford multiplication. (This statement also holds for n odd.)

In dimension $8k, 8k \pm 1$ the spin space has a real structure.
In dimensions $8k+3, 8k+4, 8k+5$ the spin space has a quaternionic structure.
In dimensions $8k \pm 2$ the spaces S^+, S^- are duals.

In low dimensions there are isomorphisms, defined by the spinor representations:

- $\mathrm{Spin}(2) = S^1$.
- $\mathrm{Spin}(3) = SU(2) = Sp(1)$
- $\mathrm{Spin}(4) = SU(2) \times SU(2) = Sp(1) \times Sp(1)$
- $\mathrm{Spin}(5) = Sp(2)$
- $\mathrm{Spin}(6) = SU(4)$.

In dimensions $7, 8$ there are special phenomena, related to exceptional holonomy $G_2 \subset SO(7), \mathrm{Spin}(7) \subset SO(8)$.

The Dirac operator

Go back to $\mathbf{R}^n$ and $\gamma_i : S \to S$ for $i = 1, \ldots, n$ with $\gamma_i^2 = -1, \gamma_i\gamma_j + \gamma_j\gamma_i = 0,\ i \neq j$. Define a differential operator on S-valued functions on $\mathbf{R}^n$ by

$$D = \sum \gamma_i \frac{\partial}{\partial x_i}.$$

Then

$$D^2 = \Delta = -\sum \frac{\partial^2}{\partial x_i^2}.$$

This is the (Euclidean) Dirac operator.

A spin structure on an oriented Riemannian n-manifold M is a principle $\mathrm{Spin}(n)$ bundle $P \to M$ such that the tangent bundle of M is associated to the representation of $\mathrm{Spin}(n)$ on $\mathbf{R}^n$. The Levi-Civita connection induces a connection on P.

Given P, we have a complex vector bundle of spinors $S \to M$ associated to the spin representation of $\mathrm{Spin}(n)$ and a Clifford multiplication map $\gamma : TM \to \mathrm{End}(S)$ which we can also write as

$$C : TM \otimes S \to S.$$

There is a covariant derivative $\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$.
The Dirac operator is the composite $D = C \circ \nabla$ (identifying TM with T^*M using the metric) which gives

$$D : \Gamma(S) \to \Gamma(S).$$

Since all the bundles involved have metrics, there is an adjoint operator

$$\nabla^* : \Gamma(T^*M \otimes S) \to \Gamma(S).$$

The Lichnerowicz formula is

$$D^2 = \nabla^*\nabla + \frac{\mathrm{scal}}{4},$$

where scal is the scalar curvature of M.

To derive the formula, choose a local orthonormal frame e_i of tangent vectors. At a given point $p \in M$ we can arrange that $\nabla e_i = 0$. Write ∇_i for the covariant derivative in the direction e_i and $\gamma_i : S \to S$ for Clifford multiplication with e_i.
Then $\nabla = \sum \nabla_i \otimes e_i$ and $\nabla_i^* = -\nabla_i + \mathrm{div}(e_i)$. So, at the point p,

$$\nabla^* \nabla = -\sum \nabla_i^2 .$$

We have $D = \sum \gamma_i \nabla_i$ so at the point p

$$D^2 = \sum \gamma_i \gamma_j \nabla_i \nabla_j$$

This is (at p):

$$D^2 = -\sum \nabla_i^2 + \sum_{i<j} \gamma_i \gamma_j [\nabla_i, \nabla_j].$$

Using again that ∇e_i vanishes at p we have

$$[\nabla_i, \nabla_j] = \rho(F_{ij})$$

where F is the curvature of P and ρ is the Lie algebra action on S.

In terms of the curvature R_{ijkl}, this is

$$D^2 = \nabla^* \nabla + \frac{1}{8} \sum R_{ijkl} \gamma_i \gamma_j \gamma_k \gamma_l .$$

Another way of saying this is that we have the curvature $\mathrm{Riem} \in \Lambda^2 \otimes \Lambda^2$ and the algebraic term is the image of this under Clifford multiplication.

Clifford multiplication on $\Lambda^2 \otimes \Lambda^2$ has three components:

- The wedge product $\Lambda^2 \otimes \Lambda^2 \to \Lambda^4$. This vanishes on Riem by the Bianchi identity.
- A component $\Lambda^2 \otimes \Lambda^2 \to \Lambda^2$. This is skew-symmetric so also vanishes on $\mathrm{Riem} \in s^2(\Lambda^2)$.
- A component $\Lambda^2 \otimes \Lambda^2 \to \Lambda^0$. Up to a factor 2, this is the contraction of Riem giving the scalar curvature.

More generally, let $E \to M$ be a Hermitian vector bundle with connection. Then we have

$$D_E : \Gamma(E \otimes S) \to \Gamma(E \otimes S)$$

and a formula (schematically)

$$D_E^2 = \nabla^* \nabla + \frac{\mathrm{scal}}{4} + F_E.$$

Section 7: The index of the Dirac operator
Subsection 7.1

From now on, suppose that M is compact and has even dimension $2m$. Then $S = S^+ \oplus S^-$ and the Dirac operator is the sum of

$$D^+ : \Gamma(S^+) \to \Gamma(S^-)$$

and its adjoint D^-. More generally we can couple to a Hermitian vector bundle E giving D_E^+.
This is an elliptic operator and by general theory has an index

$$\text{ind } D_E^+ = \dim \ker D_E^+ - \dim \ker D_E^-.$$

which is a deformation invariant, independent of the choice of metric on M and connection on E.

