A PROBLEM IN 4-MANIFOLD TOPOLOGY

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This is not a new problem, it has been well-known to 4-manifold specialists for the 20 years since the paper [1] of Fintushel and Stern , which is our basic reference. (Other good background references include [2] and [4].) The question involves a simple topological construction, *knot surgery*, introduced by Fintushel and Stern, involving a compact 4-manifold M and a knot K (i.e. an embedded circle in the 3-sphere S^3). We assume that there is an embedded 2-dimensional torus T in M with trivial normal bundle. We fix an identification of a neighbourhood N of T in M with a product $D^2 \times T$, where D^2 is the 2-dimensional disc . Thus the boundary of N is identified with the 3-dimensional torus $T^3 = S^1 \times T = S^1 \times S^1 \times S^1$. Likewise, a tubular neighbourhood ν of the knot K in S^3 can be identified with $D^2 \times K$, with boundary $S^1 \times K = S^1 \times S^1$. Thus the product $Y_K = (S^3 \setminus \nu) \times S^1$ has the same boundary, a 3-torus, as the complement $M \setminus N$ and we define a new compact 4-manifold

$$M_{K,\phi} = (M \setminus N) \cup_\phi Y_K,$$

where the notation means that the two spaces are glued along their common boundary using a diffeomorphism $\phi : \partial N \to \partial Y_K$. This map ϕ is chosen to take the circle ∂D^2 in the boundary of N, which bounds a disc in N, to the "longitude" in the boundary of ν, which is distinguished by the fact that it bounds a surface in the complement $S^3 \setminus \nu$. This condition does not completely fix ϕ but for the case of main interest here it is known that the resulting manifold is independent of the choice of ϕ, so we just write M_K. For the trivial knot K_0 the complement $S^3 \setminus \nu$ is diffeomorphic to $S^1 \times D^2$, so Y_{K_0} is the same as N and M_{K_0} is the same as M—-the construction just cuts out N and then puts it back again.

The general problem is: *for two knots K_1, K_2 , when is the 4-manifold M_{K_1} diffeomorphic to M_{K_2}?* But there is no need to be so ambitious so we can ask: *can we find interesting examples of M, K_1, K_2 such that M_{K_1} and M_{K_2} either are, or are not, diffeomorphic?*

The simplest way in which one might detect the effect of this knot surgery is through the fundamental group. For a non-trivial knot K, the fundamental group of the complement $S^3 \setminus \nu$ is a complicated nonabelian group, but it has the property that it is normally generated by the loops in the boundary 2-torus. That is, the only normal subgroup of $\pi_1(S^3 \setminus \nu)$ which contains $\pi_1(\partial\nu)$ is the whole group. It follows that if the complement $M \setminus T$ is simply connected then the same is true of M_K. In particular, this will be true if M is simply connected and there is a 2-sphere Σ in M which meets T transversely in a single point. From now on we restrict attention to the case when the 4-manifold M is the 4-manifold underlying a complex *K3 surface* X and $T \subset K$ is a complex curve. Regarded as complex manifolds there is a huge moduli space of K3 surfaces (only some of which contain complex curves) but it is known that all such pairs (X, T) are equivalent up to diffeomorphism. For one explicit model we could take X to be the quartic surface in $\mathbf{CP}^3$ defined by the equation

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0.$$

If $\kappa \in \mathbf{C}$ is a fourth root of -1 then the line L defined by the equations $z_1 = \kappa z_0, z_3 = \kappa z_2$ lies in X and for a generic plane Π through L the intersection of X with Π is the union of L and a smooth plane curve of degree 3. It is well-known that smooth plane cubics are (as differentiable manifolds) 2-dimensional tori, so this gives our torus $T \subset X$, which one can check has trivial normal bundle. Using the manifest symmetries of X we can find another line L' in X which is skew to L , and then L' meets T in just one point. A standard general result in complex algebraic geometry (the Lefschetz hyperplane theorem) shows that X is simply connected and since L' is a 2-sphere (as a differentiable manifold) we see that $X \setminus T$ is simply connected. There are many other possible models for (X, T) that one can take, for example using the “Kummer construction” via the quotient of a 4-torus by an involution.

To set our problem in context we recall that, in 1982, Freedman obtained a complete classification of simply connected 4-manifolds up to *homeomorphism*: everything is determined by the homology. At the level of homology all knot complements looks the same and it follows that all the manifolds X_K are homeomorphic to the K3 surface X. By contrast the classification up to *diffeomorphism*, which is the setting for our problem, is a complete mystery. The only tools available come from the Seiberg-Witten equations which yield the *Seiberg-Witten invariants*. Ignoring some significant technicalities, these invariants of a smooth 4-manifold M take the form of a finite number of distinguished classes (“basic classes”) in the homology $H_2(M)$, with for each basic class β a non-zero integer $SW(\beta)$. So there is a way to show that 4-manifolds are not diffeomorphic, by showing that their Seiberg-Witten invariants are different, but if the Seiberg-Witten invariants are the same one has no technique to decide if the manifolds are in fact diffeomorphic, except for constructing a diffeomorphism by hand, if such exists. The special importance of the K3 surface X appears here in the fact that it has the simplest possible non-trivial Seiberg-Witten invariant: there is just one basic class $0 \in H_2(X)$ and $SW(0) = 1$.