The *Atiyah-Singer Index Theorem* gives a formula

$$\mathrm{ind}\ D_E^+ = \int_M \hat{A}(M)\mathrm{ch}\,(E),$$

where

- $\hat{A}(M)$ is a certain power series in the Pontrayagin classes p_i of the manifold M, beginning with

$$1 - \frac{1}{24}p_1 + \left(\frac{-4p_2 + 7p_1^2}{5760}\right) + \dots$$

- $\mathrm{ch}\,(E)$ is a power series in the Chern classes $c_i(E)$, beginning with

$$1 + c_1 + \left(\frac{c_1^2}{2} - c_2\right) + \dots.$$

Taking suitable bundles E, this leads to a number of other famous formulae:

- The Gauss-Bonnet formula, for the index of the operator

$$d + d^* : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M).$$

- The Hirzebruch signature formula, expressing the signature of a manifold of dimension $4k$ in terms of Pontrayagin classes.
- The Riemann-Roch formula, for the holomorphic Euler Characteristic of a holomorphic vector bundle over a compact complex manifold.

Subsection 7.2: Applications to scalar curvature

Combined with the Lichnerowicz formula, the index theorem gives information about manifolds with positive scalar curvature. Most simply:

If M^{4k} has a spin structure and a Riemannian metric with $\mathrm{scal} > 0$ *then*

$$\int_M \hat{A} = 0.$$

Note that $\mathbf{CP}^2$ has $p_1 \neq 0$ and $\mathrm{scal} > 0$ but does not have a spin structure.

A more subtle argument (Gromov and Lawson 1980) gives a proof of the nonexistence of metrics of positive scalar curvature on T^n.

By taking a product with S^1 we can suppose that n is even. For example take $n = 4$.

The tangent bundle of T^4 is trivial, so the Pontrayagin classes vanish.

Let $L \to T^4$ be a complex line bundle with $c_1^2(L) \neq 0$. Then the index of D_L^+ is non zero.

Let $\alpha_1, \ldots \alpha_4$ be a standard basis of closed 1-forms on T^4. If $[\alpha_i] \in H^1(T^4)$ are integral classes then there is a Hermitian line bundle L with connection having curvature

$$(2\pi)^{-1}(\alpha_1\alpha_2 + \alpha_3\alpha_4),$$

and the index of D_L^+ is non-zero.

For $\nu >> 0$ let $\pi : T^4 \to T^4$ be the ν^4 covering map so that the

$$\pi^*(\nu^{-1}\alpha_i)$$

define integral classes.

Suppose that T^4 has a metric g with $\mathrm{scal} \geq \epsilon > 0$.

Then $\pi^*(g)$ also has $\mathrm{scal} \geq \epsilon$ and there is a line bundle $\tilde{L} = \pi^*(L)^{\nu^{-2}}$ with $|F| \leq C\nu^{-2}$ such that $\mathrm{ind} D^+_{\tilde{L}} \neq 0$.

Taking $\nu >> \epsilon^{-1/2}$ we get a contradiction to the Lichnerowicz formula.

7.3 The heat equation and the index theorem: background

The appearance of the Chern character in the index formula

$$\mathrm{ind}\ D_E^+ = \int_M \hat{A}(M)\mathrm{ch}\,(E),$$

is not surprising. By general results in algebraic topology, for a fixed compact manifold M any map I from complex vector bundles over M to the integers such that

$$I(E_1 \oplus E_2) = I(E_1) + I(E_2)$$

can be expressed as

$$I(E) = \int_M a(M)\mathrm{ch}\ E,$$

for some $a(M) \in H^*(M)$.

We will focus attention on the case of trivial bundles and the class $\hat{A}(M)$.

Recall the definition on $\hat{A}$. Let $f(z)$ be the function

$$f(z) = \left(\frac{z/2}{\sinh(z/2)}\right)^{1/2}.$$

This is an even function with $f(0) = 1$. Take variables λ_a and consider

$$\prod_a f(\lambda_a)^2.$$

This can be written as a function $A(\sigma_1, \sigma_2, \dots)$ of the elementary symmetric functions of the λ_a^2:

$$\sigma_1 = \sum \lambda^2 \ , \sigma_2 = \sum \lambda_a^2 \lambda_b^2 \ \dots .$$

Now set $\hat{A} = A(p_1, p_2, \dots)$ where p_i are the Pontrayagin classes.

Thus we have

$$f(z)^2 = 1 - \frac{z^2}{24} + cz^4 + \dots,$$

where $c = \frac{1}{(24)^2} - \frac{1}{2^4 5!}$, which gives

$$\prod_a f(\lambda_a)^2 = 1 - \frac{1}{24}\sum \lambda_a^2 + c(\sum \lambda_a^2)^2 + \left(\frac{1}{(24)^2} - 2c\right)\sum \lambda_a^2 \lambda_b^2 + \dots$$

which gives

$$\hat{A} = 1 - \frac{p_1}{24} - \left(\frac{-4p_2 + 7p_1^2}{5760}\right) + \dots.$$

Chern-Weil theory expresses $\hat{A}(M)$ in terms of curvature. Given a Riemannian metric, we define a differential form

$$\underline{\hat{A}} = \det f\left(\frac{R}{4\pi i}\right), \qquad (***)$$

where R is the curvature tensor. This formula is understood in the following way. For $\xi \in \mathfrak{so}(n)$ we can form $\det\, f(\xi/4\pi i)$ as an invariant function on the Lie algebra.
Now think of R as a 2-form with values in a bundle of Lie algebras and define (***) by using multiplication on 2-forms.

Then it is a fact that $\underline{\hat{A}}$ is a closed form representing $\hat{A}$.

Now we make a digression to review some theory of the heat equation on a compact Riemannian manifold M. This is the equation

$$\partial_t f = -\Delta f.$$

The solution can be written as $f = \exp(-t\Delta)f_0$ where f_0 is teh initial value.

This has an integral representation

$$f(x,t) = \int k_t(x,y) f_0(y) dy$$

where $k_t(\ ,y)$ is the fundamental solution of the heat equation with limit δ_y as $t \to 0$.

There is an orthonormal basis ϕ_λ of eigenfunctions for Δ on M

$$\Delta\phi_\lambda = \lambda\phi_\lambda$$

and

$$k_t(x,y) = \sum e^{-\lambda t}\phi_\lambda(x)\phi_\lambda(y)$$

The fundamental solution of the heat equation on $\mathbf{R}^n$ is

$$K_t(|x-y|) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}.$$

This can be used as a starting point in the construction of an asymptotic solution on the manifold M.

Given $y \in M$, let $r(x) = d(x, y)$. Then $K_t(r)$ is a solution of the heat equation on M if it happened that $\Delta r^2 = -2n$.

In fact, writing $E = \Delta r^2 + 2n$, one computes

$$(\partial_t + \Delta) K_t(r) = -\frac{1}{4t} E K_t.$$

So in general we have

$$(\partial_t + \Delta) K_t(r) = O(t^{-(n/2+1)}).$$

We seek to improve this by considering uK_t, for a function u on M.

Then
$$(\partial_t + \Delta)uK_t = \Delta u\ K_t - \frac{u}{4t}EK_t - 2\nabla u.\nabla K_t.$$

Working in polar coordinates,
$$\nabla K_t = -\frac{rK_t}{2t}\partial_r$$

so
$$-2\nabla u.\nabla K_t = \frac{rK_t}{t}\frac{\partial u}{\partial r}.$$

If we can solve the equation for u,

$$r\frac{\partial u}{\partial r} = \frac{u\,E}{4}, \qquad (*****)$$

then

$$(\partial_t + \Delta)(uK_t) = O(t^{-n/2}).$$

The geometry of the exponential map gives a solution to the equation (*****)
The formula for the Laplacian in a general coordinate system is

$$\Delta f = -\frac{1}{\sqrt{g}}\sum \partial_i(g^{ij}\partial_j f),$$

where $\sqrt{g}$ is the volume form in these coordinates.

Using geodesic polar coordinates we get

$$-\Delta r^2 = 2 + 2r\frac{1}{\sqrt{g}}\partial_r\sqrt{g}.$$

Set $J = \frac{\sqrt{g}}{r^{n-1}}$. Then

$$\frac{\partial J}{\partial r} = J\left(g^{-1/2}\partial_r g^{1/2} - \frac{n-1}{r}\right).$$

This leads to the formula

$$\partial_r J = \frac{J\,E}{2}.$$

Finally we see that $u = J^{-1/2}$ is a solution to (*****).

The function $J = J(x, y)$ is the determinant of the exponential map $\exp_y$ with respect to the volume form on TM_y and the Riemannian volume form on M, at the point $v \in TM_y$ with $\exp_y(v) = x$.

Our improved approximation to $k_t(x, y)$ is

$$J(x, y)^{1/2} K_t = J(x, y)^{-1/2} \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t}.$$

We can continue in the same way to find functions $u_p(x, y)$, such that

$$\left(u_0 + tu_1 + t^2 u_2 + \dots \right) K_t$$

is a formal solution of the heat equation.

At stage p we choose u_p to cancel the $O(t^{-n/2+p-1})$ error term.

This involves solving a radial ODE, as for $u_0 = J^{1/2}$. Any finite jet of the u_p can be found, in principle, by a mechanical algebraic procedure in terms of pointwise invariants of the Riemannian manifold (curvature and iterated covariant derivatives of curvature).

Some analysis shows that these formal solutions do give asymptotic descriptions of the true solution $k_t(x, y)$.