The main result of Fintushel and Stern in [1] is a calculation of the Seiberg-Witten invariants of the knot-surgered manifolds X_K .To explain their result we need to recall the *Alexander polynomial* of a knot K. While the knotting is invisible in the homology of the complement $S^3 \setminus \nu$ we get something interesting by passing to the infinite cyclic cover. The action of the covering transformations makes the 1-dimensional homology of this covering space a module over the group ring of $\mathbf{Z}$, which is the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$ of Laurent series with integer co-efficients. One finds that this is a torsion module Λ/I, for a principal ideal $I \subset \Lambda$ and the generator of this ideal I gives the Alexander polynomial $p_K \in \Lambda$. From this point of view p_K is defined up to multiplication by a unit in Λ but there is a way to normalise so that

$$p_K(t) = a_0 + \sum_{i=1}^{g} a_i(t^i + t^{-i}),$$

for integers a_i with $a_0 + 2\sum_{i=1}^{g} a_i = 1$.

Fintushel and Stern show that X_K has basic classes $\pm 2i[T]$, where $[T]$ is the homology class of a "parallel" copy of T in the complement $X \setminus N$ (which is contained in all X_K) and $SW(2i[T]) = a_i$. In other words, the Seiberg-Witten invariants capture exactly the Alexander polynomial of K. It is easy to construct distinct knots with same Alexander polynomial, so our question becomes: *if K_1, K_2 are knots with the same Alexander polynomial are the 4-manifolds X_{K_1}, X_{K_2} diffeomorphic?*

As we have outlined, this question is a prototype—in an explicit and elementary setting—for the fundamental mystery of four-dimensional differential topology. There are also important connections with symplectic topology. A knot is called "fibred" if there is a fibration $\pi : S^3 \setminus \nu \to S^1$, extending the standard fibration on the 2-torus boundary. The fibre S is the complement of a disc in a compact surface of genus g and in this case the Alexander polynomial is just t^{-g} times the characteristic polynomial of the action of the monodromy on $H_1(S)$. In particular the polynomial is "monic", with leading co-efficent a_g equal to ± 1. On the other hand there are knots K with monic Alexander polynomial which are not fibred and distinct fibred knots may have the same Alexander polynomial. If K is fibred then one can construct a symplectic structure ω_K on X_K. Conversely if X_K has a symplectic structure then results of Taubes on Seiberg-Witten invariants, combined with the calculation of Fintushel and Stern, show that p_K must be monic. So we have further questions such as

(1) *If p_K is monic but K is not fibred, does X_K admit a symplectic structure?*
(2) *If K_1, K_2 are fibred knots and (X_{K_1}, ω_{K_1}) is symplectomorphic to (X_{K_2}, ω_{K_2}) are K_1, K_2 equivalent ?*

Another question in the same vein as (1) is whether a 4-manifold $S^1 \times Z^3$ admits a symplectic structure if and only if the 3-manifold Z^3 fibres over the circle. This was proved by Friedl and Vidussi [3] and by Kutluhan and Taubes [5] (with an extra technical assumption).

If we take the product $X_K \times S^2$ we move into the realm of high-dimensional geometric topology: the subtleties of 4-dimensions disappear and all the manifolds are diffeomorphic. But in the symplectic theory there are still interesting questions:

- *For which fibred knots K_1, K_2 are $(X_{K_i} \times S^2, \omega_{K_i} + \omega_{S^2})$ symplectomorphic?*

It seems likely that the Alexander polynomials must be the same, using Taubes' result relating the Seiberg-Witten and Gromov-Witten invariants.

[1] R. Fintushel and R. Stern *Knots,links and 4-manifolds* Inventiones Math. 134 (1998) 363-400

[2] R, Fintushel and R. Stern *Six lectures on 4-manifolds* in *Low dimensional topoology* IAS/Park City Math. Series Vol.15 Amer. Math. Soc. 2009

[3] S. Friedl and S. Vidussi *Twisted Alexander polynomials detect fibered 3-manifolds* Annals of Math. 173 (2011) 1587-1643

[4] R. Gompf and A. Stipsicz *Four-manifolds and Kirby calculus* Grad. Studies in Math. Amer. Math. Soc. (1999)

[5] C. Kutluhan and C. Taubes *Seiberg-Witten Floer homology and symplectic forms on* $S^1 \times M^3$ Geometry and Topology 13 (2009) 493-525