The *trace* of $\exp(-\Delta t)$ has a spectral description

$$\sum e^{-\lambda t},$$

and an integral description

$$\int_M k_t(x,x)dx.$$

In particular

$$\operatorname{Tr}\exp(-t\Delta) \sim (4\pi t)^{-n/2}(I_0 + t\ I_1 + t^2\ I_2 \dots)$$

where $I_p = \int_M u_p(x,x)dx$.

These two descriptions were used by Weyl to get an asymptotic formula for the distribution of eigenvalues λ.

This concludes our digression on the heat equation.

Now go back to our Dirac operator

$$D^+ : \Gamma(S^+) \to \Gamma(S^-),$$

over a compact manifold M^{2m}. The adjoint is D^- and $D = D^+ + D^-$. Write $\Delta^\pm$ for the restrictions of D^2 to $\Gamma(S^\pm)$.

These are operators of the Laplace type and the same theory applies to the associated heat operators $\exp(t\Delta^\pm)$.

If $\Delta^+\phi = \lambda\phi$ then $\Delta^- D^+\phi = \lambda D^+\phi$ and if λ is not zero $D^+\phi \neq 0$.

It follows that the strictly positive spectra of Δ^+, Δ^- are the same.

This means that for any $t > 0$

$$\mathrm{Tr}(\exp(-\Delta^+ t)) - \mathrm{Tr}(\exp(-\Delta^- t)) = \mathrm{index}\ D^+.$$

On the other hand the operators $\exp(-\Delta^\pm t)$ have integral representations with kernels $k_t^\pm(x, y)$ where now

$$k_t^\pm(x, y) \in \mathrm{Hom}\,(S_y^\pm, S_x^\pm)$$

and

$$\mathrm{Tr}(\exp(-\Delta^\pm t) = \int_M \mathrm{tr}\ k_t^\pm(x, x) dx.$$

The right hand side has an asymptotic description in terms of local invariants.
We conclude that

$$\mathrm{ind} D^+ = \int_M \Theta(x) dx,$$

where the index density $\Theta(x)$ is the difference of the t^0 terms in the asymptotics of $\mathrm{tr} k^+(x, x), \mathrm{tr} k^-(x, x)$.

The *Local Index theorem* is the statement that $\Theta \, dx = [\, \underline{\hat{A}} \,]_{2m}$.

Here dx is shorthand for the Riemannian volume form and $[\ ]_{2m}$ is the component in Λ^{2m}. This is only interesting when m is even. For m odd the index of the Dirac operator is zero because the kernel of D^- is the complex conjugate of the kernel of D^+.

Example On a Riemann surface the Dirac operator is the $\bar{\partial}$ operator on $K^{1/2}$ and the vector spaces $H^0(K^{1/2}), H^1(K^{1/2})$ are Serre duals.

The local version implies the global Atiyah-Singer formula, but is a stronger statement.

The first proof of the index theorem was announced by Atiyah and Singer in 1963. That used cobordism theory.

The heat equation approach goes back to McKean and Singer (1967). It was developed into proofs by Patodi (1971), Gilkey (1973), Atiyah-Bott-Patodi (1973). These proofs were rather long and indirect.

A shorter proof, with new insights, was found by Getzler (1986), related to ideas from quantum field theory and supersymmetry.

We will discuss the proof from Chapter 5 of the book *Heat Kernels and Dirac operators* Berline, Getzler, Vergne Springer 2004, which uses an idea of Berline and Vergne (1986).

A new insight is that the function $f(z) = \left(\frac{z/2}{\sinh(z/2)}\right)^{1/2}$ arises from the classical differential geometry of the frame bundle of M

Section 8: Proof of the local index theorem

Set up

M^{2m} is a Riemannian manifold with spin structure.

$\pi : P \to M$ is the $\mathrm{Spin}(2m)$ frame bundle.

$\mathbf{S}_{2m}$ is the spin space (complex dimension 2^m) and
$\rho : \mathrm{Spin}(2m) \to \mathrm{End}\, \mathbf{S}_{2m}$ is the spin representation.

We have a bundle $S \to M$ given by $P \times_{\mathrm{Spin}(2m)} \mathbf{S}_{2m}$.

A linear map $L : \mathbf{S}_{2m} \to \mathbf{S}_{2m}$ has four components $\mathbf{S}^{\pm}_{2m} \to \mathbf{S}^{\pm}$.

The *supertrace* $\mathrm{Str}(L)$ is the trace of $L : \mathbf{S}^{+}_{2m} \to \mathbf{S}^{+}_{2m}$ minus the trace of $L : \mathbf{S}^{-}_{2m} \to \mathbf{S}^{-}_{2m}$.

Similarly for a bundle endomorphism of $S \to M$.

The operator $\exp(-tD^2)$ on $\Gamma(S)$ is represented by a heat kernel

$$k_t^S(x,y) \in \mathrm{Hom}(S_y, S_x).$$

The index density $\Theta(x)$ is the t^0 term in the asymptotic expansion of $\mathrm{Str}\ k_t^S(x,x)$.

Lifting to P, this heat kernel defines, for $p, q \in P$, $\tilde{k}_t^S(p,q) \in \mathrm{End}(\mathbf{S}_{2m})$ with the transformation property

$$\tilde{k}_t^S(pg_1, qg_2) = \rho(g_1^{-1}) \circ \tilde{k}_t^S \circ \rho(g_2),$$

and for any $p \in \pi^{-1}(x)$ we have $\mathrm{Str}\ k_t^S(x,x) = \mathrm{Str}\ \tilde{k}_t^S(p,p)$.

The connection on P defines a standard Riemannian metric on the total space, hence a Laplace operator Δ_P on functions on P and a heat kernel $k_t^P(p, q)$.

For simplicity, we will make the assumption that M has constant scalar curvature $4b$.
Write $\nu = m(2m - 1)$, the dimension of $\mathrm{Spin}(2m)$.

Proposition

$$\tilde{k}_t^S(p, q) = e^{-ct} \int_{\mathrm{Spin}(2m)} k_t^P(p, qg)\, \rho(g^{-1})\, dg,$$

where $c = \nu - b$.

Proof of the Proposition

Step 1 The Lichnerowicz formula gives $D^2 = \nabla_S^* \nabla_S + b$.

Step 2 Write the derivative ∇ on the functions on the total space of P as $\nabla_H + \nabla_V$. We get $\Delta_P = \nabla_H^* \nabla_H + \nabla_V^* \nabla_V$.

Step 3 Identify sections s of $S \to M$ with equivariant $\mathbf{S}_{2m}$-valued functions

$$\tilde{s} : P \to \mathbf{S}_{2m} \qquad \tilde{s}(pg) = \rho(g)^{-1} \tilde{s}(p).$$

Under this identification $\nabla_S^* \nabla_S$ corresponds to $\nabla_H^* \nabla_H$, restricted to equivariant functions.

Step 4
$\nabla_V^*\nabla_V = \nu$ on equivariant functions.

To see this, let X_{ij} be the left-invariant vector fields on $\mathrm{Spin}(2m)$ corresponding to the standard orthonomal basis $e_i \wedge e_j$ of the Lie algebra. Then, identifying a fibre of P with the group,

$$\Delta_V = -\sum \nabla^2_{X_{ij}}.$$

The action of $\nabla_{X_{ij}}$ on an equivariant function is just the Lie algebra action of $(d\rho)(e_i \wedge e_j)$ on $\mathbf{S}_{2m}$. This is $\gamma_i\gamma_j$. So the action of $\nabla^2_{X_{ij}}$ is $\gamma_i\gamma_j\gamma_i\gamma_j = -1$.

(This is an instance of the general theory of the *Casimir operator* associated to any compact Lie group.)

Step 5

So the operator D^2 on $\Gamma(S)$ corresponds to $\Delta_P + b - \nu = \Delta_P + c$ acting on equivariant functions and $\exp(-tD^2)$ corresponds to $e^{-tc}\exp(-t\Delta_P)$.

Step 6

Let s be a section of S supported on a set on which P has a local section $y \mapsto q(y)$. Let $\tilde{s}$ be the equivariant function on P corresponding to s and $\tilde{\sigma}$ the equivariant function corresponding to $\sigma = \exp(-tD^2)s$.

Then

$$\tilde{\sigma}(p) = \int_M \tilde{k}_t^S(p, q(y))\, \tilde{s}(q(y))\, dy, \qquad (**)$$

(and this is independent of the choice of $q(y)$).

The result in Step 5 implies that

$$\tilde{\sigma}(p) = e^{-ct} \int_P k_t^P(p, q)\; \tilde{s}(q)\; dq.$$

Writing $q = q(y)g$, this is

$$\tilde{\sigma}(p) = e^{-ct} \int_M \int_{\mathrm{Spin}(2m)} k_t^P(p, q(y)g)\; \tilde{s}(q(y)g)\; dg\; dy.$$

Using the equivariance of $\tilde{s}$ we get

$$\tilde{\sigma}(p) = e^{-ct} \int_M \left(\int_{\mathrm{Spin}(2m)} k_t^P(p, q(y)g) \rho(g^{-1}) dg \right) \tilde{s}(q(y))\; dy,$$

and, comparing with (**), this establishes the Proposition.

Fix $x \in M$ and $p \in \pi^{-1}(x)$.

Corollary to Proposition

$\Theta(x)$ is the t^0 term in the asymptotic expansion of

$$\int_{\mathrm{Spin}(2m)} e^{-ct}\, k_t^P(p, pg)\, \mathrm{Str}\left(\rho(g^{-1})\right)\, dg.$$

Define $\Theta_0(x)$ by replacing k_t^P by its approximation, so $\Theta_0(x)$ is the t^0 term in the expansion of

$$\int_{\mathrm{Spin}(2m)} e^{-ct}\, J^{-1/2}(p, pg) K_t(r(g))\, \mathrm{Str}\left(\rho(g^{-1})\right)\, dg, \qquad (*)$$

where $r(g)$ is the distance from p to pg.

We will calculate Θ_0 and then explain why it is equal to Θ.

For any fixed $\delta > 0$, the contribution to the integral (*) from the region where $r > \delta$ is $o(t^\mu)$ for all μ so does not affect the asymptotic expansion. Thus we can pull the integral back to the Lie algebra, writing $g = \exp(\xi)$. We will show below that the fibres of P are totally geodesic so $r(\exp(\xi)) = |\xi|$. Write $j(\xi)$ for the determinant of the derivative of the exponential map at ξ and $J(\xi) = J(p, p\exp(\xi))$. So now we are considering

$$\frac{e^{-ct}}{(4\pi t)^{m+\nu/2}} \int_{\Lambda^2} e^{-|\xi|^2/4t} J(\xi)^{-1/2} j(\xi)\, \mathrm{Str}\left(\rho \exp(-\xi)\right) \; d\xi \qquad (**)$$

Consider an integral

$$I(f,t) = \int_{\mathbf{R}^N} e^{-|x|^2/4t} f(x)dx.$$

Then

1. If f is a polynomial of pure degree d then $I(f,t) = I(f,1)t^{N/2+d/2}$;
2. If $f(x) = O(|x|^d)$ then $I(f,t) = O(t^{N/2+d/2})$.

It follows that we can find the asymptotic expansion of $(**)$ by replacing the functions in the integrand by suitable truncations of their Taylor series. So we are reduced to calculating integrals $I(f,t)$ for polynomials f, and hence to algebra.

The geometry of the manifold M only enters (in the formula for Θ_0) through the function $J(\xi)$.

We now consider the term $\mathrm{Str}\,(\rho\exp(-\xi))$.
For a Euclidean vector space V the exterior square Λ^2 is contained in three different algebras:

- The exterior algebra $\Lambda^* V$;
- The Clifford algebra $\mathrm{Cl}(V)$:
- The endomorphism algebra $\mathrm{End}(V)$

Each of these will play a role in our discussions. We will write

$$\tau : \Lambda^2 \to \mathfrak{so}(V) \subset \mathrm{End}(V),$$

normalised so that in dimension 2

$$\tau(e_1 \wedge e_2) = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Recall that we have $\gamma : \mathrm{Cl}(V) \to \mathrm{End}(S)$. We fix the vector space identification of $\mathrm{Cl}(V)$ with Λ^*. Define

$$T : \Lambda^* \to \mathbf{R}$$

to be the projection onto the top-dimensional component, normalised by

$$T(e_1 \wedge \cdots \wedge e_{2m}) = 1.$$

Then one finds that , for $\alpha \in \mathrm{Cl}(V) = \Lambda^*$,

$$\mathrm{Str}(\gamma(\alpha)) = (-2i)^m T(\alpha).$$

Hence

$$\operatorname{Str}(\rho \exp(-\xi)) = (-2i)^m T(\exp_{Cl}(-\xi)),$$

where $\exp_{Cl}$ is the exponential series computed in the Clifford algebra.

We want to relate this to the simpler $\exp_\Lambda(-\xi)$, where $\exp_\Lambda$ is the exponential series computed in the exterior algebra.

The relation we need is the formula

$$T(\exp_{Cl}(\xi)) = T(\exp_{\Lambda}(\xi)) \quad f(\tau(\xi))^{1/2}, \qquad (***)$$

where, for an endormorphism A;

$$f(A) = \det\left(\frac{\sinh(A)}{A}\right).$$

To establish this formula it suffices, by $SO(2m)$-invariance, to consider

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_m,$$

where $\xi_1 = \lambda_1 e_1 \wedge e_2, \xi_2 = \lambda_2 e_3 \wedge e_4, \ldots$. The ξ_i commute in the Clifford algebra so $\exp_{Cl}(\xi) = \exp_{Cl}(\xi_1)\exp_{Cl}(\xi_2)\ldots\exp_{Cl}(\xi_m)$ and similarly for the exterior algebra.

This reduces the problem to the case $m = 1$ and $\xi = \lambda e_1 \wedge e_2$.
Then, in the Clifford algebra, $\xi^2 = -\lambda^2$ and
$\exp_{Cl}(\xi) = \cos\lambda + \sin\lambda\ e_1\wedge_2$ so

$$T(\exp_{Cl}(\xi)) = \sin\lambda,$$

while $T(\exp_\Lambda(\xi)) = T(1+\xi) = \lambda$.
Now $\tau = \tau(\xi) = \lambda J$ for

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with $J^2 = -1$. So

$$\frac{\sinh(\tau)}{\tau} = 1+\tau^2/3!+\tau^4/5!+\cdots = 1-\lambda^2/3!+\lambda^4/5!+\cdots = \frac{\sin\lambda}{\lambda},$$

and

$$\det\left(\frac{\sinh(\tau)}{\tau}\right) = \left(\frac{\sin\lambda}{\lambda}\right)^2.$$

Notice that $T(\exp_\Lambda(\xi)) = T(\xi^m/m!)$ is the Pfaffian $\mathrm{Pfaff}(\xi)$.

For a function $F(\xi)$ on Λ^2 define

$$I_t(F) = \frac{1}{(4\pi t)^{m+\nu/2}} \int_{\Lambda^2} e^{-|\xi|^2/4t} F(\xi)\mathrm{Pfaff}(\xi) d\xi \quad .$$

So if $F_0(\xi) = J(\xi)^{-1/2} j(\xi) f(\xi)^{1/2}$ the number Θ_0 is the t^0 term in the expansion of $(-2i)^m e^{-ct} I_t(F_0)$.

The Pfaffian has degree m so if F is a polynomial of degree d then $I_t(F) = O(t^{d-m}/2)$. Thus for a general function F only terms of order $\leq m$ contribute to the t^0 term in $I_t(F)$.

Observation 1

In standard co-ordinates x_i on $\mathbf{R}^N$, if G is a polynomial of degree $< \mu$ then

$$\int_{\mathbf{R}^N} e^{-|x|^2/4} G(x)\, x_1 \dots x_\mu \, dx = 0.$$

On Λ^2 we have co-ordinates ξ_{ij} for $i < j$. The Pfaffian is a sum of monomials

$$\pm \xi_{i_1 j_1} \xi_{i_2 j_2} \dots \xi_{i_m j_m},$$

over partitions $(i_1 j_1)(i_2 j_2) \dots (i_m j_m)$ of $(1 \dots 2m)$.
It follows that $I_t(F) = 0$ for polynomials F of degree $< m$.

So for any F the expansion of $I_t(F)$ has no terms in negative powers of t.

Thus

$$\Theta_0 = \lim_{t \to 0} (-2i)^m \;\; I_t(F_0).$$

Now go back to the statement: $\Theta(x)$ is the t^0 term in the asymptotic expansion of

$$\int_{\mathrm{Spin}(2m)} e^{-ct} \, k_t^P(p, pg) \, \mathrm{Str}\left(\rho(g^{-1})\right) \, dg.$$

We have an asymptotic expansion of $k_t^P(p, pg)$ in powers of t:

$$\left(J^{-1/2}(p, pg) + tU_1(g) + t^2 U_2(g) + \dots\right) K_t(r(g)).$$

The whole discussion above can be applied to each term and we see that the higher terms U_i *do not contribute to the* t^0 *term in expansion of the integral*.

So $\Theta(x) = \Theta_0(x)$.

To summarise progress so far, we have a formula

$$\Theta = \frac{(-2i)^m}{(4\pi)^{\nu/2+m}} \int_{\Lambda^2} e^{-|\xi|^2/4} F_i(\xi) \mathrm{Pfaff}(\xi) d\xi,$$

where $F_1(\xi)$ is the degree m term in the Taylor series of $F_0(\xi) = J^{-1/2}(\xi) j(\xi) f^{1/2}(\xi)$.

Observation 2
In standard co-ordinates x_i on $\mathbf{R}^N$, if G is a standard monomial of degree μ then

$$\int_{\mathbf{R}^N} e^{-|x|^2/4} G(x)\ x_1 \dots x_\mu\ dx$$

is zero unless $G = x_1 \dots x_\mu$ in which case the integral is $2^\mu (4\pi)^{N/2}$.

The polynomial functions on Λ^2 can be identified with the symmetric powers $s^*(\Lambda^2)$. On the other hand we have a ring homomorphism induced by wedge product

$$s^*(\Lambda^2) \to \Lambda^{\text{even}}.$$

So for any polynomial function $F(\xi)$ we have a $Q(F) \in \Lambda^{\text{even}}$. This extends in an obvious way to general smooth functions on Λ^2.

Explicitly, we have standard coordinates ξ_{ij} on Λ^2 and Q is defined by substituting $e_i \wedge e_j$ for ξ_{ij}.
For example

- $Q(\xi_{12}^2) = 0$.
- $Q(\xi_{12}\xi_{13}) = 0$
- $Q(\xi_{12}\xi_{34}) = e_1 \wedge e_2 \wedge e_3 \wedge e_4$

Now Observations 1,2 and a little thought show that for a polynomial F

$$\int_{\Lambda^2} e^{-|\xi|^2/4} F(\xi) \mathrm{Pfaff}(\xi) d\xi = \mathrm{Const.} TQ(F), \qquad (****)$$

(which only involves the degree m part of F).

For example consider the case $m = 2$. Then

$$\mathrm{Pfaff} = \xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23}.$$

If F is a degree 2 polynomial, in multi-index notation

$$\sum_I a_I \xi^I$$

then the integral in (****) is

$$a_{12,34} - a_{13,24} + a_{14,23}$$

the 4-form $Q(F)$ is

$$a_{12,34}(e_1 \wedge e_2) \wedge (e_3 \wedge e_4) + a_{13,24}(e_1 \wedge e_3) \wedge (e_2 \wedge e_4) + a_{14,23}(e_1 \wedge e_4) \wedge (e_2 \wedge e$$

which is $(a_{12,34} - a_{13,24} + a_{14,23})e_1 \wedge e_2 \wedge e_3 \wedge e_4$.

So now we have

$$\Theta = \mathrm{Const.}\, T\ Q(F)$$

where $F(\xi) = J(\xi)^{-1/2} j(\xi) f^{1/2}(\xi)$.

This is simplified by the following observation. The group $O(2m)$ acts on Λ^2 and preserves the functions $j, f^{1/2}$. So $Q(j), Q(f^{1/2})$ are forms in Λ^* invariant under $O(2m)$. But the only such forms are multiples of 1 (easy proof by induction on dimension). It follows that $Q(j) = Q(f^{1/2}) = 1$ and since Q is a ring homomorphism we have

$$\Theta = \mathrm{Const.}\, T\left(Q(J^{-1/2})\right).$$

Our problem is reduced to finding the function $J(\xi)$ on Λ^2 and computing $Q(J^{-1/2})$.

The Riemannian geometry of the frame bundle P

The connection gives a decomposition $TP = V \oplus H$ where $H = \pi^*(TM)$.

Recall the formula for the Levi-Civita connection on a Riemannian manifold in terms of orthonormal vector field X, Y, Z:

$$\nabla_X Y.Z = \frac{1}{2}\left([X, Y].Z - [X, Z].Y + [Y, Z].X\right).$$

On the bundle P we consider orthonormal vector fields

- vertical fields v generating the group action, corresponding to elements $\xi \in \mathfrak{g} = \mathfrak{so}(2m)$;
- horizontal field $\tilde{h}$; horizontal lifts of vector fields h on M, with respect to the connection

Then we have

- $[v_1, v_2]$ is vertical, given by the bracket in the Lie group $\mathfrak{g}$;
- $[v, \tilde{h}] = 0$;
- $[\tilde{h}_1, \tilde{h}_2]$ has a horizontal component equal to the horizontal lift of $[h_1, h_2]$ and a vertical component given by the curvature R.

In this way one can compute the Levi-Civita connection on the total space P.

One finds that for any such vertical field v the covariant derivative $\nabla_v v$ is zero, which shows that the fibres are totally geodesic submanifolds, as we mentioned earlier.
Fix $p \in P$ as before and $\xi \in \mathfrak{g}$ corresponding to a vertical vector field v.
The derivative of the Riemannian exponential map is a linear map

$$L_\xi : TP_p \to TP_{p\exp(\xi)}.$$

The totally geodesic property of the fibres implies that L_ξ preserves the horizontal and vertical subspaces.

The determinant of L_ξ in the vertical component is the function $j(\xi)$ that we saw before.

We need to compute the determinant J_H of the horizontal component of L_ξ.
The Riemann curvature R_M of M lies in $\mathfrak{g} \otimes \Lambda^2$.
The inner product $R_M.\xi$ lies in Λ^2. Let $A = A_\xi$ be the corresponding element $\tau(R_M.\xi) \in \mathfrak{g} \subset \mathrm{End}\ TM$.
One finds that

$$\nabla_v \tilde{h} = A(\tilde{h}).$$

Write Riem_P for the Riemann curvature tensor of the total space P.
We have

$$\mathrm{Riem}_P(v, \tilde{h})v = [\nabla_v, \nabla_{\tilde{h}}]v,$$

because $[v, \tilde{h}] = 0$.
Since $\nabla_v v = 0$ and, again, $[\tilde{h}, v] = 0$ this gives

$$\mathrm{Riem}_P(v, \tilde{h})v = \nabla_v \nabla_v \tilde{h},$$

which implies that

$$\mathrm{Riem}_P(v, \tilde{h})v = A^2(\tilde{h}).$$

Conclusion The Jacobi equation along the geodesic $p\exp(t\xi)$ in P, restricted to horizontal variations and with respect to a covariant constant frame along the geodesic is

$$\frac{d^2}{dt^2}w = A^2\ w,$$

where $A = R_M.\xi$

This is a constant co-efficient ODE and we have explicit solutions. One finds that

$$J_H(\xi) = \det\left(\frac{\sinh A}{A}\right),$$

with $A = A_\xi = \tau(R_M.\xi)$.

Last step

For the same reason as before we can drop the vertical component $j(\xi)$ in $J(\xi)$ and we have

$$\Theta = \mathrm{Const.}\ T\ Q(J_H^{-1/2})$$

where

$$J_H(\xi) = \det\left(\frac{\sinh A_\xi}{A_\xi}\right),$$

and $A_\xi = \tau(R_M.\xi)$.
We want to show that $\Theta = [\ \hat{A}\ ]_{2m} \qquad (*)$.
Th equality follows from the symmetry of the Riemann curvature tensor $R_M \in \Lambda^2 \otimes \Lambda^2$.

Let $\sigma(\xi)$ be the degree m polynomial function on Λ^2 defined by the order m term in the Taylor series of the function

$$\det^{1/2}\left(\frac{A}{\sinh A}\right),$$

where $A(\xi) = \tau(\xi)$. Then given $\rho \in \Lambda^2 \otimes \Lambda^2$ we can form $\sigma_1(\rho) \in \mathbf{R} \otimes \Lambda^{2m}$ by applying σ to the first factor and wedge product in the second factor. Or we can form $\sigma_2(\rho) \in \Lambda^{2m} \otimes \mathbf{R}$ by switching the factors.

With $\rho = R_M$ the two constructions give the two sides of (*).

Since R_M is in $s^2(\Lambda^2)$ they give the same result.